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# Cauchy problem of the generalized Zakharov type system in $\mathbf{R}^2$

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## Abstract

In this paper, we consider the initial value problem for a two-dimensional generalized Zakharov system with quantum effects. We prove the existence and uniqueness of global smooth solutions to the initial value problem in the Sobolev space through making *a priori* integral estimates and the Galerkin method.

**Keywords:** Zakharov system; Cauchy problem; existence; uniqueness

## 1 Introduction

In the recent years, special interest has been devoted to quantum corrections to the Zakharov equations for Langmuir waves in a plasma [1]. By use of a quantum fluid approach, the following modified Zakharov equations are obtained:

$$iE_t + E_{xx} - H^2 E_{xxxx} = nE, \quad (1)$$

$$n_{tt} - n_{xx} + H^2 n_{xxxx} = |E|_{xx}^2, \quad (2)$$

where  $H$  is the dimensionless quantum parameter given by the ratio of the ion plasmon and electron thermal energies. For  $H = 0$ , this system was derived by Zakharov in [2] to model a Langmuir wave in plasma. The Zakharov system attracted many scientists' wide interest and attention [3–14].

In this paper, we deal with the following generalized Zakharov system:

$$iE_t + \Delta E - H^2 \Delta^2 E - nE = 0, \quad (3)$$

$$n_{tt} - \Delta n + H^2 \Delta^2 n - \Delta |E|^2 = 0, \quad (4)$$

where  $(E, n) : (x, t) \in \mathbf{R}^2 \times \mathbf{R}$  and the initial data are taken to be

$$E|_{t=0} = E_0(x), \quad n|_{t=0} = n_0(x), \quad n_t|_{t=0} = n_1(x). \quad (5)$$

To study a smooth solution of the generalized Zakharov system, we transform it into the following form:

$$iE_t + \Delta E - H^2 \Delta^2 E - nE = 0, \quad (6)$$

$$n_t - \Delta\varphi = 0, \tag{7}$$

$$\varphi_t - n + H^2 \Delta n - |E|^2 = 0, \tag{8}$$

with initial data

$$E|_{t=0} = E_0(x), \quad n|_{t=0} = n_0(x), \quad \varphi|_{t=0} = \varphi_0(x). \tag{9}$$

Now we state the main results of the paper.

**Theorem 1.1** *Suppose that  $E_0(x) \in H^{l+4}(\mathbf{R}^2)$ ,  $n_0(x) \in H^{l+2}(\mathbf{R}^2)$ ,  $n_1(x) \in H^l(\mathbf{R}^2)$ ,  $l \geq 0$ . Then there exists a unique global smooth solution of the initial value problem (3)-(5).*

$$\begin{aligned} E(x, t) &\in L^\infty(0, T; H^{l+4}(\mathbf{R}^2)), & E_t(x, t) &\in L^\infty(0, T; H^l(\mathbf{R}^2)) \\ n(x, t) &\in L^\infty(0, T; H^{l+2}(\mathbf{R}^2)), & n_t(x, t) &\in L^\infty(0, T; H^l(\mathbf{R}^2)) \\ n_{tt}(x, t) &\in L^\infty(0, T; H^{l-2}(\mathbf{R}^2)). \end{aligned}$$

The obtained results may be useful for better understanding the nonlinear coupling between the ion-acoustic waves and the Langmuir waves in a two-dimensional space.

**2 A priori estimates**

**Lemma 2.1** *Suppose that  $E_0(x) \in L^2(\mathbf{R}^2)$ . Then, for the solution of problem (6)~(9), we have*

$$\|E\|_{L^2(\mathbf{R}^2)}^2 = \|E_0(x)\|_{L^2(\mathbf{R}^2)}^2.$$

*Proof* Taking the inner product of (6) and  $E$ , then taking the imaginary part, we have

$$\begin{aligned} \text{Im}(iE_t, E) &= \text{Re}(E_t, E) = \frac{1}{2} \frac{d}{dt} \|E\|_{L^2}^2, \\ \text{Im}(\Delta E - H^2 \Delta^2 E - nE, E) &= 0. \end{aligned}$$

Hence, we get

$$\frac{d}{dt} \|E\|_{L^2}^2 = 0.$$

We thus get Lemma 2.1. □

**Lemma 2.2** (Sobolev’s estimations) *Assume that  $u \in L^q(\mathbf{R}^n)$ ,  $D^m u \in L^r(\mathbf{R}^n)$ ,  $1 \leq q, r \leq \infty$ ,  $0 \leq j \leq m$ , we have the estimations*

$$\|D^j u\|_{L^p(\mathbf{R}^n)} \leq C \|D^m u\|_{L^r(\mathbf{R}^n)}^\alpha \|u\|_{L^q(\mathbf{R}^n)}^{1-\alpha},$$

where  $C$  is a positive constant,  $0 \leq \frac{j}{m} \leq \alpha \leq 1$ ,

$$\frac{1}{p} = \frac{j}{n} + \alpha \left( \frac{1}{r} - \frac{m}{n} \right) + (1 - \alpha) \frac{1}{q}.$$

