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Generalized contractions with triangular α-orbital admissible mapping on Branciari metric spaces

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Abstract

The purpose of this paper is to generalize fixed point theorems introduced by Jleli *et al.* (J. Inequal. Appl. 2014:38, 2014) by using the concept of triangular α-orbital admissible mappings established in Popescu (Fixed Point Theory Appl. 2014:190, 2014). Some examples are given here to illustrate the usability of the obtained results.

MSC: 46S40; 47H10; 54H25

Keywords: generalized metric space; fixed point; triangular α-orbital admissible mapping; α-orbital attractive mapping

1 Introduction

Recently, Branciari [3] refined the notion of metric to get a new distance function by substituting the triangle inequality with the quadrilateral inequality. This refined metric function was called general metric in some sources, rectangular metric in some others. Throughout the manuscript, we use the Branciari metric for this new function. In a pioneering work, the author [3] successfully defined an open ball and hence a topology for the Branciari metric. On the other hand, the topology of the Branciari metric is quite different from the usual metric topology. For more details, see *e.g.* the Branciari metric [4–6] and the related references therein. Besides the interesting topological properties induced by the Branciari metric, the author of [3] reported the analogous celebrated Banach contraction mapping principle which has been generalized, extended, and improved in several ways; see *e.g.* [1–5, 7–34]. Although Branciari [3] correctly stated the analog of Banach contraction mapping principle in the setting of Branciari metric space, proofs has gaps which was removed by a number of authors; see *e.g.* [5, 12, 19, 31].

In this paper we extend the results introduced by Jleli *et al.* [1, 18] by using the concept of triangular α-orbital admissible mappings obtained in [2]. Throughout the article $\mathbb{N}$, $\mathbb{R}$ shall denote the set of natural and real numbers, respectively.

Definition 1 [3] Let X be a non-empty set and $d : X \times X \longrightarrow [0,\infty)$ be a mapping such that, for all $x, y \in X$ and all distinct points $u, v \in X$, each of them different from x and y, one has

(i) $d(x,y) = 0 \Longleftrightarrow x = y$,

(ii) $d(x,y) = d(y,x)$,
(iii) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$.

Then (X,d) is called a Branciari metric space (or for short BMS). As mentioned above, such spaces are called also generalized metric space, rectangular metric space in the literature. We assert that the Branciari metric space is more suitable regarding the fact that several extensions of the metric are called general metrics.

Definition 2 Let (X,d) be a BMS, $\{x_n\}$ be a sequence in X, and $x \in X$, we say that $\{x_n\}$ is convergent to x if and only if $d(x_n,x) \longrightarrow 0$ as $n \longrightarrow \infty$. We denote this by $x_n \longrightarrow x$.

Definition 3 Let (X,d) be a BMS and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n,x_m) \longrightarrow 0$ as $n,m \longrightarrow \infty$.

Definition 4 Let (X,d) be a BMS. We say that (X,d) is complete if and only if every Cauchy sequence in X converges to some element in X.

Definition 5 [32] Let $T : X \to X$ be a map and $\alpha : X \times X \to [0,+\infty)$ be a function. We say that T is α-admissible if $x,y \in X$, $\alpha(x,y) \geq 1$ implies that $\alpha(Tx,Ty) \geq 1$.

Definition 6 [11] A map $T : X \to X$ is said to be triangular α-admissible if:
(T1) T is α-admissible,
(T2) $\alpha(x,u) \geq 1$ and $\alpha(u,y) \geq 1$ implies that $\alpha(x,y) \geq 1$, $x,u,y \in X$.

Definition 7 [2] Let $T : X \to X$ be a map and $\alpha : X \times X \to [0,+\infty)$ be a function. Then T is said to be α-orbital admissible if
(T3) $x \in X$, $\alpha(x,Tx) \geq 1$ implies that $\alpha(Tx,T^2x) \geq 1$.

Definition 8 [2] Let $T : X \to X$ be a map and $\alpha : X \times X \to [0,+\infty)$ be a function. Then T is said to be triangular α-orbital admissible if it is α-orbital admissible and
(T4) $x,y \in X$, $\alpha(x,y) \geq 1$, and $\alpha(y,Ty) \geq 1$ implies that $\alpha(x,Ty) \geq 1$.

Example 9 [2] Let $X = \{0,1,2,3\}$, $d : X \times X \longrightarrow \mathbb{R}$, $d(x,y) = |x-y|$, $T : X \to X$ such that $T(0) = 0$, $T(1) = 2$, $T(2) = 1$, $T(3) = 3$, and $\alpha : X \times X \to [0,+\infty)$,

$$\alpha(x,y) = \begin{cases} 1, & \text{if } (x,y) \in A, \\ 0, & \text{otherwise,} \end{cases}$$

where $A = \{(0,1),(0,2),(1,1),(2,2),(1,2),(2,1),(1,3),(2,3)\}$. Clearly, T is triangular α-orbital admissible, T is α-orbital admissible, but T is not triangular α-admissible.

Definition 10 [2] Let $T : X \to X$ be a map and $\alpha : X \times X \to [0,+\infty)$ be a function. Then T is said to be α-orbital attractive if

$$x \in X, \quad \alpha(x,Tx) \geq 1 \quad \text{implies that} \quad \alpha(x,y) \text{ or } \alpha(y,Tx) \geq 1,$$

for every $y \in X$.

We denote by Θ the set of functions $\theta : (0,\infty) \longrightarrow (1,\infty)$ satisfying the following conditions:

(Θ1) θ is non-decreasing,
(Θ2) for each sequence $\{t_n\} \subset (0,\infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \quad \text{if and only if} \quad \lim_{n\to\infty} t_n = 0^+,$$

(Θ3) there exists $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim_{t\longrightarrow 0^+} \frac{\theta(t)-1}{t^r} = \ell$.

Very recently Jleli *et al.* [1] established the following generalization of the Banach fixed point theorem in the setting of the Branciari metric space.

Theorem 11 [1] *Let (X,d) be a complete BMS and $T : X \longrightarrow X$ be a given mapping. Suppose that there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \quad \Longrightarrow \quad \theta\big(d(Tx, Ty)\big) \leq \big[\theta\big(d(x,y)\big)\big]^k.$$

Then T has a unique fixed point.

Example 12 [1] The functions $\theta : (0,\infty) \longrightarrow (1,\infty)$ are elements of Θ:
(1) $\theta(t) = e^{\sqrt{t}}$,
(2) $\theta(t) = e^{\sqrt{te^t}}$,
(3) $\theta(t) = 2 - \frac{2}{\pi}\arctan(\frac{1}{t^\gamma})$, $0 < \gamma < 1$, $t > 0$.

Theorem 13 [18] *Let (X,d) be a complete BMS and $T : X \longrightarrow X$ be a given mapping. Suppose that there exist $\theta \in \Theta$ that is continuous and $k \in (0,1)$ such that*

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \quad \Longrightarrow \quad \theta\big(d(Tx, Ty)\big) \leq \big[\theta\big(M(x,y)\big)\big]^k,$$

where

$$M(x,y) = \max\big\{d(x,y), d(x,Tx), d(y,Ty)\big\}.$$

Then T has a unique fixed point.

The following lemmas will be needed in the sequel.

Lemma 14 [5] *Let (X,d) be a BMS and $\{x_n\}$ be a Cauchy sequence in (X,d) such that $d(x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$ for some $x \in X$. Then $d(x_n, y) \longrightarrow d(x,y)$ as $n \longrightarrow \infty$ for all $y \in X$. In particular, $\{x_n\}$ does not converge to y if $y \neq x$.*

Lemma 15 [19] *Let (X,d) be a BMS and $\{x_n\}$ be a Cauchy sequence in (X,d) and $x, y \in X$. Suppose that there exists a positive integer N such that*
(i) *$x_n \neq x_m$ for all $n, m > N$;*
(ii) *x_n and x are distinct points in X for all $n > N$;*
(iii) *x_n and y are distinct points in X for all $n > N$;*
(iv) $\lim_{n\longrightarrow\infty} d(x_n, x) = \lim_{n\longrightarrow\infty} d(x_n, y)$.
Then we have $x = y$.

