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Research Article

Dynamics in Bank Crisis Model

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Bank crisis is grabbing more serious attention as several financial turmoils have broken out in the past several decades, which leads to a number of researches in this field. Comparing with researches carried out on basis of degree distribution in complex networks, this paper puts forward a mathematical model constructed upon dynamic systems, for which we mainly focus on the stability of critical point. After the model is constructed to describe the evolution of the banking market system, we devoted ourselves to find out the critical point and analyze its stability. However, to refine the stability of the critical point, we add some impulsive terms in the former model. And we discover that the bank crisis can be controlled according to the analysis of equilibrium points of the modified model, which implies the interference from outside may modify the robustness of the bank network.

1. Introduction

The subprime mortgage crisis firstly occurred in 2007, which triggered a severe global recession and caused significant disruption in the finance industry [1]. The financial crisis exposed the existing problems in the financial supervision theory and practice. Being the backbone of the financing industry, banking industry not only imposes significant influence on the economy and society of the country, but also shares an intimate relationship to the lives of ordinary people. And since asset allocation is the main business of financial institutions like commercial banks, owing to this business, complex debt relationships have formed among financial institutions, so the asset allocation risk is highly infectious. When a financial institution makes a mistake on asset allocation and cannot guarantee normal liquidity position, then partial financial difficulty may evolve into a big financial catastrophe. Therefore, the bank crisis has a domino effect [2].

Consequently, exploring banking crisis diffusion and transmission mechanism will be greatly helpful to control banking crisis. Going over the development of banking, after the crisis took place, immediate and accurate crisis appraisal and crisis discovery is exceedingly critical for successful recovery from the crisis. As a global theoretical issue, banking crisis has caught the attention of economists all around the world. But it is not always feasible to use dynamic system and complex network theories to cope with banking crisis [3, 4].

Plenty of systems in the real world, like the metabolic and protein interaction networks, scientific collaboration networks, food chains, and chemical reaction networks, can be regarded as a complex network, in which nodes represent individuals or organizations and edges represent the interactions among them [5, 6]. Moreover, the result can be precise according to some empirical study. The research of dynamics and asymptotical behavior of spreading process conducted in complex networks is of practical importance for the purpose of controlling them, such as computer viruses [7] and sexually transmitted diseases [8]. It is reasonable to suppose that the bank network can be regarded as a complex network based on some empirical research. In fact, structure of bank network has already been considered in the work of many scholars. Boss et al. studied the bank in Austria [9] and Upper and Worms studied Germany's [10].

In our opinion, topology properties are set aside firstly and the whole network will be regarded as homogeneous. In the following part, we always assume that models of bank crisis are similar to models of how infectious diseases spread [11]. There have been many achievements in the field of epidemic model [12, 13]. And the studies mainly focus on system of ordinary differential equations or delay differential equations. Based on known models under dynamical system and considering the characteristics of bank network, we establish epidemic model of bank crisis. All banks N are divided into three parts which are normally functioned banks S, banks with possibility of crisis E, and banks in crisis represented by I. There is an assumption that normally functioned banks have a rate to change into banks with possibility of crisis after getting influence of crisis. At the same time, with the help of better management and adjusting macroeconomic policy, part of E will be cured into normally functioned banks S or banks in crisis I. Moreover, part of banks in crisis will be cleaned out or cured into normally functioned banks. And ones assume that the sum of normally functioned banks $S(t)$, banks with possibility of crisis $E(t)$, and banks in crisis $I(t)$ equals the number of all the banks $N(t)$ at time t.

In this paper, much attention is paid to the system of impulsive differential equation. Firstly, we analyze the characteristic of the spread of bank crisis among homogeneous network and get the condition to ensure the stability of the critical point. Then the impulse is added into the model, which may be a new way to control the banking markets. The analysis is similar to the system without impulse, in which the existence and stability of the considered model are discussed. Impulse can be interpreted as intervention of government or other related bank supervisors in order to control the occurrence or spread of bank crisis. Only periodically, control of banking industry is considered for convenience. In the end, we get the primary consequences through the comparison between the properties of equilibrium points in two different situations and make a plan about future work.

The rest of the paper is organized as follows. And Section 2 is the analysis of the stability of the equilibrium of the original model. Then we consider the critical point of the new model with impulse. In Section 3, the difference between the two situations is discussed, and some strategies are suggested.