**Lemma 2.3** *Suppose that  $E_0(x) \in H^2(\mathbf{R}^2)$ ,  $n_0(x) \in H^1(\mathbf{R}^2)$ ,  $\varphi_0(x) \in H^1(\mathbf{R}^2)$ . Then we have*

$$\mathcal{F}(t) = \mathcal{F}(0),$$

where

$$\mathcal{F}(t) = \|\nabla E\|_{L^2}^2 + H^2\|\Delta E\|_{L^2}^2 + \int_{\mathbf{R}^2} n|E|^2 \, dx + \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\|n\|_{L^2}^2 + \frac{H^2}{2}\|\nabla n\|_{L^2}^2.$$

*Proof* Take the inner products of (6) and  $-E_t$ . Since

$$\operatorname{Re}(iE_t, -E_t) = 0, \quad \operatorname{Re}(\Delta E, -E_t) = \frac{1}{2} \frac{d}{dt} \|\nabla E\|_{L^2}^2,$$

$$\operatorname{Re}(-H^2 \Delta^2 E, -E_t) = \frac{H^2}{2} \frac{d}{dt} \|\Delta E\|_{L^2}^2,$$

$$\begin{aligned} \operatorname{Re}(-nE, -E_t) &= \frac{1}{2} \int_{\mathbf{R}^2} n|E_t|^2 \, dx \\ &= \frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^2} n|E|^2 \, dx - \frac{1}{2} \int_{\mathbf{R}^2} n_t |E|^2 \, dx \\ &= \frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^2} n|E|^2 \, dx - \frac{1}{2} \int_{\mathbf{R}^2} n_t (\varphi_t - n + H^2 \Delta n) \, dx \\ &= \frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^2} n|E|^2 \, dx - \frac{1}{2} \int_{\mathbf{R}^2} n_t \varphi_t \, dx + \frac{1}{4} \frac{d}{dt} \|n\|_{L^2}^2 + \frac{H^2}{4} \frac{d}{dt} \|\nabla n\|_{L^2}^2 \\ &= \frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^2} n|E|^2 \, dx - \frac{1}{2} \int_{\mathbf{R}^2} \Delta\varphi \varphi_t \, dx + \frac{1}{4} \frac{d}{dt} \|n\|_{L^2}^2 + \frac{H^2}{4} \frac{d}{dt} \|\nabla n\|_{L^2}^2 \\ &= \frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^2} n|E|^2 \, dx + \frac{1}{4} \frac{d}{dt} \|\nabla\varphi\|_{L^2}^2 + \frac{1}{4} \frac{d}{dt} \|n\|_{L^2}^2 + \frac{H^2}{4} \frac{d}{dt} \|\nabla n\|_{L^2}^2, \end{aligned}$$

thus it follows that

$$\begin{aligned} &\frac{d}{dt} \left[ \|\nabla E\|_{L^2}^2 + H^2\|\Delta E\|_{L^2}^2 + \int_{\mathbf{R}^2} n|E|^2 \, dx + \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\|n\|_{L^2}^2 + \frac{H^2}{2}\|\nabla n\|_{L^2}^2 \right] \\ &= 0. \end{aligned} \tag{10}$$

Letting

$$\mathcal{F}(t) = \|\nabla E\|_{L^2}^2 + H^2\|\Delta E\|_{L^2}^2 + \int_{\mathbf{R}^2} n|E|^2 \, dx + \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\|n\|_{L^2}^2 + \frac{H^2}{2}\|\nabla n\|_{L^2}^2,$$

and noticing (10), we obtain

$$\mathcal{F}(t) = \mathcal{F}(0). \quad \square$$

**Lemma 2.4** *Suppose that  $E_0(x) \in H^2(\mathbf{R}^2)$ ,  $n_0(x) \in H^1(\mathbf{R}^2)$ ,  $\varphi_0(x) \in H^1(\mathbf{R}^2)$ . Then we have*

$$\sup_{0 \leq t \leq T} (\|E\|_{H^2} + \|n\|_{H^1} + \|\varphi\|_{H^1}) \leq C.$$

*Proof* By Hölder’s inequality, Young’s inequality and Lemma 2.2, it follows that

$$\begin{aligned} \left| \int_{\mathbf{R}^2} n|E|^2 \, dx \right| &\leq \|n\|_{L^2} \|E\|_{L^4}^2 \\ &\leq \frac{1}{4} \|n\|_{L^2}^2 + \|E\|_{L^4}^4 \\ &\leq \frac{1}{4} \|n\|_{L^2}^2 + C \|\Delta E\|_{L^2} \|E\|_{L^2}^3 \\ &\leq \frac{1}{4} \|n\|_{L^2}^2 + \frac{H^2}{2} \|\Delta E\|_{L^2}^2 + C. \end{aligned}$$

From Lemma 2.3 we get

$$\|\nabla E\|_{L^2}^2 + \frac{H^2}{2} \|\Delta E\|_{L^2}^2 + \frac{1}{2} \|\nabla \varphi\|_{L^2}^2 + \frac{1}{4} \|n\|_{L^2}^2 + \frac{H^2}{2} \|\nabla n\|_{L^2}^2 \leq \mathcal{F}(0) + C.$$