Lemma 16 [2] *Let $T : X \longrightarrow X$ be a triangular α-orbital admissible mapping. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$.*

2 Main results

In this section, we state and prove our main result.

Theorem 17 *Let (X, d) be a complete BMS, $T : X \longrightarrow X$ be a given map and let $\alpha : X \times X \longrightarrow [0, \infty)$ be a mapping. Suppose that the following conditions hold:*

(1) *there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that*

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \quad \Longrightarrow \quad \alpha(x, y) \cdot \theta\bigl(d(Tx, Ty)\bigr) \leq \bigl[\theta\bigl(R(x, y)\bigr)\bigr]^k,$$

where

$$R(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}\right\},$$

(2) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$ and $\alpha(x_1, T^2x_1) \geq 1$,*

(3) *T is a triangular α-orbital admissible mapping,*

(4) *T is continuous.*

Then T has a fixed point $x_* \in X$ and $\{T^n x_1\}$ converges to x_*.

Proof Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq 1$ and $\alpha(x_1, T^2x_1) \geq 1$. We define the iterative sequence $\{x_n\}$ in X by the rule $x_n = Tx_{n-1} = T^n x_1$ for all $n \geq 1$. Obviously, if there exists $n_0 \geq 1$ for which $T^{n_0}x_1 = T^{n_0+1}x_1$ then $T^{n_0}x_1$ shall be a fixed point of T. Thus, we suppose that $T^n x_1 \neq T^{n+1}x_1$ for every $n \geq 1$. Now from Lemma 16, we get

$$\alpha\bigl(T^n x_1, T^{n+1}x_1\bigr) \geq 1 \quad \text{for all } n \geq 1, \tag{2.1}$$

also

$$\alpha\bigl(T^n x_1, T^{n+2}x_1\bigr) \geq 1 \quad \text{for all } n \geq 1. \tag{2.2}$$

From condition (1) and (2.1), for every $n \geq 1$, we write

$$\begin{aligned}
&\theta\bigl(d\bigl(T^n x_1, T^{n+1}x_1\bigr)\bigr) \\
&\quad \leq \alpha\bigl(T^{n-1}x_1, T^n x_1\bigr) \cdot \theta\bigl(d\bigl(T^{n-1}x_1, T^n x_1\bigr)\bigr) \\
&\quad \leq \left[\theta\left(\max\left\{\begin{array}{c} d(T^{n-1}x_1, T^n x_1), d(T^{n-1}x_1, TT^{n-1}x_1), \\ d(T^n x_1, TT^n x_1), \frac{d(T^{n-1}x_1, TT^{n-1}x_1)d(T^n x_1, TT^n x_1)}{1+d(T^{n-1}x_1, T^n x_1)} \end{array}\right\}\right)\right]^k \\
&\quad = \left[\theta\left(\max\left\{\begin{array}{c} d(T^{n-1}x_1, T^n x_1), d(T^n x_1, T^{n+1}x_1), \\ \frac{d(T^{n-1}x_1, T^n x_1)d(T^n x_1, T^{n+1}x_1)}{1+d(T^{n-1}x_1, T^n x_1)} \end{array}\right\}\right)\right]^k \\
&\quad = \bigl[\theta\bigl(\max\bigl\{d\bigl(T^{n-1}x_1, T^n x_1\bigr), d\bigl(T^n x_1, T^{n+1}x_1\bigr)\bigr\}\bigr)\bigr]^k.
\end{aligned} \tag{2.3}$$

If there exists $n \geq 1$ such that $\max\{d(T^{n-1}x_1, T^n x_1), d(T^n x_1, T^{n+1}x_1)\} = d(T^n x_1, T^{n+1}x_1)$, then inequality (2.3) turns into

$$\theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big) \leq \big[\theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big)\big]^k,$$

this implies

$$\ln\big[\theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big)\big] \leq k \ln\big[\theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big)\big],$$

which is a contradiction with $k \in (0,1)$. Therefore $\max\{d(T^{n-1}x_1, T^n x_1), d(T^n x_1, T^{n+1}x_1)\} = d(T^{n-1}x_1, T^n x_1)$ for all $n \geq 1$. Thus, from (2.3), we have

$$\theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big) \leq \big[\theta\big(d\big(T^{n-1}x_1, T^n x_1\big)\big)\big]^k \quad \text{for all } n \geq 1.$$

This implies

$$\begin{aligned}\theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big) &\leq \big[\theta\big(d\big(T^{n-1}x_1, T^n x_1\big)\big)\big]^k \\ &\leq \big[\theta\big(d\big(T^{n-2}x_1, T^{n-1}x_1\big)\big)\big]^{k^2} \leq \cdots \leq \big[\theta\big(d(x_1, Tx_1)\big)\big]^{k^n}.\end{aligned}$$

Thus we have

$$1 \leq \theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big) \leq \big[\theta\big(d(x_1, Tx_1)\big)\big]^{k^n} \quad \text{for all } n \geq 1. \tag{2.4}$$

Letting $n \longrightarrow \infty$, we obtain

$$\lim_{n\longrightarrow\infty} \theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big) = 1, \tag{2.5}$$

which together with $(\Theta 2)$ gives as

$$\lim_{n\longrightarrow\infty} d\big(T^n x_1, T^{n+1}x_1\big) = 0.$$

From condition $(\Theta 3)$, there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that

$$\lim_{n\longrightarrow\infty} \frac{\theta(d(T^n x_1, T^{n+1}x_1)) - 1}{[d(T^n x_1, T^{n+1}x_1)]^r} = \ell.$$

Suppose that $\ell < \infty$. In this case, let $B = \frac{\ell}{2} > 0$. From the definition of the limit, there exists $n_0 \geq 1$ such that

$$\left|\frac{\theta(d(T^n x_1, T^{n+1}x_1)) - 1}{[d(T^n x_1, T^{n+1}x_1)]^r} - \ell\right| \leq B \quad \text{for all } n \geq n_0.$$

This implies

$$\frac{\theta(d(T^n x_1, T^{n+1}x_1)) - 1}{[d(T^n x_1, T^{n+1}x_1)]^r} \geq \ell - B = B \quad \text{for all } n \geq n_0.$$

Then

$$n\big[d\big(T^n x_1, T^{n+1}x_1\big)\big]^r \le An\big[\theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big)-1\big] \quad \text{for all } n \ge n_0,$$

where $A = \frac{1}{B}$. Suppose now that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \ge 1$ such that

$$\frac{\theta(d(T^n x_1, T^{n+1}x_1)) - 1}{[d(T^n x_1, T^{n+1}x_1)]^r} \ge B \quad \text{for all } n \ge n_0.$$

This implies

$$n\big[d\big(T^n x_1, T^{n+1}x_1\big)\big]^r \le An\big[\theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big)-1\big] \quad \text{for all } n \ge n_0,$$

where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_0 \ge 1$ such that

$$n\big[d\big(T^n x_1, T^{n+1}x_1\big)\big]^r \le An\big[\theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big)-1\big] \quad \text{for all } n \ge n_0.$$

By using (2.4), we get

$$n\big[d\big(T^n x_1, T^{n+1}x_1\big)\big]^r \le An\big(\big[\theta\big(d(x_1, Tx_1)\big)\big]^{k^n} - 1\big) \quad \text{for all } n \ge n_0. \tag{2.6}$$

Letting $n \longrightarrow \infty$ in the inequality (2.6), we obtain

$$\lim_{n\longrightarrow\infty} n\big[d\big(T^n x_1, T^{n+1}x_1\big)\big]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$d\big(T^n x_1, T^{n+1}x_1\big) \le \frac{1}{n^{\frac{1}{r}}} \quad \text{for all } n \ge n_1. \tag{2.7}$$

