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On some recent coincidence and immediate consequences in partially ordered b-metric spaces

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Abstract
The aim of this article is to present, improve and generalize some recent coincidence and coupled coincidence point results from several papers in the framework of partially ordered b-metric spaces. Two examples are also provided to support the superiority of the obtained results.

MSC: 47H10; 54H25

Keywords: coincidence point; coupled coincidence point; partially ordered b-metric space; weakly compatible; mixed g-monotone property

1 Introduction and preliminaries

It is well known that the Banach contraction principle (see [1]) plays an important role in various fields of applied mathematical analysis and scientific applications, and it has been generalized and improved in many different directions. Some of such generalizations are obtained via rational metric spaces, such as ordered Banach spaces, partially ordered metric spaces, 2-metric spaces, fuzzy metric spaces, probabilistic metric spaces, G-metric spaces, cone metric spaces, cone Banach spaces, b-metric spaces or metric type spaces, *etc.* (see [2–27]). One of the most influential spaces is b-metric space, also called metric type space by some authors, introduced by Bakhtin (see [16]) in 1989. Since then, a large number of papers on fixed point results in the setting of b-metric spaces have appeared (see [17–27]).

The following definitions and results will be needed in what follows.

Definition 1.1 ([28]) Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is called a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:

(b1) $d(x,y) = 0$ if and only if $x = y$;
(b2) $d(x,y) = d(y,x)$;
(b3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

In this case, the pair (X,d) is called a b-metric space. If $(X,\preceq)$ is still a partially ordered set, then $(X,\preceq,d)$ is called a partially ordered b-metric space.

Otherwise, for more concepts such as b-convergence, b-completeness, b-Cauchy sequence and b-closed set in b-metric spaces, we refer the reader to [20–29] and the references mentioned therein.

Definition 1.2 ([28, 29]) Let $(X, \preceq)$ be a partially ordered set and f, g, h be three self-maps on X such that $f(X) \cup g(X) \subseteq h(X)$. Then

(1) elements $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds;
(2) f is called monotone g-nondecreasing w.r.t. $\preceq$ if $gx \preceq gy$ implies $fx \preceq fy$. In particular, f is called nondecreasing w.r.t. $\preceq$ if $x \preceq y$ implies $fx \preceq fy$;
(3) the pair (f, g) is said to be weakly increasing if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$;
(4) the pair (f, g) is said to be partially weakly increasing if $fx \preceq gfx$ for all $x \in X$;
(5) f is said to be g-weakly isotone increasing if $fx \preceq gfx \preceq fgfx$ for all $x \in X$;
(6) the pair (f, g) is said to be weakly increasing with respect to h if and only if for all $x \in X$, $fx \preceq gy$ for all $y \in h^{-1}(fx)$, and $gx \preceq fy$ for all $y \in h^{-1}(gx)$;
(7) the ordered pair (f, g) is said to be partially weakly increasing with respect to h if $fx \preceq gy$ for all $y \in h^{-1}(fx)$;
(8) a partially ordered b-metric space $(X, \preceq, d)$ is said to be regular if the following conditions hold:
 (i) if a nondecreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,
 (ii) if a nonincreasing sequence $y_n \to y$, then $y_n \succeq y$ for all n;
(9) the pair (f, g) is said to be compatible if and only if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$;
(10) the pair (f, g) is said to be weakly compatible if f and g commute at their coincidence points (*i.e.*, $fgx = gfx$, whenever $fx = gx$).

Fixed point results in partially ordered metric spaces were firstly obtained by Ran and Reurings (see [30]) and then by Nieto and López (see [31, 32]). Subsequently, many authors presented numerous interesting and significant results in ordered metric and ordered b-metric spaces (see [3, 28, 29, 33–38]).

Throughout this paper, we introduce the denotations Ψ, Υ, Φ, Θ as follows.

Let Ψ be the family of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(a1) ψ is continuous,
(a2) ψ is nondecreasing,
(a3) $\psi(0) = 0 < \psi(t)$ for every $t > 0$.

In this case, ψ is said to be an altering distance function.

Let Υ be the family of all functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(b1) φ is right continuous,
(b2) φ is nondecreasing,
(b3) $\varphi(t) < t$ for every $t > 0$.

Let Φ be the family of all functions $\phi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(c1) ϕ is lower semi-continuous,
(c2) $\phi(t) = 0$ if and only if $t = 0$.

Let Θ be the family of all continuous functions $\theta : [0,\infty) \to [0,\infty)$ with $\theta(t) = 0$ if and only if $t = 0$.

In [29] authors introduced and proved two theorems as follows.