Take the inner products of Eq. (8) and  $\varphi$ . It follows that

$$(\varphi_t - n + H^2 \Delta n - |E|^2, \varphi) = 0 \tag{11}$$

since

$$\begin{aligned} (\varphi_t, \varphi) &= \frac{1}{2} \frac{d}{dt} \|\varphi\|_{L^2}^2, \\ (n + |E|^2, \varphi) &\leq (\|n\|_{L^2} + \|E\|_{L^4}^2) \|\varphi\|_{L^2} \leq \frac{1}{2} \|\varphi\|_{L^2}^2 + C, \end{aligned}$$

where

$$\begin{aligned} \|E\|_{L^4}^2 &\leq C \|\nabla E\|_{L^2} \|E\|_{L^2} \leq C. \\ (H^2 \Delta n, \varphi) &= H^2(n, \Delta \varphi) = H^2(n, n_t) = \frac{H^2}{2} \frac{d}{dt} \|n\|_{L^2}^2. \end{aligned}$$

Hence, from Eq. (11) we get

$$\frac{d}{dt} (\|\varphi\|_{L^2}^2 + H^2 \|n\|_{L^2}^2) \leq \|\varphi\|_{L^2}^2 + C.$$

Using Gronwall’s inequality, we obtain that

$$\sup_{0 \leq t \leq T} (\|\varphi\|_{L^2}^2 + H^2 \|n\|_{L^2}^2) \leq C.$$

We thus get Lemma 2.4. □

**Lemma 2.5** *Suppose that  $E_0(x) \in H^4(\mathbf{R}^2)$ ,  $n_0(x) \in H^2(\mathbf{R}^2)$ ,  $\varphi_0(x) \in H^2(\mathbf{R}^2)$ . Then we have*

$$\sup_{0 \leq t \leq T} (\|E\|_{H^4} + \|n\|_{H^2} + \|\varphi\|_{H^2} + \|E_t\|_{L^2} + \|n_t\|_{L^2} + \|\varphi_t\|_{L^2}) \leq C.$$

*Proof* Differentiating (6) with respect to  $t$ , then taking the inner products of the resulting equation and  $E_t$ , we have

$$(iE_{tt} + \Delta E_t - H^2 \Delta^2 E_t - (nE)_t, E_t) = 0 \tag{12}$$

since

$$\begin{aligned} \operatorname{Im}(iE_{tt}, E_t) &= \frac{1}{2} \frac{d}{dt} \|E_t\|_{L^2}^2, & \operatorname{Im}(\Delta E_t - H^2 \Delta^2 E_t - nE_t, E_t) &= 0, \\ |\operatorname{Im}(-n_t E, E_t)| &\leq C \|E\|_{L^\infty} \|n_t\|_{L^2} \|E_t\|_{L^2} \leq C (\|E_t\|_{L^2}^2 + \|n_t\|_{L^2}^2). \end{aligned}$$

By Lemma 2.2, it follows that

$$\|E\|_{L^\infty} \leq C \|\Delta E\|_{L^2}^{\frac{1}{2}} \|E\|_{L^2}^{\frac{1}{2}};$$

thus from Eq. (12) we get

$$\frac{d}{dt} \|E_t\|_{L^2}^2 \leq C (\|E_t\|_{L^2}^2 + \|n_t\|_{L^2}^2). \tag{13}$$

Differentiating Eq. (7) with respect to  $t$ , then taking the inner products of the resulting equation and  $n_t$ , we have

$$(n_{tt} - \Delta \varphi_t, n_t) = 0 \tag{14}$$

since

$$\begin{aligned} (n_{tt}, n_t) &= \frac{1}{2} \frac{d}{dt} \|n_t\|_{L^2}^2, \\ (-\Delta \varphi_t, n_t) &= (-\Delta n + H^2 \Delta^2 n - \Delta |E|^2, n_t) \\ &= \frac{1}{2} \frac{d}{dt} \|\nabla n\|_{L^2}^2 + \frac{H^2}{2} \frac{d}{dt} \|\Delta n\|_{L^2}^2 - (\Delta |E|^2, n_t). \end{aligned}$$

Noting that

$$|(\Delta |E|^2, n_t)| \leq C \|E\|_{L^\infty} \|\Delta E\|_{L^2} \|n_t\|_{L^2} \leq C (\|n_t\|_{L^2}^2 + 1),$$

from Eq. (14) we get

$$\frac{d}{dt} [\|n_t\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + H^2 \|\Delta n\|_{L^2}^2] \leq C (\|n_t\|_{L^2}^2 + 1). \tag{15}$$

From Eq. (13) and (15) we get

$$\frac{d}{dt} [\|E_t\|_{L^2}^2 + \|n_t\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + H^2 \|\Delta n\|_{L^2}^2] \leq C (\|E_t\|_{L^2}^2 + \|n_t\|_{L^2}^2 + 1).$$

By Gronwall's inequality, it follows that

$$\sup_{0 \leq t \leq T} [\|E_t\|_{L^2}^2 + \|n_t\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 + H^2 \|\Delta n\|_{L^2}^2] \leq C. \tag{16}$$