Now, we will prove that T has a periodic point. Suppose that it is not the case, then $T^n x_1 \ne T^m x_1$ for all $n, m \ge 1$ such that $n \ne m$. Using condition (1) and (2.2), we get

$$\begin{aligned}
&\theta\big(d\big(T^n x_1, T^{n+2}x_1\big)\big) \\
&\quad\le \alpha\big(T^{n-1}x_1, T^{n+1}x_1\big)\cdot\theta\big(d\big(T^{n-1}x_1, T^{n+1}x_1\big)\big) \\
&\quad\le \left[\theta\left(\max\left\{\begin{array}{c} d(T^{n-1}x_1, T^{n+1}x_1), d(T^{n-1}x_1, TT^{n-1}x_1), \\ d(T^{n+1}x_1, TT^{n+1}x_1), \frac{d(T^{n-1}x_1, TT^{n-1}x_1)d(T^{n+1}x_1, TT^{n+1}x_1)}{1+d(T^{n-1}x_1, T^{n+1}x_1)} \end{array}\right\}\right)\right]^k \\
&\quad= \left[\theta\left(\max\left\{\begin{array}{c} d(T^{n-1}x_1, T^{n+1}x_1), d(T^{n-1}x_1, T^n x_1), \\ d(T^{n+1}x_1, T^{n+2}x_1), \frac{d(T^{n-1}x_1, T^n x_1)d(T^{n+1}x_1, T^{n+2}x_1)}{1+d(T^{n-1}x_1, T^{n+1}x_1)} \end{array}\right\}\right)\right]^k \\
&\quad= \left[\theta\left(\max\left\{\begin{array}{c} d(T^{n-1}x_1, T^{n+1}x_1), d(T^{n-1}x_1, T^n x_1), \\ d(T^{n+1}x_1, T^{n+2}x_1) \end{array}\right\}\right)\right]^k.
\end{aligned} \tag{2.8}$$

Since θ is non-decreasing, we obtain from (2.8)

$$\theta\big(d\big(T^n x_1, T^{n+2}x_1\big)\big) \le \left[\max\left\{\begin{array}{c} \theta(d(T^{n-1}x_1, T^{n+1}x_1)), \theta(d(T^{n-1}x_1, T^n x_1)), \\ \theta(d(T^{n+1}x_1, T^{n+2}x_1)) \end{array}\right\}\right]^k. \tag{2.9}$$

Let I be the set of $n \in \mathbb{N}$ such that

$$\begin{aligned} u_n &= \max\{\theta(d(T^{n-1}x_1, T^{n+1}x_1)), \theta(d(T^{n-1}x_1, T^n x_1)), \theta(d(T^{n+1}x_1, T^{n+2}x_1))\} \\ &= \theta(d(T^{n-1}x_1, T^{n+1}x_1)). \end{aligned}$$

If $|I| < \infty$ then there is $N \geq 1$ such that, for all $n \geq N$,

$$\begin{aligned} &\max\{\theta(d(T^{n-1}x_1, T^{n+1}x_1)), \theta(d(T^{n-1}x_1, T^n x_1)), \theta(d(T^{n+1}x_1, T^{n+2}x_1))\} \\ &\quad = \max\{\theta(d(T^{n-1}x_1, T^n x_1)), \theta(d(T^{n+1}x_1, T^{n+2}x_1))\}. \end{aligned}$$

In this case, we get from (2.9)

$$1 \leq \theta(d(T^n x_1, T^{n+2}x_1)) \leq [\max\{\theta(d(T^{n-1}x_1, T^n x_1)), \theta(d(T^{n+1}x_1, T^{n+2}x_1))\}]^k$$

for all $n \geq N$. Letting $n \longrightarrow \infty$ in the above inequality and using (2.5), we obtain

$$\lim_{n\longrightarrow\infty} \theta(d(T^n x_1, T^{n+2}x_1)) = 1.$$

If $|I| = \infty$, we can find a subsequence of $\{u_n\}$, then we denote also by $\{u_n\}$, such that

$$u_n = \theta(d(T^{n-1}x_1, T^{n+1}x_1)) \quad \text{for } n \text{ large enough.}$$

In this case, we obtain from (2.9)

$$\begin{aligned} 1 &\leq \theta(d(T^n x_1, T^{n+2}x_1)) \leq [\theta(d(T^{n-1}x_1, T^{n+1}x_1))]^k \\ &\leq [\theta(d(T^{n-2}x_1, T^n x_1))]^{k^2} \leq \cdots \leq [\theta(d(x_1, T^2 x_1))]^{k^n} \end{aligned}$$

for n large. Letting $n \longrightarrow \infty$ in the above inequality, we obtain

$$\lim_{n\longrightarrow\infty} \theta(d(T^n x_1, T^{n+2}x_1)) = 1. \tag{2.10}$$

Then in all cases, (2.10) holds. Using (2.10) and (Θ2), we have

$$\lim_{n\longrightarrow\infty} \theta(d(T^n x_1, T^{n+2}x_1)) = 0.$$

Similarly from (Θ3) there exists $n_2 \geq 1$ such that

$$d(T^n x_1, T^{n+2}x_1) \leq \frac{1}{n^{\frac{1}{r}}} \quad \text{for all } n \geq n_2. \tag{2.11}$$

Let $h = \max\{n_0, n_1\}$. we consider two cases.

Case 1: If $m > 2$ is odd, then writing $m = 2L + 1$, $L \geq 1$, using (2.7), for all $n \geq h$, we obtain

$$\begin{aligned} d(T^n x_1, T^{n+m}x_1) \leq{}& d(T^n x_1, T^{n+1}x_1) + d(T^{n+1}x_1, T^{n+2}x_1) + \cdots \\ &+ d(T^{n+2L}x_1, T^{n+2L+1}x_1) \end{aligned}$$

$$\begin{aligned}&\leq \frac{1}{n^{\frac{1}{r}}}+\frac{1}{(n+1)^{\frac{1}{r}}}+\cdots+\frac{1}{(n+2L)^{\frac{1}{r}}}\\&\leq \sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{r}}}.\end{aligned}$$

Case 2: If $m > 2$ is even, then writing $m = 2L$, $L \geq 2$, using (2.7) and (2.11), for all $n \geq h$, we have

$$\begin{aligned}d\bigl(T^n x_1, T^{n+m}x_1\bigr) &\leq d\bigl(T^n x_1, T^{n+2}x_1\bigr)+d\bigl(T^{n+2}x_1, T^{n+3}x_1\bigr)+\cdots\\&\quad + d\bigl(T^{n+2L-1}x_1, T^{n+2L}x_1\bigr)\\&\leq \frac{1}{n^{\frac{1}{r}}}+\frac{1}{(n+2)^{\frac{1}{r}}}+\cdots+\frac{1}{(n+2L-1)^{\frac{1}{r}}}\\&\leq \sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{r}}}.\end{aligned}$$

Thus, combining all cases, we have

$$d\bigl(T^n x_1, T^{n+m}x_1\bigr) \leq \sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{r}}} \quad \text{for all } n \geq h, m \geq 1.$$

Since the series $\sum_{i=n}^{\infty}\frac{1}{i^{\frac{1}{r}}}$ is convergent (since $\frac{1}{r} > 1$), we deduce that $\{T^n x_1\}$ is a Cauchy sequence. From the completeness of X, there is $x_* \in X$ such that $T^n x_1 \longrightarrow x_*$ as $n \longrightarrow \infty$. Now, since T is continuous we have

$$x_* = \lim_{n\longrightarrow\infty} T^{n+1}x_1 = \lim_{n\longrightarrow\infty} T\bigl(T^n x_1\bigr) = T\Bigl(\lim_{n\longrightarrow\infty} T^n x_1\Bigr) = Tx_*.$$

We obtain $x_* = Tx_*$, which is a contradiction with the assumption that T does not have a periodic point. Thus T has a periodic point, say x_* of period q. Suppose that the set of fixed points of T is empty. Then we have

$$q > 1 \quad \text{and} \quad d(x_*, Tx_*) > 0.$$

By using condition (1) and (2.1), we get

$$\begin{aligned}\theta\bigl(d(x_*, Tx_*)\bigr) &= \theta\bigl(d\bigl(T^q x_*, T^{q+1}x_*\bigr)\bigr)\\&\leq \alpha\bigl(T^{q-1}x_*, T^q x_*\bigr)\cdot\theta\bigl(d\bigl(T^q x_*, T^{q+1}x_*\bigr)\bigr)\\&\leq \bigl[\theta\bigl(d(x_*, Tx_*)\bigr)\bigr]^{k^q} < \theta\bigl(d(x_*, Tx_*)\bigr),\end{aligned}$$

which is a contradiction. Thus the set of fixed points of T is non-empty (that is, T has at least one fixed point). □

Since a metric space is a Branciari metric space, we can obtain the following result from Theorem 17.