2. The *SEI* Model without Protection Mechanisms

Take the following model into account [14]:

$$\begin{aligned} S' &= bN - dS - K\frac{SI}{N} + \gamma I, \\ E' &= K\frac{SI}{N} - dE - \epsilon\frac{EI}{N}, \\ I' &= \epsilon\frac{EI}{N} - (d + \gamma + \alpha) I. \end{aligned} \tag{1}$$

In (1), the meaning of S, E, and I has been explained above. b is the proportion of new-born banks and d is the proportion of the bank suffering bank failure. K is interpreted as the contact rate between S and I, while e is defined as the contact rate between E and I. S/N and E/N are the proportion of normal banks and banks in condition E among all the banks, respectively. $K(SI/N)$ denoted the average number of banks that are infected to be in condition E by all banks in condition I in a unit time, and $\epsilon(EI/N)$ denote the figure for banks that are infected to be in condition I. γ is the proportion of banks in condition I changing into condition S. α is the proportion of banks that is shift out of the system due to bank crisis. The total number of all banks meets the following:

$$N' = (b - d) N - \alpha I. \tag{2}$$

Let $s = S/N$, $e = E/N$, $i = I/N$, and the original system can be transferred into

$$\begin{aligned} s' &= (\alpha - K) si + \gamma i - bs + b \\ e' &= Ksi - be + (\alpha - \epsilon) ei \\ i' &= \alpha i^2 + \epsilon ei - (\gamma + \alpha + b) i \end{aligned} \tag{3}$$

in which s, e, i represent the ratio of banks under normal conditions, banks in period of incubation, and banks under crisis, respectively.

Adding some protection mechanism to the original system simultaneously, we will achieve the following impulsive differential equations:

$$\begin{aligned} s' &= (\alpha - K) si + \gamma i - bs + b \\ e' &= Ksi - be + (\alpha - \epsilon) ei \\ i' &= \alpha i^2 + \epsilon ei - (\gamma + \alpha + b) i \end{aligned} \tag{4}$$

when $t = t_n$

$$\begin{aligned} s(t^+) &= s(t) + (1 - p) e(t) \\ e(t^+) &= pe(t) \\ i(t^+) &= i(t), \end{aligned} \tag{5}$$

where $1 - p$ represents the proportion of the banks diverted from E to I due to financial aid given by some organization. Then it is obvious to verify that $0 < p < 1$. t^+ is a abbreviation of right-hand limit; that is, $f(t^+) = \lim_{x \to t^+} f(x)$.

Here, t_n represents time needed when implementing the protection mechanism. Then the equilibrium points of these two systems are discussed, respectively, and the condition of the stability of the equilibrium is derived. After comparing the discrepancies between these two situations, we draw some consequential conclusions.

Apparently, this system has a non-crisis-state critical point, which is $(1, 0, 0)$. Under some certain conditions, there exists an endemic crisis-state critical point (s^*, e^*, i^*).

Here, we define

$$R_0 = \frac{b(\gamma + \alpha + b)}{K\epsilon}. \tag{6}$$

Theorem 1. *If $R_0 > 1$, then the non-crisis-state critical point of system (3) is locally asymptotically stable.*

Proof. We calculate the corresponding Jacobian matrix of system (3) at the endemic equilibrium point $(1, 0, 0)$,

$$\mathbf{J} = \begin{pmatrix} -b & 0 & \alpha - K + \gamma \\ 0 & -b & K \\ 0 & 0 & -\gamma - \alpha - b \end{pmatrix}. \quad (7)$$

In consideration of its characteristic polynomial

$$(\lambda + b)^2 (\lambda + \gamma + \alpha + b) = 0. \quad (8)$$

Provided that the roots of the polynomial are all negative, the system is locally asymptotically stable. It is easy to achieve the following equations and notice that $R_0 > 1$; we have

$$\begin{aligned} \lambda_1 &= -b \\ \lambda_2 &= -b < 0 \\ \lambda_3 &= -\gamma - \alpha - b < 0. \end{aligned} \quad (9)$$

This completes the proof of Theorem 1. □

Theorem 2. *If $R_1 < 1$, then the endemic crisis-state equilibrium exists. Besides, under the condition $2\alpha - K - \epsilon > 0$ and $(\gamma + \alpha + b)(2\alpha - K - \epsilon) < \alpha b$, the critical point of system (3) is locally asymptotically stable if $F((\gamma + \alpha + b)/\alpha)F(b/(2\alpha - K - \epsilon)) < 0$. (The definition of F is contained in the proof.)*

Proof. When it turns to endemic crisis-state equilibrium, the corresponding polynomial is

$$\begin{aligned} -(b - i\alpha)\Big(i^2\left(-K\alpha + \alpha^2 + K\epsilon - \alpha\epsilon\right) & \\ + i\Big(bK - 2b\alpha + K\alpha - \alpha^2 + K\gamma - \alpha\gamma & \\ + b\epsilon - K\epsilon + \alpha\epsilon + \gamma\epsilon\Big) & \\ + b^2 + b\alpha + b\gamma\Big) = 0. & \end{aligned} \quad (10)$$

The numerical solutions of this equation can be found with the help of the Mathematic software. However, the solutions are too much complex and most of them are of no value. Therefore, we just discuss the sufficient condition to guarantee that the equation is solvable on the interval $[0, 1]$.