Take the inner products of Eq. (8) and  $\varphi_t$ . It follows that

$$(\varphi_t - n + H^2 \Delta n - |E|^2, \varphi_t) = 0 \tag{17}$$

since

$$\begin{aligned} (\varphi_t, \varphi_t) &= \|\varphi_t\|_{L^2}^2, \\ (-n + H^2 \Delta n - |E|^2, \varphi_t) &\leq (\|n\|_{L^2} + H^2 \|\Delta n\|_{L^2} + \|E\|_{L^4}^2) \|\varphi_t\|_{L^2} \\ &\leq C + \frac{1}{2} \|\varphi_t\|_{L^2}^2. \end{aligned}$$

From Eq. (17) we get

$$\|\varphi_t\|_{L^2}^2 \leq C.$$

Take the inner products of Eq. (6) and  $\Delta E$ . It follows that

$$(iE_t + \Delta E - H^2 \Delta^2 E - nE, \Delta E) = 0 \tag{18}$$

since

$$\begin{aligned} (iE_t - nE, \Delta E) &\leq (\|E_t\|_{L^2} + \|E\|_{L^\infty} \|n\|_{L^2}) \|\Delta E\|_{L^2} \leq C, \\ (\Delta E - H^2 \Delta^2 E, \Delta E) &= \|\Delta E\|_{L^2}^2 + H^2 \|\nabla^3 E\|_{L^2}^2. \end{aligned}$$

From Eq. (18) we get

$$\|\nabla^3 E\|_{L^2}^2 \leq C.$$

From (6) we obtain

$$H^2 \|\Delta^2 E\|_{L^2} \leq \|E_t\|_{L^2} + \|\Delta E\|_{L^2} + \|nE\|_{L^2} \leq C,$$

where

$$\|nE\|_{L^2} \leq \|n\|_{L^4} \|E\|_{L^4} \leq C \|\nabla n\|_{L^2}^{\frac{1}{2}} \|n\|_{L^2}^{\frac{1}{2}} \|\nabla E\|_{L^2}^{\frac{1}{2}} \|E\|_{L^2}^{\frac{1}{2}} \leq C.$$

From (7) we obtain

$$\|\Delta \varphi\|_{L^2} = \|n_t\|_{L^2} \leq C.$$

We thus get Lemma 2.5. □

**Lemma 2.6** *Suppose that  $f_1, f_2 \in H^s(\Omega)$ ,  $\Omega \subseteq \mathbf{R}^n$ . Then we have*

$$\|D^s(f_1 \cdot f_2)\|_{L^2} \leq C_s (\|f_1\|_{L^2} \|D^s f_2\|_{L^2} + \|D^s f_1\|_{L^2} \|f_2\|_{L^2}),$$

where the constant  $C_s$  is independent of  $f_1$  and  $f_2$ .

**Lemma 2.7** *Suppose that  $E_0(x) \in H^{m+4}(\mathbb{R}^2)$ ,  $n_0(x) \in H^{m+2}(\mathbb{R}^2)$ ,  $\varphi_0(x) \in H^{m+2}(\mathbb{R}^2)$ ,  $m \geq 0$ . Then we have*

$$\begin{aligned} \sup_{0 \leq t \leq T} (\|E(x, t)\|_{H^{m+4}} + \|n(x, t)\|_{H^{m+2}} + \|\varphi(x, t)\|_{H^{m+2}}) &\leq C \\ \sup_{0 \leq t \leq T} (\|E_t(x, t)\|_{H^m} + \|n_t(x, t)\|_{H^m} + \|\varphi_t\|_{H^m}) &\leq C. \end{aligned}$$

*Proof* Lemma 2.7 is true when  $m = 0$  (Lemma 2.5). Suppose that Lemma 2.7 is true when  $m = k, (k \geq 0)$ . Take the inner products of (8) and  $(-1)^{k+1} \Delta^{k+3} \varphi$ . It follows that

$$(\varphi_t - n + H^2 \Delta n - |E|^2, (-1)^{k+1} \Delta^{k+3} \varphi) = 0 \tag{19}$$

since

$$\begin{aligned} (\varphi_t, (-1)^{k+1} \Delta^{k+3} \varphi) &= \frac{1}{2} \frac{d}{dt} \|\nabla^{k+3} \varphi\|_{L^2}^2, \\ (-n, (-1)^{k+1} \Delta^{k+3} \varphi) &= (-n, (-1)^{k+1} \Delta^{k+2} n_t) = \frac{1}{2} \frac{d}{dt} \|\nabla^{k+2} n\|_{L^2}^2, \\ (H^2 \Delta n, (-1)^{k+1} \Delta^{k+3} \varphi) &= H^2 (\Delta n, (-1)^{k+1} \Delta^{k+2} n_t) = \frac{H^2}{2} \frac{d}{dt} \|\nabla^{k+3} n\|_{L^2}^2, \\ |(-|E|^2, (-1)^{k+1} \Delta^{k+3} \varphi)| &= |(\nabla^{k+3} |E|^2, \nabla^{k+3} \varphi)| \leq \|\nabla^{k+3} |E|^2\|_{L^2} \|\nabla^{k+3} \varphi\|_{L^2} \\ &\leq C \|E\|_{L^2} \|\nabla^{k+3} E\|_{L^2} \|\nabla^{k+3} \varphi\|_{L^2} \\ &\leq C (\|\nabla^{k+3} \varphi\|_{L^2}^2 + 1), \end{aligned}$$