Corollary 18 *Let (X,d) be a complete metric space, $T : X \longrightarrow X$ be a given map and let $\alpha : X \times X \longrightarrow [0,\infty)$ be a mapping. Suppose that the following conditions hold*:

(1) *there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$x,y \in X, \quad d(Tx,Ty) \neq 0 \quad \Longrightarrow \quad \alpha(x,y) \cdot \theta\big(d(Tx,Ty)\big) \leq \big[\theta\big(R(x,y)\big)\big]^k,$$

where

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\},$$

(2) *there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$ and $\alpha(x_1,T^2x_1) \geq 1$,*

(3) *T is a triangular α-orbital admissible mapping,*

(4) *T is continuous.*

Then T has a fixed point $x_* \in X$ and $\{T^n x_1\}$ converges to x_*.

In the next theorem we omit the continuity hypothesis of T.

Theorem 19 *Let (X,d) be a complete BMS, $T : X \longrightarrow X$ be a given map and let $\alpha : X \times X \longrightarrow [0,\infty)$ be a mapping. Suppose that the following conditions hold*:

(1) *there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$x,y \in X, \quad d(Tx,Ty) \neq 0 \quad \Longrightarrow \quad \alpha(x,y) \cdot \theta\big(d(Tx,Ty)\big) \leq \big[\theta\big(R(x,y)\big)\big]^k,$$

where

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\},$$

(2) *there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$ and $\alpha(x_1,T^2x_1) \geq 1$,*

(3) *T is a triangular α-orbital admissible mapping,*

(4) *if $\{T^n x_1\}$ is a sequence in X such that $\alpha(T^n x_1, T^{n+1}x_1) \geq 1$ for all n and $x_n \longrightarrow x \in X$ as $n \longrightarrow \infty$, then there exists a subsequence $\{T^{n(k)}x_1\}$ of $\{T^n x_1\}$ such that $\alpha(T^{n(k)}x_1, x) \geq 1$ for all k,*

(5) *θ is continuous.*

Then T has a fixed point $x_* \in X$ and $\{T^n x_1\}$ converges to x_*.

Proof Let $x_1 \in X$ be such that $\alpha(x_1,Tx_1) \geq 1$ and $\alpha(x_1,T^2x_1) \geq 1$. Following the proof of Theorem 17, we see that the sequence $\{T^n x_1\}$ defined by $x_n = Tx_{n-1} = T^n x_1$ for all $n \geq 1$ converges to $x_* \in X$. From condition (4), we see that there exists a subsequence $\{T^{n(k)}x_1\}$ of $\{T^n x_1\}$ such that $\alpha(T^{n(k)}x_1, x_*) \geq 1$ for all k. We can suppose $T^{n(k)+1}x_1 \neq Tx_*$, then, from condition (1), we have

$$\begin{aligned}
&\theta\big(d\big(T^{n(k)+1}x_1, Tx_*\big)\big) \\
&\quad = \theta\big(d\big(T\big(T^{n(k)}x_1\big), Tx_*\big)\big) \\
&\quad \leq \alpha\big(T^{n(k)}x_1, x_*\big) \cdot \theta\big(d\big(T\big(T^{n(k)}x_1\big), Tx_*\big)\big)
\end{aligned}$$

$$\leq \left[\theta\left(\max\begin{Bmatrix} d(T^{n(k)}x_1, x_*), d(T^{n(k)}x_1, T(T^{n(k)}x_1)), \\ d(x_*, Tx_*), \frac{d(T^{n(k)}x_1, T(T^{n(k)}x_1))d(x_*, Tx_*)}{1+(dT^{n(k)}x_1, x_*)} \end{Bmatrix}\right)\right]^k$$

$$= \left[\theta\left(\max\begin{Bmatrix} d(T^{n(k)}x_1, x_*), d(T^{n(k)}x_1, T^{n(k)+1}x_1), \\ d(x_*, Tx_*), \frac{d(T^{n(k)}x_1, T^{n(k)+1}x_1))d(x_*, Tx_*)}{1+(dT^{n(k)}x_1, x_*)} \end{Bmatrix}\right)\right]^k. \quad (2.12)$$

Now, we suppose that $d(x_*, Tx_*) > 0$. Taking the limit as $k \longrightarrow \infty$ in (2.12), and by using the continuity of θ, and Lemma 14, we obtain

$$\theta\big(d(x_*, Tx_*)\big) \leq \big[\theta\big(d(x_*, Tx_*)\big)\big]^k < \theta\big(d(x_*, Tx_*)\big),$$

which is a contradiction. Thus we have $x_* = Tx_*$, which is also a contradiction with the assumption that T does not have a periodic point. Thus T has a periodic point, say x_* of period q. Suppose that the set of fixed points of T is empty. Then we have

$$q > 1 \quad \text{and} \quad d(x_*, Tx_*) > 0.$$

By using condition (1) and (2.1), we get

$$\begin{aligned}\theta\big(d(x_*, Tx_*)\big) &= \theta\big(d\big(T^q x_*, T^{q+1}x_*\big)\big) \leq \alpha\big(T^{q-1}x_*, T^q x_*\big)\cdot\theta\big(d\big(T^q x_*, T^{q+1}x_*\big)\big) \\ &\leq \big[\theta\big(d(x_*, Tx_*)\big)\big]^{kq} < \theta\big(d(x_*, Tx_*)\big),\end{aligned}$$

which is a contradiction. Thus the set of fixed points of T is non-empty (that is, T has at least one fixed point). □

Example 20 Let $X = [-2,-1] \cup \{0\} \cup [1,2]$. Define $d : X \times X \longrightarrow [0,\infty)$ as follows:

$$d(x,x) = 0, \quad \text{for all } x \in X,$$

$$d(1,2) = d(2,1) = 3,$$

$$d(1,-1) = d(-1,1) = d(-1,2) = d(2,-1) = 1,$$

$$d(x,y) = |x-y|, \quad \text{otherwise}.$$

It is clear that (X,d) is a complete BMS, but it is not metric space because d does not satisfy triangle inequality on X. Indeed,

$$3 = d(1,2) > d(1,-1) + d(-1,2) = 1 + 1 = 2.$$

Let $T : X \longrightarrow X$ be the mapping defined by

$$Tx = \begin{cases} -x & \text{if } x \in [-2,-1) \cup (1,2], \\ 0 & \text{if } x \in \{-1,0,1\}. \end{cases}$$

Let $\alpha : X \times X \longrightarrow [0,\infty)$ be given by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } xy \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$

Also define $\theta : (0,\infty) \longrightarrow (1,\infty)$ by

$$\theta(t) = e^{\sqrt{te^t}}.$$

Obviously, T is triangular α-orbital admissible mapping. Also the hypotheses of Theorem 19 are satisfied by T and, hence, T has a fixed point.