Define F as the function corresponding to (10) (with respect to i). Since

$$F(0) = -b^2 (\gamma + \alpha + b) < 0 \quad (11)$$

$$F(1) = -(\gamma + b)\left[(b - \alpha + K)(b - \alpha + \epsilon) - K\epsilon\right]. \quad (12)$$

Now if $F(1) > 0$, this cubic equation is solvable on the interval $(0, 1)$. In the expression of $F(1)$, there exists a concave parabola with respect to b; namely, $g(b) = (b-\alpha+K)(b-\alpha+\epsilon)-K\epsilon$. Notice that $b \in (0, 1)$; it can only achieve the maximum value on the two endpoints of the interval. In fact, we just require the maximum value to be negative. The numerical value of the equation on 0 and 1 is $(\alpha - K)(\alpha - \epsilon) - K\epsilon$ and $(1 - \alpha + K)(1 - \alpha + \epsilon) - K\epsilon$, so we define

$$R_1 = \max\left\{\frac{(\alpha - K)(\alpha - \epsilon)}{K\epsilon}, \frac{(1 - \alpha + K)(1 - \alpha + \epsilon)}{K\epsilon}\right\}. \quad (13)$$

If $R_1 < 1$, then the endemic crisis-state equilibrium exists.

The next target is to discuss the stability of the endemic crisis-state equilibrium point. With the restriction of $s + e + i = 1$, we can simplify the model to two-dimensional one

$$\begin{aligned} e' &= Ki + (\alpha - K - \epsilon) ei - Ki^2 - be \\ i' &= \alpha i^2 + \epsilon ei - (\gamma + \alpha + b) i. \end{aligned} \quad (14)$$

The Jacobi matrix at the critical point (e^*, i^*) is

$$\mathbf{J} = \begin{pmatrix} (\alpha - K - \epsilon) i^* - b & K + (\alpha - K - \epsilon) e^* - 2Ki^* \\ \epsilon i^* & 2\alpha i^* + \epsilon e^* - (\gamma + \alpha + b) \end{pmatrix}. \quad (15)$$

It is easy to verify that

$$\begin{aligned} \operatorname{trace}(J) &= (\alpha - K - \epsilon) i^* - b + 2\alpha i^* + \epsilon e^* - (\gamma + \alpha + b) \\ \det(J) &= \left[(\alpha - K - \epsilon) i^* - b\right]\left[2\alpha i^* + \epsilon e^* - (\gamma + \alpha + b)\right] \\ &\quad - \epsilon i^* \left[K + (\alpha - K - \epsilon) e^* - 2Ki^*\right]. \end{aligned} \quad (16)$$

And to make the critical point stable, the following conditions are satisfied:

$$\begin{aligned} (2\alpha - K - \epsilon) i^* - b &< 0 \\ \left[\alpha(\alpha - K - \epsilon) + K\epsilon\right] i^{*2} - \epsilon b e^* &> 0. \end{aligned} \quad (17)$$

In fact, with union of the stable conditions of system (14) and the two conditions given above, we can get that $\operatorname{trace}(J) < 0$ and $\det(J) > 0$, which ensure that the critical point is stable.

There exists a root of $F(i) = 0$ according to the existence theorem of zero points under the condition $F((\gamma + \alpha + b)/\alpha)F(b/(2\alpha-K-\epsilon)) < 0$. To get a rough range of the solution explicitly, conditions $2\alpha-K-\epsilon > 0$ and $(\gamma+\alpha+b)(2\alpha-K-\epsilon) < \alpha b$ are needed. If they hold, the zero of F can be bounded from (17). This completes the proof of Theorem 2. □

Remark 3. It is reasonable to assume that $2\alpha > K + \epsilon$ if we have found that K and ϵ represent the spreading rate of the crisis. In fact, we make an assumption that the spreading rate of the crisis is quite small to make it more suitable to describe the realistic world.