thus from Eq. (19) it follows that

$$\frac{d}{dt} (\|\nabla^{k+3} \varphi\|_{L^2}^2 + \|\nabla^{k+2} n\|_{L^2}^2 + H^2 \|\nabla^{k+3} n\|_{L^2}^2) \leq C (\|\nabla^{k+3} \varphi\|_{L^2}^2 + 1). \tag{20}$$

By using Gronwall’s inequality, we have

$$\sup_{0 \leq t \leq T} (\|\nabla^{k+3} \varphi\|_{L^2}^2 + \|\nabla^{k+2} n\|_{L^2}^2 + \|\nabla^{k+3} n\|_{L^2}^2) \leq C.$$

From (7) and (8) we get

$$\begin{aligned} \|\nabla^{k+1} n_t\|_{L^2} &= \|\nabla^{k+3} \varphi\|_{L^2} \leq C, \\ \|\nabla^{k+1} \varphi_t\|_{L^2} &\leq C (\|\nabla^{k+1} n\|_{L^2} + \|\nabla^{k+3} n\|_{L^2} + \|E\|_{L^2} \|\nabla^{k+1} E\|_{L^2}) \leq C. \end{aligned}$$

Differentiating (6) with respect to  $t$ , then taking the inner products of the resulting equation and  $(-1)^{k+1} \Delta^{k+1} E_t$ , we obtain

$$(iE_{tt} + \Delta E_t - H^2 \Delta^2 E_t - (nE)_t, (-1)^{k+1} \Delta^{k+1} E_t) = 0. \tag{21}$$

Since

$$\text{Im}(iE_{tt}, (-1)^{k+1} \Delta^{k+1} E_t) = \frac{1}{2} \frac{d}{dt} \|\nabla^{k+1} E_t\|_{L^2}^2,$$

$$\begin{aligned} \operatorname{Im}(\Delta E_t - H^2 \Delta^2 E_t, (-1)^{k+1} \Delta^{k+1} E_t) &= 0, \\ |\operatorname{Im}(-n_t E, (-1)^{k+1} \Delta^{k+1} E_t)| &\leq |(\nabla^{k+1}(n_t E), \nabla^{k+1} E_t)| \\ &\leq C(\|\nabla^{k+1} n_t\|_{L^2} \|E\|_{L^2} + \|\nabla^{k+1} E\|_{L^2} \|n_t\|_{L^2}) \|\nabla^{k+1} E_t\|_{L^2} \\ &\leq C(\|\nabla^{k+1} E_t\|_{L^2}^2 + 1), \\ |\operatorname{Im}(-n E_t, (-1)^{k+1} \Delta^{k+1} E_t)| &\leq |(\nabla^{k+1}(n E_t), \nabla^{k+1} E_t)| \\ &\leq C(\|\nabla^{k+1} n\|_{L^2} \|E_t\|_{L^2} + \|\nabla^{k+1} E_t\|_{L^2} \|n\|_{L^2}) \|\nabla^{k+1} E_t\|_{L^2} \\ &\leq C(\|\nabla^{k+1} E_t\|_{L^2}^2 + 1), \end{aligned}$$

thus from Eq. (21) we get

$$\frac{d}{dt} \|\nabla^{k+1} E_t\|_{L^2}^2 \leq C(\|\nabla^{k+1} E_t\|_{L^2}^2 + 1). \tag{22}$$

By using Gronwall’s inequality, we get

$$\sup_{0 \leq t \leq T} \|\nabla^{k+1} E_t\|_{L^2}^2 \leq C.$$

From (6) we obtain

$$\|\nabla^{k+5} E\|_{L^2} \leq C(\|\nabla^{k+1} E_t\|_{L^2} + \|\nabla^{k+3} E\|_{L^2} + \|\nabla^{k+1} n\|_{L^2} \|E\|_{L^2} + \|n\|_{L^2} \|\nabla^{k+1} E\|_{L^2}) \leq C.$$

Hence

$$\begin{aligned} \sup_{0 \leq t \leq T} (\|E(x, t)\|_{H^{k+5}} + \|n(x, t)\|_{H^{k+3}} + \|\varphi(x, t)\|_{H^{k+3}}) &\leq C, \\ \sup_{0 \leq t \leq T} (\|E_t(x, t)\|_{H^{k+1}} + \|n_t(x, t)\|_{H^{k+1}} + \|\varphi_t\|_{H^{k+1}}) &\leq C. \end{aligned}$$

This means Lemma 2.7 is true when  $m = k + 1$ . Thus Lemma 2.7 is proved completely.  $\square$

### 3 Existence and uniqueness of solution

Now, with these lemmas, we are able to prove Theorem 1.1. First we obtain the existence and uniqueness of the global generalized solution of problem (6)-(9).