Corollary 21 *Let (X,d) be a complete metric space, $T : X \longrightarrow X$ be a given map and let $\alpha : X \times X \longrightarrow [0,\infty)$ be a mapping. Suppose that the following conditions hold*:

(1) *there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \quad \Longrightarrow \quad \alpha(x,y) \cdot \theta\bigl(d(Tx,Ty)\bigr) \leq \bigl[\theta\bigl(R(x,y)\bigr)\bigr]^k,$$

where

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\},$$

(2) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$ and $\alpha(x_1, T^2x_1) \geq 1$,*

(3) *T is a triangular α-orbital admissible mapping,*

(4) *if $\{T^n x_1\}$ is a sequence in X such that $\alpha(T^n x_1, T^{n+1}x_1) \geq 1$ for all n and $x_n \longrightarrow x \in X$ as $n \longrightarrow \infty$, then there exists a subsequence $\{T^{n(k)}x_1\}$ of $\{T^n x_1\}$ such that $\alpha(T^{n(k)}x_1, x) \geq 1$ for all k,*

(5) *θ is continuous.*

Then T has a fixed point $x_* \in X$ and $\{T^n x_1\}$ converges to x_*.

To ensure the uniqueness of the fixed point, we shall consider the following hypothesis.

(H) for all $x \neq y \in X$, there exists $v \in X$ such that $\alpha(x,v) \geq 1$, $\alpha(y,v) \geq 1$, and $\alpha(v,Tv) \geq 1$.

Theorem 22 *Adding condition* (H) *to the hypothesis of Theorem* 17 *or Corollary* 18 (*respectively, Theorem* 19 *or Corollary* 21) *the uniqueness of the fixed point is obtained.*

Proof Suppose that x_* and y_* are two fixed points of T such that $x_* \neq y_*$. Then by (H), there exists $v \in X$ such that

$$\alpha(x_*, v) \geq 1, \qquad \alpha(y_*, v) \geq 1 \quad \text{and} \quad \alpha(v, Tv) \geq 1.$$

Since T is a triangular α-orbital admissible mapping, we see that

$$\alpha\bigl(x_*, T^n v\bigr) \geq 1, \qquad \alpha\bigl(y_*, T^n v\bigr) \geq 1 \quad \text{for all } n \geq 1.$$

By Theorem 17 (respectively, Theorem 19) we deduce that the sequence $\{T^n v\}$ converges to a fixed point z_* of T. We can suppose that $x_* \neq T^{n+1}v$ for all $n \geq 1$, then from condition (1), we have

$$\begin{aligned}
\theta\bigl(d\bigl(x_*, T^{n+1}v\bigr)\bigr) &= \theta\bigl(d\bigl(Tx_*, T^{n+1}v\bigr)\bigr) \leq \alpha\bigl(x_*, T^n v\bigr) \cdot \theta\bigl(d\bigl(Tx_*, T^{n+1}v\bigr)\bigr) \\
&\leq \left[\theta\left(\max\left\{\begin{array}{c} d(x_*, T^n v), d(x_*, Tx_*), \\ d(T^n v, T^{n+1}v), \frac{d(x_*, Tx_*)d(T^n v, T^{n+1}v)}{1+(x_*, T^n v)} \end{array}\right\}\right)\right]^k.
\end{aligned}$$

This implies

$$\theta\bigl(d\bigl(x_*, T^{n+1}v\bigr)\bigr) < \theta\left(\max\left\{\begin{matrix} d(x_*, T^n v), d(x_*, Tx_*), \\ d(T^n v, T^{n+1}v), \frac{d(x_*, Tx_*)d(T^n v, T^{n+1}v)}{1+(x_*, T^n v)} \end{matrix}\right\}\right).$$

Letting $n \longrightarrow \infty$ in the above equality, if $x_* \neq z_*$, then we get

$$d(x_*, z_*) < d(x_*, z_*),$$

which is a contradiction. Therefore, $x_* = z_*$. Similarly, we get $y_* = z_*$. Hence, $x_* = y_*$, which is a contradiction. □

Corollary 23 *Let (X,d) be a complete BMS and $T : X \longrightarrow X$ be a given mapping. Suppose that there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \quad \Longrightarrow \quad \theta\bigl(d(Tx, Ty)\bigr) \leq \bigl[\theta\bigl(R(x,y)\bigr)\bigr]^k,$$

where

$$R(x,y) = \max\left\{d(x,y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x,y)}\right\}.$$

Then T has a unique fixed point.

Proof Setting $\alpha(x,y) = 1$ for all $x, y \in X$ in Theorem 22, we get this result. □

Corollary 24 [18] *Let (X,d) be a complete BMS and $T : X \longrightarrow X$ be a given mapping. Suppose that there exist $\theta \in \Theta$ that is continuous and $k \in (0,1)$ such that*

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \quad \Longrightarrow \quad \theta\bigl(d(Tx, Ty)\bigr) \leq \bigl[\theta\bigl(M(x,y)\bigr)\bigr]^k,$$

where

$$M(x,y) = \max\bigl\{d(x,y), d(x, Tx), d(y, Ty)\bigr\}.$$

Then T has a unique fixed point.

Corollary 25 [1] *Let (X,d) be a complete BMS and $T : X \longrightarrow X$ be a given mapping. Suppose that there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \quad \Longrightarrow \quad \theta\bigl(d(Tx, Ty)\bigr) \leq \bigl[\theta\bigl(d(x,y)\bigr)\bigr]^k.$$

Then T has a unique fixed point.

Example 26 Let $X = \{0,1,2\}$ endow with the metric d given by $d(x,y) = |x - y|$ for all $x, y \in X$. It is easy to show that (X,d) is a complete metric space. Let $T : X \longrightarrow X$ be the mapping defined by

$$T(0) = 0, \qquad T(1) = 2, \qquad T(2) = 1,$$

and $\alpha : X \times X \longrightarrow [0,\infty)$ be given by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \left\{ \begin{array}{c} (0,1),(0,2),(1,1),(2,2), \\ (1,2),(2,1) \end{array} \right\}, \\ 0 & \text{otherwise.} \end{cases}$$

Also define $\theta : (0,\infty) \longrightarrow (1,\infty)$ by

$$\theta(t) = e^{\sqrt{t}}.$$

It is not difficult to show that T is triangular α-orbital admissible mapping. Also the hypotheses of Theorem 22 are satisfied by T and hence, T has a fixed point. But the result of Jleli *et al.* (Corollary 25) cannot be applied to T. Indeed for $x = 1$, $y = 0$, we have

$$\begin{aligned} \theta\bigl(d\bigl(T(1),T(0)\bigr)\bigr) &= \theta\bigl(d(2,0)\bigr) = e^{\sqrt{2}} \\ &\nleq [e]^k = \bigl[\theta\bigl(d(1,0)\bigr)\bigr]^k, \quad \text{for all } k \in (0,1). \end{aligned}$$

Now we will use the concept of an α-orbital attractive mapping introduced in [2].

Theorem 27 *Let (X,d) be a complete BMS, $T : X \longrightarrow X$ be a given map and let $\alpha : X \times X \longrightarrow [0,\infty)$ be a mapping. Suppose that the following conditions hold*:

(1) *there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$x,y \in X, \quad d(Tx,Ty) \neq 0 \quad \Longrightarrow \quad \alpha(x,y) \cdot \theta\bigl(d(Tx,Ty)\bigr) \leq \bigl[\theta\bigl(R(x,y)\bigr)\bigr]^k,$$

where

$$R(x,y) = \max\left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1 + d(x,y)} \right\},$$

(2) *there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$ and $\alpha(x_1,T^2x_1) \geq 1$,*

(3) *T is an α-orbital admissible mapping,*

(4) *T is an α-orbital attractive mapping,*

(5) *θ is continuous.*

Then T has a unique fixed point $x_* \in X$ and $\{T^n x_1\}$ converges to x_*.