3. The Model with Impulsive Control

What we shall consider in the next part is the change of the stability of the critical point when there is no crisis, which is caused by the addition of impulse. And we could get the main conclusion of this paper from the new result. Because of the restriction of $s + e + i = 1$, in the later analysis the system consisting of only two variables e and i is considered:

$$\begin{aligned} e' &= Ki + (\alpha - K - \epsilon) ei - Ki^2 - be \\ i' &= \alpha i^2 + \epsilon ei - (\gamma + \alpha + b) i \\ e(t^+) &= pe(t) \\ i(t^+) &= i(t). \end{aligned} \quad (18)$$

Since what we consider is the situation where there is no crisis, let $i = 0$. Then the system is simplified to

$$\begin{aligned} e' &= -be \\ e(t^+) &= pe(t). \end{aligned} \tag{19}$$

Theorem 4. *The non-crisis-state periodic solution of the impulsive system (19) is locally asymptotically stable.*

Proof. Integrating on the interval where there is no impulse, we have

$$e(t) = e(nT^+)e^{-b(t-nT)}, \quad nT < t \le (n+1)T. \tag{20}$$

And then we can construct a stroboscopic map during every impulsive interval, which would be named G:

$$e((n+1)T^+) = G(e(nT^+)) = pe(nT^+)e^{-bT}. \tag{21}$$

G has a singular fixed point, which implies that e has a periodic solution of T. In fact, any solution having the fixed point as initial condition of (19) will be pulled back to the value at the beginning of the period $(nT, (n+1)T)$, which means the behavior of the solution will repeat itself in the last period. In other words, it is a periodic solution. And because the following inequality holds

$$\frac{dG(e(nT^+))}{de(nT^+)} = pe^{-bT} < 1 \tag{22}$$

therefore the only fixed point is stable, which corresponds the fact that the periodic solution is stable.

Considering the local stability of the system composed of e and i, we set $e = e^* + u$ and $i = i^* + v$, where (e^*, i^*) is the equilibrium solution. As $e^* = 0$ and $i^* = 0$, here $e = u$ and $i = v$ and e, i satisfies (18).

When $t \ne t_n$, the linearized system of the equation at the periodic solution is

$$\begin{pmatrix} e' \\ i' \end{pmatrix} = \begin{pmatrix} -b & K \\ 0 & -(\gamma+\alpha+b) \end{pmatrix} \begin{pmatrix} e \\ i \end{pmatrix}. \tag{23}$$

Set $\Psi(t)$ as its fundamental solution matrix, which satisfies

$$\frac{d\Psi(t)}{dt} = \begin{pmatrix} -b & K \\ 0 & -(\gamma+\alpha+b) \end{pmatrix} \Psi(t). \tag{24}$$

Solve (24) and we have

$$\Psi(t) = \begin{pmatrix} e^{-bt} & e^{-bt} - \dfrac{Ke^{-(\gamma+\alpha+b)t}}{\gamma+\alpha} \\ 0 & e^{-(\gamma+\alpha+b)t} \end{pmatrix}. \tag{25}$$

When $t = t_n$, the impulsive equation is

$$\begin{pmatrix} e(t^+) \\ i(t^+) \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e(t^-) \\ i(t^-) \end{pmatrix}. \tag{26}$$

Set

$$M = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Psi(t). \tag{27}$$

Then

$$M = \begin{pmatrix} pe^{-bT} & pe^{-bT} - \dfrac{pKe^{-(\gamma+\alpha+b)T}}{\gamma+\alpha} \\ 0 & e^{-(\gamma+\alpha+b)T} \end{pmatrix}. \tag{28}$$

The eigenvalues of M are $\lambda_1 = pe^{-bT}$ and $\lambda_2 = e^{-(\gamma+\alpha+b)T}$ whose module lengths are less than 1. According to the Floquet theory of impulsive differential equation, the periodic solution is local stable and thus locally asymptotically stable. □

Remark 5. The result presented above shows that the crisis is under our control because the equilibrium point is unconditionally stable for the system with impulse, which acts quite differently from the original situation, in which it requires some extra conditions (see the definition of R_0 and R_1). Then we have the confidence to say that the interference actually has an effect on the control of crisis.

4. Conclusion and Future Work

The model discussed in this paper is a *SEI* model with periodic impulse. In the first section, our attention has been paid to the model without impulse and analyzed the stability of its critical point. In the second section, considering its meaning in the realistic world, we add the impulse at the fixed time. And the stability of the new system has been studied. According to those analysis of equilibriums under different situation, we have already drawn a conclusion that with impulsive method the crisis is under control (see Remark 5). Considering its realistic meaning, the "impulsive control" could be offered by the government, for example, macroeconomic regulation and control. Then according to our model the crisis will be under control. This indicates the whole system will reach the equilibrium faster, which contributes to the stability of the whole market. In the future, the topology of banking market complex network will be considered in the model in order to make it more practical.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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