**Definition 3.1** The functions  $E \in L^\infty(0, T; H^4) \cap W^{1,\infty}(0, T; L^2)$ ,  $n \in L^\infty(0, T; H^2) \cap W^{1,\infty}(0, T; L^2)$  and  $\varphi \in L^\infty(0, T; H^2) \cap W^{1,\infty}(0, T; L^2)$  are called a generalized solution of problem (6)-(9) if for any  $\omega \in L^2$  they satisfy the integral equality

$$\begin{aligned} (iE_{jt}, \omega) + (\Delta E_j, \omega) &= H^2(\Delta^2 E_j, \omega) + (nE_j, \omega), \quad j = 1, 2, \dots, N, \\ (n_t, \omega) &= (\Delta \varphi, \omega), \\ (\varphi_t, \omega) + H^2(\Delta n, \omega) &= (n, \omega) + (|E|^2, \omega) \end{aligned}$$

with initial data

$$E(x, 0) = E_0(x), \quad n(x, 0) = n_0(x), \quad \varphi(x, 0) = \varphi_0(x).$$



Now, one can estimate the following theorem.

**Theorem 3.1** *Suppose that  $E_0(x) \in H^{l+4}(\mathbf{R}^2)$ ,  $n_0(x) \in H^{l+2}(\mathbf{R}^2)$ ,  $\varphi_0(x) \in H^{l+2}(\mathbf{R}^2)$ ,  $l \geq 0$ . Then there exists a global smooth solution of the initial value problem (6)-(9).*

$$\begin{aligned} E(x, t) &\in L^\infty(0, T; H^{l+4}(\mathbf{R}^2)), & E_t(x, t) &\in L^\infty(0, T; H^l(\mathbf{R}^2)), \\ n(x, t) &\in L^\infty(0, T; H^{l+2}(\mathbf{R}^2)), & n_t(x, t) &\in L^\infty(0, T; H^l(\mathbf{R}^2)), \\ \varphi(x, t) &\in L^\infty(0, T; H^{l+2}(\mathbf{R}^2)), & \varphi_t(x, t) &\in L^\infty(0, T; H^l(\mathbf{R}^2)). \end{aligned}$$

*Proof* By using the Galerkin method, choose the basic periodic functions  $\{\omega_\kappa(x)\}$  as follows:

$$-\Delta\omega_\kappa(x) = \lambda_\kappa\omega_\kappa(x), \quad \omega_\kappa(x) \in H^2(\Omega), \kappa = 1, \dots, l.$$

The approximate solution of problem (6)-(9) can be written as

$$\begin{aligned} E^l(x, t) &= \sum_{\kappa=1}^l \alpha_\kappa^l(t)\omega_\kappa(x), & n^l(x, t) &= \sum_{\kappa=1}^l \beta_\kappa^l(t)\omega_\kappa(x), \\ \varphi^l(x, t) &= \sum_{\kappa=1}^l \gamma_\kappa^l(t)\omega_\kappa(x), \end{aligned}$$

where

$$E^l(x, t) = (E_1^l, E_2^l, \dots, E_N^l), \quad \alpha_\kappa^l(t) = (\alpha_{\kappa 1}^l, \alpha_{\kappa 2}^l, \dots, \alpha_{\kappa N}^l).$$

$\Omega$  is a two-dimensional cube with  $2D$  in each direction, that is,  $\overline{\Omega} = \{x = (x_1, x_2) \mid |x_j| \leq 2D, j = 1, 2\}$ . According to Galerkin’s method, these undetermined coefficients  $\alpha_\kappa^l(t)$ ,  $\beta_\kappa^l(t)$  and  $\gamma_\kappa^l(t)$  need to satisfy the following initial value problem of the system of ordinary differential equations:

$$(iE_{jt}^l, \omega_\kappa) + (\Delta E_j^l, \omega_\kappa) = H^2(\Delta^2 E_j^l, \omega_\kappa) + (n^l E_j^l, \omega_\kappa), \quad j = 1, 2, \dots, N, \tag{23}$$

$$(n_t^l, \omega_\kappa) = (\Delta \varphi^l, \omega_\kappa), \tag{24}$$

$$(\varphi_t^l, \omega_\kappa) + H^2(\Delta n^l, \omega_\kappa) = (n^l, \omega_\kappa) + (|E^l|^2, \omega_\kappa) \tag{25}$$

with initial data

$$E^l(x, 0) = E_0^l(x), \quad n^l(x, 0) = n_0^l(x), \quad \varphi^l(x, 0) = \varphi_0^l(x), \tag{26}$$

where

$$E_0^l(x) \xrightarrow{H^4} E_0(x), \quad n_0^l(x) \xrightarrow{H^2} n_0(x), \quad \varphi_0^l(x) \xrightarrow{H^2} \varphi_0(x), \quad l \rightarrow \infty.$$