Proof Let $x_1 \in X$ be such that $\alpha(x_1,Tx_1) \geq 1$ and $\alpha(x_1,T^2x_1) \geq 1$. We define the iterative sequence $\{x_n\}$ in X by the rule $x_n = Tx_{n-1} = T^n x_1$ for all $n \geq 1$. Obviously, if there exists $n_0 \geq 1$ for which $T^{n_0}x_1 = T^{n_0+1}x_1$ then $T^{n_0}x_1$ shall be a fixed point of T. Thus, we suppose that $T^n x_1 \neq T^{n+1}x_1$ for every $n \geq 1$. Since T is α -orbital admissible, we have

$$\alpha(x_1,Tx_1) \geq 1 \quad \text{implies} \quad \alpha\bigl(Tx_1,T^2x_1\bigr) \geq 1$$

and

$$\alpha\bigl(x_1,T^2x_1\bigr) \geq 1 \quad \text{implies} \quad \alpha\bigl(Tx_1,T^3x_1\bigr) \geq 1.$$

By continuing this process, we get

$$\alpha\big(T^n x_1, T^{n+1}x_1\big) \geq 1 \quad \text{for all } n \geq 1 \tag{2.13}$$

and

$$\alpha\big(T^n x_1, T^{n+2}x_1\big) \geq 1 \quad \text{for all } n \geq 1. \tag{2.14}$$

From condition (1) and (2.13), then for every $n \geq 1$, we write

$$\begin{aligned}
&\theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big) \\
&\quad \leq \alpha\big(T^{n-1}x_1, T^n x_1\big) \cdot \theta\big(d\big(T^{n-1}x_1, T^n x_1\big)\big) \\
&\quad \leq \left[\theta\left(\max\left\{\begin{matrix} d(T^{n-1}x_1, T^n x_1), d(T^{n-1}x_1, TT^{n-1}x_1), \\ d(T^n x_1, TT^n x_1), \frac{d(T^{n-1}x_1, TT^{n-1}x_1)d(T^n x_1, TT^n x_1)}{1+d(T^{n-1}x_1, T^n x_1)} \end{matrix}\right\}\right)\right]^k \\
&\quad = \left[\theta\left(\max\left\{\begin{matrix} d(T^{n-1}x_1, T^n x_1), d(T^n x_1, T^{n+1}x_1), \\ \frac{d(T^{n-1}x_1, T^n x_1)d(T^n x_1, T^{n+1}x_1)}{1+d(T^{n-1}x_1, T^n x_1)} \end{matrix}\right\}\right)\right]^k \\
&\quad = \big[\theta\big(\max\big\{d\big(T^{n-1}x_1, T^n x_1\big), d\big(T^n x_1, T^{n+1}x_1\big)\big\}\big)\big]^k.
\end{aligned} \tag{2.15}$$

If there exists $n \geq 1$ such that $\max\{d(T^{n-1}x_1, T^n x_1), d(T^n x_1, T^{n+1}x_1)\} = d(T^n x_1, T^{n+1}x_1)$, then inequality (2.15) turns into

$$\theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big) \leq \big[\theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big)\big]^k,$$

this implies

$$\ln\big[\theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big)\big] \leq k \ln\big[\theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big)\big],$$

which is a contradiction with $k \in (0,1)$. Therefore $\max\{d(T^{n-1}x_1, T^n x_1), d(T^n x_1, T^{n+1}x_1)\} = d(T^{n-1}x_1, T^n x_1)$ for all $n \geq 1$. Thus, from (2.15), we have

$$\theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big) \leq \big[\theta\big(d\big(T^{n-1}x_1, T^n x_1\big)\big)\big]^k \quad \text{for all } n \geq 1.$$

This implies

$$\begin{aligned}
\theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big) &\leq \big[\theta\big(d\big(T^{n-1}x_1, T^n x_1\big)\big)\big]^k \\
&\leq \big[\theta\big(d\big(T^{n-2}x_1, T^{n-1}x_1\big)\big)\big]^{k^2} \leq \cdots \leq \big[\theta\big(d(x_1, Tx_1)\big)\big]^{k^n}.
\end{aligned}$$

Thus we have

$$1 \leq \theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big) \leq \big[\theta\big(d(x_1, Tx_1)\big)\big]^{k^n} \quad \text{for all } n \geq 1. \tag{2.16}$$

Letting $n \longrightarrow \infty$, we obtain

$$\lim_{n\longrightarrow\infty} \theta\big(d\big(T^n x_1, T^{n+1}x_1\big)\big) = 1, \tag{2.17}$$

which together with (Θ2) gives

$$\lim_{n\longrightarrow\infty} d\bigl(T^n x_1, T^{n+1}x_1\bigr) = 0.$$

From condition (Θ3), there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that

$$\lim_{n\longrightarrow\infty} \frac{\theta(d(T^n x_1, T^{n+1}x_1)) - 1}{[d(T^n x_1, T^{n+1}x_1)]^r} = \ell.$$

Suppose that $\ell < \infty$. In this case, let $B = \frac{\ell}{2} > 0$. From the definition of the limit, there exists $n_0 \geq 1$ such that

$$\left|\frac{\theta(d(T^n x_1, T^{n+1}x_1)) - 1}{[d(T^n x_1, T^{n+1}x_1)]^r} - \ell\right| \leq B \quad \text{for all } n \geq n_0.$$

This implies

$$\frac{\theta(d(T^n x_1, T^{n+1}x_1)) - 1}{[d(T^n x_1, T^{n+1}x_1)]^r} \geq \ell - B = B \quad \text{for all } n \geq n_0.$$

Then

$$n\bigl[d\bigl(T^n x_1, T^{n+1}x_1\bigr)\bigr]^r \leq An\bigl[\theta\bigl(d\bigl(T^n x_1, T^{n+1}x_1\bigr)\bigr) - 1\bigr] \quad \text{for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Suppose now that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \geq 1$ such that

$$\frac{\theta(d(T^n x_1, T^{n+1}x_1)) - 1}{[d(T^n x_1, T^{n+1}x_1)]^r} \geq B \quad \text{for all } n \geq n_0.$$

This implies

$$n\bigl[d\bigl(T^n x_1, T^{n+1}x_1\bigr)\bigr]^r \leq An\bigl[\theta\bigl(d\bigl(T^n x_1, T^{n+1}x_1\bigr)\bigr) - 1\bigr] \quad \text{for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_0 \geq 1$ such that

$$n\bigl[d\bigl(T^n x_1, T^{n+1}x_1\bigr)\bigr]^r \leq An\bigl[\theta\bigl(d\bigl(T^n x_1, T^{n+1}x_1\bigr)\bigr) - 1\bigr] \quad \text{for all } n \geq n_0.$$

By using (2.16), we get

$$n\bigl[d\bigl(T^n x_1, T^{n+1}x_1\bigr)\bigr]^r \leq An\bigl(\bigl[\theta\bigl(d(x_1, Tx_1)\bigr)\bigr]^{k^n} - 1\bigr) \quad \text{for all } n \geq n_0. \tag{2.18}$$

Letting $n \longrightarrow \infty$ in the inequality (2.18), we obtain

$$\lim_{n\longrightarrow\infty} n\bigl[d\bigl(T^n x_1, T^{n+1}x_1\bigr)\bigr]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$d\bigl(T^n x_1, T^{n+1}x_1\bigr) \leq \frac{1}{n^{\frac{1}{r}}} \quad \text{for all } n \geq n_1. \tag{2.19}$$

Now, we will prove that T has a periodic point. Suppose that it is not the case, then $T^n x_1 \neq T^m x_1$ for all $m, n \geq 1$ such that $n \neq m$. Using condition (1) and (2.14), we get

$$\begin{aligned}
&\theta\big(d\big(T^n x_1, T^{n+2} x_1\big)\big) \\
&\quad \leq \alpha\big(T^{n-1} x_1, T^{n+1} x_1\big) \cdot \theta\big(d\big(T^{n-1} x_1, T^{n+1} x_1\big)\big) \\
&\quad \leq \left[\theta\left(\max\left\{\begin{matrix} d(T^{n-1}x_1, T^{n+1}x_1), d(T^{n-1}x_1, TT^{n-1}x_1), \\ d(T^{n+1}x_1, TT^{n+1}x_1), \frac{d(T^{n-1}x_1, TT^{n-1}x_1) d(T^{n+1}x_1, TT^{n+1}x_1)}{1+d(T^{n-1}x_1, T^{n+1}x_1)} \end{matrix}\right\}\right)\right]^k \\
&\quad = \left[\theta\left(\max\left\{\begin{matrix} d(T^{n-1}x_1, T^{n+1}x_1), d(T^{n-1}x_1, T^{n}x_1), \\ d(T^{n+1}x_1, T^{n+2}x_1), \frac{d(T^{n-1}x_1, T^{n}x_1) d(T^{n+1}x_1, T^{n+2}x_1)}{1+d(T^{n-1}x_1, T^{n+1}x_1)} \end{matrix}\right\}\right)\right]^k \\
&\quad = \left[\theta\left(\max\left\{\begin{matrix} d(T^{n-1}x_1, T^{n+1}x_1), d(T^{n-1}x_1, T^{n}x_1), \\ d(T^{n+1}x_1, T^{n+2}x_1) \end{matrix}\right\}\right)\right]^k.
\end{aligned} \tag{2.20}$$

Since θ is non-decreasing, we obtain from (2.20)

$$\theta\big(d\big(T^n x_1, T^{n+2} x_1\big)\big) \leq \left[\max\left\{\begin{matrix} \theta(d(T^{n-1}x_1, T^{n+1}x_1)), \theta(d(T^{n-1}x_1, T^{n}x_1)), \\ \theta(d(T^{n+1}x_1, T^{n+2}x_1)) \end{matrix}\right\}\right]^k. \tag{2.21}$$

Let I be the set of $n \in \mathbb{N}$ such that

$$\begin{aligned}
u_n &= \max\big\{\theta\big(d\big(T^{n-1}x_1, T^{n+1}x_1\big)\big), \theta\big(d\big(T^{n-1}x_1, T^{n}x_1\big)\big), \theta\big(d\big(T^{n+1}x_1, T^{n+2}x_1\big)\big)\big\} \\
&= \theta\big(d\big(T^{n-1}x_1, T^{n+1}x_1\big)\big).
\end{aligned}$$

If $|I| < \infty$ then there is $N \geq 1$ such that, for all $n \geq N$,

$$\begin{aligned}
&\max\big\{\theta\big(d\big(T^{n-1}x_1, T^{n+1}x_1\big)\big), \theta\big(d\big(T^{n-1}x_1, T^{n}x_1\big)\big), \theta\big(d\big(T^{n+1}x_1, T^{n+2}x_1\big)\big)\big\} \\
&\quad = \max\big\{\theta\big(d\big(T^{n-1}x_1, T^{n}x_1\big)\big), \theta\big(d\big(T^{n+1}x_1, T^{n+2}x_1\big)\big)\big\}.
\end{aligned}$$

In this case, we get from (2.21)

$$1 \leq \theta\big(d\big(T^n x_1, T^{n+2} x_1\big)\big) \leq \big[\max\big\{\theta\big(d\big(T^{n-1}x_1, T^{n}x_1\big)\big), \theta\big(d\big(T^{n+1}x_1, T^{n+2}x_1\big)\big)\big\}\big]^k$$

for all $n \geq N$. Letting $n \longrightarrow \infty$ in the above inequality and using (2.17), we obtain

$$\lim_{n \longrightarrow \infty} \theta\big(d\big(T^n x_1, T^{n+2} x_1\big)\big) = 1.$$

If $|I| = \infty$, we can find a subsequence of $\{u_n\}$, then we denote also by $\{u_n\}$, such that

$$u_n = \theta\big(d\big(T^{n-1}x_1, T^{n+1}x_1\big)\big) \quad \text{for } n \text{ large enough.}$$

In this case, we obtain from (2.21)

$$\begin{aligned}
1 &\leq \theta\big(d\big(T^n x_1, T^{n+2} x_1\big)\big) \leq \big[\theta\big(d\big(T^{n-1}x_1, T^{n+1}x_1\big)\big)\big]^k \\
&\leq \big[\theta\big(d\big(T^{n-2}x_1, T^{n}x_1\big)\big)\big]^{k^2} \leq \cdots \leq \big[\theta\big(d\big(x_1, T^2 x_1\big)\big)\big]^{k^n}
\end{aligned}$$

for n large. Letting $n \longrightarrow \infty$ in the above inequality, we obtain

$$\lim_{n\longrightarrow\infty} \theta\bigl(d\bigl(T^n x_1, T^{n+2}x_1\bigr)\bigr) = 1. \tag{2.22}$$

Then in all cases, (2.22) holds. Using (2.22) and (Θ2), we have

$$\lim_{n\longrightarrow\infty} \theta\bigl(d\bigl(T^n x_1, T^{n+2}x_1\bigr)\bigr) = 0.$$

Similarly from (Θ3) there exists $n_2 \geq 1$ such that

$$d\bigl(T^n x_1, T^{n+2}x_1\bigr) \leq \frac{1}{n^{\frac{1}{r}}} \quad \text{for all } n \geq n_2. \tag{2.23}$$

Let $h = \max\{n_0, n_1\}$. We consider two cases.

Case 1: If $m > 2$ is odd, then writing $m = 2L+1$, $L \geq 1$, using (2.19), for all $n \geq h$, we obtain

$$\begin{aligned} d\bigl(T^n x_1, T^{n+m}x_1\bigr) &\leq d\bigl(T^n x_1, T^{n+1}x_1\bigr) + d\bigl(T^{n+1}x_1, T^{n+2}x_1\bigr) + \cdots \\ &\quad + d\bigl(T^{n+2L}x_1, T^{n+2L+1}x_1\bigr) \\ &\leq \frac{1}{n^{\frac{1}{r}}} + \frac{1}{(n+1)^{\frac{1}{r}}} + \cdots + \frac{1}{(n+2L)^{\frac{1}{r}}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}. \end{aligned}$$

Case 2: If $m > 2$ is even, then writing $m = 2L$, $L \geq 2$, using (2.19) and (2.23), for all $n \geq h$, we have

$$\begin{aligned} d\bigl(T^n x_1, T^{n+m}x_1\bigr) &\leq d\bigl(T^n x_1, T^{n+2}x_1\bigr) + d\bigl(T^{n+2}x_1, T^{n+3}x_1\bigr) + \cdots \\ &\quad + d\bigl(T^{n+2L-1}x_1, T^{n+2L}x_1\bigr) \\ &\leq \frac{1}{n^{\frac{1}{r}}} + \frac{1}{(n+2)^{\frac{1}{r}}} + \cdots + \frac{1}{(n+2L-1)^{\frac{1}{r}}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}. \end{aligned}$$

Thus, combining all cases, we have

$$d\bigl(T^n x_1, T^{n+m}x_1\bigr) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}} \quad \text{for all } n \geq h, m \geq 1.$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}$ is convergent (since $\frac{1}{r} > 1$), we deduce that $\{T^n x_1\}$ is a Cauchy sequence. From the completeness of X, there $x_* \in X$ such that $T^n x_1 \longrightarrow x_*$ as $n \longrightarrow \infty$. Now, we prove that $x_* = Tx_*$. Since T is α-orbital attractive, we have for all $n \geq 1$

$$\alpha\bigl(T^n x_1, x_*\bigr) \geq 1 \quad \text{or} \quad \alpha\bigl(x_*, T^{n+1}x_1\bigr) \geq 1.$$

Hence there exists a subsequence $\{T^{n(k)}x_1\}$ of $\{T^n x_1\}$ such that

$$\alpha\big(T^{n(k)}x_1, x_*\big) \geq 1 \quad \text{or} \quad \alpha\big(x_*, T^{n(k)}x_1\big) \geq 1 \quad \text{for all } k \geq 1.$$