Similarly to the proof of Lemmas 2.1-2.5, for the solution  $E^l(x, t)$ ,  $n^l(x, t)$ ,  $\varphi^l(x, t)$  of problem (23)-(26), we can establish the following estimates:

$$\sup_{0 \leq t \leq T} (\|E^l\|_{H^4} + \|n^l\|_{H^2} + \|\varphi^l\|_{H^2} + \|E_t^l\|_{L^2} + \|n_t^l\|_{L^2} + \|\varphi_t^l\|_{L^2}) \leq C,$$

where the constants  $C$  are independent of  $l$  and  $D$ . By compact argument, some subsequence of  $(E^l, n^l, \varphi^l)$ , also labeled by  $l$ , has a weak limit  $(E, n, \varphi)$ . More precisely

$$\begin{aligned} E^l(x, t) &\rightharpoonup E(x, t) \quad \text{in } L^\infty(0, T; H^4) \text{ weakly star,} \\ n^l(x, t) &\rightharpoonup n(x, t) \quad \text{in } L^\infty(0, T; H^2) \text{ weakly star,} \\ \varphi^l(x, t) &\rightharpoonup \varphi(x, t) \quad \text{in } L^\infty(0, T; H^2) \text{ weakly star} \end{aligned}$$

and

$$\begin{aligned} E_t^l &\rightharpoonup E_t \quad \text{in } L^\infty(0, T; L^2) \text{ weakly star,} \\ n_t^l &\rightharpoonup n_t \quad \text{in } L^\infty(0, T; L^2) \text{ weakly star,} \\ \varphi_t^l &\rightharpoonup \varphi_t \quad \text{in } L^\infty(0, T; L^2) \text{ weakly star.} \end{aligned}$$

By using Guo and Shen’s method [15], one can prove the existence of a local solution for the periodic initial problem (6)-(9). Similarly to Zhou and Guo’s proof [16], letting  $D \rightarrow \infty$ , the existence of a local solution for the initial value problem (6)-(9) can be obtained. By the continuation extension principle, from the conditions of the theorem and *a priori* estimates in Section 2, we can get the existence of a global generalized solution for problem (6)-(9). By Lemma 2.7 and the Sobolev imbedding theorem, Theorem 3.1 is proved.  $\square$

Next, we prove the uniqueness of a solution for problem (6)-(9).

**Theorem 3.2** *Suppose that  $E_0(x) \in H^{l+4}(\mathbf{R}^2)$ ,  $n_0(x) \in H^{l+2}(\mathbf{R}^2)$ ,  $\varphi_0(x) \in H^{l+2}(\mathbf{R}^2)$ ,  $l \geq 0$ . Then the global solution of the initial value problem (6)-(9) is unique.*

*Proof* Suppose that there are two solutions  $E_1, n_1, \varphi_1$  and  $E_2, n_2, \varphi_2$ . Let

$$E = E_1 - E_2, \quad n = n_1 - n_2, \quad \varphi = \varphi_1 - \varphi_2.$$

From (6)-(9) we get

$$iE_t + \Delta E - H^2 \Delta^2 E - n_1 E_1 + n_2 E_2 = 0, \tag{27}$$

$$n_t - \Delta \varphi = 0, \tag{28}$$

$$\varphi_t - n + H^2 \Delta n - |E_1|^2 + |E_2|^2 = 0, \tag{29}$$

with initial data

$$E|_{t=0} = 0, \quad n|_{t=0} = 0, \quad \varphi|_{t=0} = 0, \quad x \in \mathbf{R}^2. \tag{30}$$

Take the inner product of (27) and  $E$ . Since

$$\text{Im}(iE_t, E) = \frac{1}{2} \frac{d}{dt} \|E\|_{L^2}^2,$$

$$\text{Im}(\Delta E - H^2 \Delta^2 E, E) = 0,$$

$$\begin{aligned} |\operatorname{Im}(n_1 E_1 - n_2 E_2, E)| &\leq |(n E_1 + n_2 E, E)| \\ &\leq C(\|E_1\|_{L^\infty} \|n\|_{L^2} + \|n_2\|_{L^\infty} \|E\|_{L^2}) \|E\|_{L^2} \\ &\leq C(\|n\|_{L^2}^2 + \|E\|_{L^2}^2), \end{aligned}$$

thus we obtain

$$\frac{d}{dt} \|E\|_{L^2}^2 \leq C(\|n\|_{L^2}^2 + \|E\|_{L^2}^2). \tag{31}$$

Take the inner product of (29) and  $\varphi$ . Since

$$\begin{aligned} (\varphi_t, \varphi) &= \frac{1}{2} \frac{d}{dt} \|\varphi\|_{L^2}^2, \quad |(-n, \varphi)| \leq C(\|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2), \\ (H^2 \Delta n, \varphi) &= (H^2 n, \Delta \varphi) = (H^2 n, n_t) = \frac{H^2}{2} \frac{d}{dt} \|n\|_{L^2}^2, \\ |(-|E_1|^2 + |E_2|^2, \varphi)| &= |((E_1 - E_2)\overline{E_1} + E_2(\overline{E_1} - \overline{E_2}), \varphi)| \\ &\leq C(\|E_1\|_{L^\infty} \|E\|_{L^2} + \|E_2\|_{L^\infty} \|E\|_{L^2}) \|\varphi\|_{L^2} \\ &\leq C(\|E\|_{L^2}^2 + \|\varphi\|_{L^2}^2), \end{aligned}$$

thus we get

$$\frac{d}{dt} (\|\varphi\|_{L^2}^2 + H^2 \|n\|_{L^2}^2) \leq C(\|E\|_{L^2}^2 + \|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2). \tag{32}$$

Hence from (31) and (32) we get

$$\frac{d}{dt} (\|E\|_{L^2}^2 + \|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2) \leq C(\|E\|_{L^2}^2 + \|n\|_{L^2}^2 + \|\varphi\|_{L^2}^2). \tag{33}$$

By using Gronwall’s inequality and noticing (30), we arrive at

$$E \equiv 0, \quad n \equiv 0, \quad \varphi \equiv 0.$$

Theorem 3.2 is proved. This completes the proof of Theorem 1.1. □

#### 4 Results and discussion

One can regard (3)-(4) as the Langmuir turbulence parameterized by  $H$  and study the asymptotic behavior of systems (3)-(4) when  $H$  goes to zero.

#### 5 Conclusions

By *a priori* integral estimates and the Galerkin method, we have the following conclusion.

Suppose that  $E_0(x) \in H^{l+4}(\mathbf{R}^2)$ ,  $n_0(x) \in H^{l+2}(\mathbf{R}^2)$ ,  $n_1(x) \in H^l(\mathbf{R}^2)$ ,  $l \geq 0$ . Then there exists a unique global smooth solution of the initial value problem (3)-(5).

$$\begin{aligned} E(x, t) &\in L^\infty(0, T; H^{l+4}(\mathbf{R}^2)), & E_t(x, t) &\in L^\infty(0, T; H^l(\mathbf{R}^2)) \\ n(x, t) &\in L^\infty(0, T; H^{l+2}(\mathbf{R}^2)), & n_t(x, t) &\in L^\infty(0, T; H^l(\mathbf{R}^2)) \\ n_{tt}(x, t) &\in L^\infty(0, T; H^{l-2}(\mathbf{R}^2)). \end{aligned}$$

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

SY carried out the existence studies and drafted the manuscript. XN carried out the uniqueness of the solution and helped to draft the manuscript. All authors read and approved the final manuscript.

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**References**

1. García, LG, Haas, F, de Oliveira, LPL, Goedert, J: Modified Zakharov equations for plasmas with a quantum correction. *Phys. Plasmas* **12**, 012302 (2005)
2. Zakharov, VE: Collapse of Langmuir waves. *Sov. Phys. JETP* **35**, 908-914 (1972)
3. Holmer, J: Local ill-posedness of the 1D Zakharov system. *Electron. J. Differ. Equ.* **2007**, 24 (2007)
4. Pecher, H: An improved local well-posedness result for the one-dimensional Zakharov system. *J. Math. Anal. Appl.* **342**, 1440-1454 (2008)
5. Guo, B, Zhang, J, Pu, X: On the existence and uniqueness of smooth solution for a generalized Zakharov equation. *J. Math. Anal. Appl.* **365**, 238-253 (2010)
6. Linares, F, Matheus, C: Well posedness for the 1D Zakharov-Rubenchik system. *Adv. Differ. Equ.* **2009**, 14 (2009)
7. Linares, F, Saut, JC: The Cauchy problem for the 3D Zakharov-Kuznetsov equation. *Discrete Contin. Dyn. Syst.* **24**, 547-565 (2009)
8. You, S: The posedness of the periodic initial value problem for generalized Zakharov equations. *Nonlinear Anal. TMA* **71**, 3571-3584 (2009)
9. You, S, Guo, B, Ning, X: Initial boundary value problem for a generalized Zakharov equations. *Appl. Math.* **57**, 581-599 (2012)
10. Masmoudi, N, Nakanishi, K: Energy convergence for singular limits of Zakharov type systems. *Invent. Math.* **172**, 535-583 (2008)
11. Seadawy, AR: Stability analysis for Zakharov-Kuznetsov equation of weakly nonlinear ion-acoustic waves in a plasma. *Comput. Math. Appl.* **67**, 172-180 (2014)
12. Ismail, A: The first integral method for constructing exact and explicit solutions to nonlinear evolution equations. *Math. Methods Appl. Sci.* **35**, 716-722 (2012)
13. Morris, R, Kara, AH, Biswas, A: Soliton solution and conservation laws of the Zakharov equation in plasmas with power law nonlinearity. *Nonlinear Anal., Model. Control* **18**, 153-159 (2013)
14. You, S, Ning, X: On global smooth solution for generalized Zakharov equations. *Comput. Math. Appl.* **72**, 64-75 (2016)
15. Guo, B, Shen, L: The existence and uniqueness of the classical solution and the periodic initial value problem for Zakharov equation. *Acta Math. Appl. Sin.* **3**, 310-324 (1982)
16. Zhou, Y, Guo, B: The periodic boundary value problem and the initial value problem for the generalized Korteweg-de Vries system of higher order. *Acta Math. Sin.* **27**, 154-176 (1984)

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