In the first case, without restriction of the generality, we can suppose that $T^{n(k)}x_1 \neq x_*$ for all k. Using condition (1), we have

$$\begin{aligned}
\theta\big(d\big(T^{n(k)+1}x_1, Tx_*\big)\big) &= \theta\big(d\big(TT^{n(k)}x_1, Tx_*\big)\big) \\
&\leq \alpha\big(T^{n(k)}x_1, x_*\big) \cdot \theta\big(d\big(TT^{n(k)}x_1, Tx_*\big)\big) \\
&\leq \left[\theta\left(\max\left\{\begin{matrix} d(T^{n(k)}x_1, x_*), d(T^{n(k)}x_1, T^{n(k)+1}x_1), \\ d(x_*, Tx_*), \frac{d(T^{n(k)}x_1, T^{n(k)+1}x_1)d(x_*, Tx_*)}{1+d(T^{n(k)}x_1, x_*)} \end{matrix}\right\}\right)\right]^k.
\end{aligned}$$

This implies

$$\theta\big(d\big(T^{n(k)+1}x_1, Tx_*\big)\big) \leq \left[\theta\left(\max\left\{\begin{matrix} d(T^{n(k)}x_1, x_*), d(T^{n(k)}x_1, T^{n(k)+1}x_1), \\ d(x_*, Tx_*), \frac{d(T^{n(k)}x_1, T^{n(k)+1}x_1)d(x_*, Tx_*)}{1+d(T^{n(k)}x_1, x_*)} \end{matrix}\right\}\right)\right]^k.$$

Letting $k \longrightarrow \infty$ in the above equality, using the continuity of θ and Lemma 14, we get

$$\theta\big(d(x_*, Tx_*)\big) \leq \big[\theta\big(d(x_*, Tx_*)\big)\big]^k < \theta\big(d(x_*, Tx_*)\big),$$

which is a contradiction. The second case is similar. Therefore, $x_* = Tx_*$, which is also a contradiction with the assumption that T does not have a periodic point. Thus T has a periodic point, say x_* of period q. Suppose that the set of fixed points of T is empty. Then we have

$$q > 1 \quad \text{and} \quad d(x_*, Tx_*) > 0.$$

By using condition (1) and (2.13), we get

$$\begin{aligned}
\theta\big(d(x_*, Tx_*)\big) &= \theta\big(d\big(T^q x_*, T^{q+1}x_*\big)\big) \leq \alpha\big(T^{q-1}x_*, T^q x_*\big) \cdot \theta\big(d\big(T^q x_*, T^{q+1}x_*\big)\big) \\
&\leq \big[\theta\big(d(x_*, Tx_*)\big)\big]^{kq} < \theta\big(d(x_*, Tx_*)\big),
\end{aligned}$$

which is a contradiction. Thus the set of fixed points of T is non-empty (that is, T has at least one fixed point).

If y_* is another fixed point of T such that $x_* \neq y_*$, since T is α-orbital attractive, we deduce that

$$\alpha\big(T^n x_1, y_*\big) \geq 1 \quad \text{or} \quad \alpha\big(y_*, T^{n+1}x_1\big) \geq 1.$$

Hence there exists a subsequence $\{T^{n(k)}x_1\}$ of $\{T^n x_1\}$ such that

$$\alpha\big(T^{n(k)}x_1, y_*\big) \geq 1 \quad \text{or} \quad \alpha\big(y_*, T^{n(k)}x_1\big) \geq 1 \quad \text{for all } k \geq 1.$$

In the first case, we have

$$\begin{aligned}
\theta\big(d\big(T^{n(k)+1}x_1, y_*\big)\big) &= \theta\big(d\big(T^{n(k)+1}x_1, Ty_*\big)\big) = \theta\big(d\big(TT^{n(k)}x_1, Ty_*\big)\big) \\
&\leq \alpha\big(T^{n(k)}x_1, y_*\big) \cdot \theta\big(d\big(TT^{n(k)}x_1, Ty_*\big)\big) \\
&\leq \left[\theta\left(\max\left\{\begin{matrix} d(T^{n(k)}x_1, y_*), d(T^{n(k)}x_1, T^{n(k)+1}x_1), \\ d(y_*, Ty_*), \frac{d(T^{n(k)}x_1, T^{n(k)+1}x_1)d(y_*, Ty_*)}{1+d(T^{n(k)}x_1, y_*)} \end{matrix}\right\}\right)\right]^k \\
&= \left[\theta\left(\max\left\{\begin{matrix} d(T^{n(k)}x_1, y_*), d(T^{n(k)}x_1, T^{n(k)+1}x_1), \\ d(y_*, Ty_*), \frac{d(T^{n(k)}x_1, T^{n(k)+1}x_1)d(y_*, Ty_*)}{1+d(T^{n(k)}x_1, y_*)} \end{matrix}\right\}\right)\right]^k \\
&< \theta\left(\max\left\{\begin{matrix} d(T^{n(k)}x_1, y_*), d(T^{n(k)}x_1, T^{n(k)+1}x_1), \\ d(y_*, Ty_*), \frac{d(T^{n(k)}x_1, T^{n(k)+1}x_1)d(y_*, Ty_*)}{1+d(T^{n(k)}x_1, y_*)} \end{matrix}\right\}\right).
\end{aligned}$$

This implies

$$\theta\big(d\big(T^{n(k)+1}x_1, y_*\big)\big) < \theta\left(\max\left\{\begin{matrix} d(T^{n(k)}x_1, y_*), d(T^{n(k)}x_1, T^{n(k)+1}x_1), \\ d(y_*, Ty_*), \frac{d(T^{n(k)}x_1, T^{n(k)+1}x_1)d(y_*, Ty_*)}{1+d(T^{n(k)}x_1, y_*)} \end{matrix}\right\}\right).$$

Letting $k \longrightarrow \infty$ in the above equality, we get

$$\theta\big(d(x_*, y_*)\big) < \theta\big(d(x_*, y_*)\big).$$

This is a contradiction. The second case is similar. □

Corollary 28 *Let (X,d) be a complete metric space, $T : X \longrightarrow X$ be a given map, and let $\alpha : X \times X \longrightarrow [0,\infty)$ be a mapping. Suppose that the following conditions hold*:

(1) *there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \quad \Longrightarrow \quad \alpha(x,y) \cdot \theta\big(d(Tx, Ty)\big) \leq \big[\theta\big(R(x,y)\big)\big]^k,$$

where

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\right\},$$

(2) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$ and $\alpha(x_1, T^2x_1) \geq 1$,*

(3) *T is an α-orbital admissible mapping,*

(4) *T is an α-orbital attractive mapping.*

Then T has a unique fixed point $x_* \in X$ and $\{T^n x_1\}$ converges to x_*.

Example 29 Let $X = \{0,6,7,8\}$ endow with the metric d given by $d(x,y) = |x-y|$ for all $x,y \in X$. It is easy to show that (X,d) is a complete metric space. Let $T : X \longrightarrow X$ be the mapping defined by

$$T(0) = T(6) = 7 \quad \text{and} \quad T(7) = T(8) = 8,$$

and $\alpha : X \times X \longrightarrow [0,\infty)$ be given by

$$\alpha(x,y) = \begin{cases} 0 & \text{if } (x,y) \in \{(6,7),(7,6)\}, \\ 1 & \text{otherwise.} \end{cases}$$

Also define $\theta : (0,\infty) \longrightarrow (1,\infty)$ by

$$\theta(t) = e^{t\sqrt{t}}.$$

It is easy to show that T is an α-orbital admissible and α-orbital attractive mapping. Also the hypotheses of Theorem 27 (Corollary 28) are satisfied by T, and hence T has a fixed point. But the result of Jleli *et al.* (Corollary 25) cannot be applied to T. Indeed for $x = 6$, $y = 7$, we have

$$\begin{aligned} \theta\bigl(d\bigl(T(6),T(7)\bigr)\bigr) &= \theta\bigl(d(7,8)\bigr) = e \\ &\nleq [e]^k = \bigl[\theta\bigl(d(6,7)\bigr)\bigr]^k, \quad \text{for all } k \in (0,1). \end{aligned}$$

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