Let $(X,\preceq,d)$ be an ordered b-metric space with $s > 1$, and let $f,g,R,S : X \to X$ be four mappings. For all $x,y \in X$, set

$$M_s(x,y) = \max\left\{d(Sx,Ry), d(Sx,fx), d(Ry,gy), \frac{d(Sx,gy) + d(Ry,fx)}{2s}\right\}. \tag{1.1}$$

Theorem 1.3 ([29]) *Let $(X,\preceq,d)$ be a partially ordered complete b-metric space with $s > 1$. Let $f,g,R,S : X \to X$ be four mappings such that $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$. Suppose that for every two comparable elements $Sx, Ry \in X$, we have*

$$\psi\bigl(s^2 d(fx,gy)\bigr) \le \psi\bigl(M_s(x,y)\bigr) - \varphi\bigl(M_s(x,y)\bigr), \tag{1.2}$$

where $\psi,\varphi : [0,\infty) \to [0,\infty)$ are altering distance functions. Let f, g, R and S be continuous, the pairs (f,S) and (g,R) be compatible and the pairs (f,g) and (g,f) be partially weakly increasing with respect to R and S, respectively. Then the pairs (f,S) and (g,R) have a coincidence point z in X. Moreover, if Rz and Sz are comparable, then z is a coincidence point of f, g, R and S.

Theorem 1.4 ([29]) *Let $(X,\preceq,d)$ be a regular partially ordered complete b-metric space with $s > 1$, $f,g,R,S : X \to X$ be four mappings such that $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$ and $R(X)$ and $S(X)$ are b-closed subsets of X. Suppose that for every two comparable elements $Sx, Ry \in X$, we have*

$$\psi\bigl(s^2 d(fx,gy)\bigr) \le \psi\bigl(M_s(x,y)\bigr) - \varphi\bigl(M_s(x,y)\bigr), \tag{1.3}$$

where $\psi,\varphi : [0,\infty) \to [0,\infty)$ are altering distance functions. Then the pairs (f,S) and (g,R) have a coincidence point z in X provided that the pairs (f,S) and (g,R) are weakly compatible and the pairs (f,g) and (g,f) are partially weakly increasing with respect to R and S, respectively. Moreover, if Rz and Sz are comparable, then z is a coincidence point of f, g, R and S.

Similarly, in [35] authors introduced and proved the following results.

Let $(X,\preceq,d)$ be a partially ordered b-metric space with $s > 1$ and $T : X \to X$ and $g : X \to X$ be two mappings. For all $x,y \in X$, put

$$M(x,y) = \max\left\{d(gx,gy), d(gx,Tx), d(gy,Ty), \frac{d(gx,Ty) + d(gy,Tx)}{2s}\right\} \tag{1.4}$$

and

$$N(x,y) = \min\bigl\{d(gx,Tx), d(gy,Ty), d(gx,Ty), d(gy,Tx)\bigr\}. \tag{1.5}$$

The mapping T is called an almost generalized (ψ,φ,L)-contractive mapping with respect to g for some $\psi \in \Psi$, $\varphi \in \Upsilon$, and $L \ge 0$ if

$$\psi\bigl(s^3 d(Tx,Ty)\bigr) \le \varphi\bigl(\psi\bigl(M(x,y)\bigr)\bigr) + L\psi\bigl(N(x,y)\bigr) \tag{1.6}$$

for all $x,y \in X$ with $gx \preceq gy$.

Theorem 1.5 ([35]) *Suppose that $(X,\preceq,d)$ is a partially ordered complete b-metric space with $s>1$. Let $T:X\to X$ be an almost generalized (ψ,φ,L)-contractive mapping with respect to $g:X\to X$, and T and g be continuous such that T is a monotone g-nondecreasing mapping, commutative with g and $T(X)\subseteq g(X)$. If there exists $x_0\in X$ such that $gx_0\preceq Tx_0$, then T and g have a coincidence point in X.*

Theorem 1.6 ([35]) *Suppose that $(X,\preceq,d)$ is a partially ordered complete b-metric space with $s>1$. Let $T:X\to X$ be an almost generalized (ψ,φ,L)-contractive mapping with respect to $g:X\to X$, T be a monotone g-nondecreasing mapping and $T(X)\subseteq g(X)$. Also suppose that if $\{gx_n\}\subset X$ is a nondecreasing sequence with $gx_n\to gz$ in gX, then $gx_n\preceq gz$, $gz\preceq g(gz)$ for all n hold. Also suppose that gX is b-closed. If there exists $x_0\in X$ such that $gx_0\preceq Tx_0$, then T and g have a coincidence point. Further, if T and g commute at their coincidence points, then T and g have a common fixed point.*

In [34] authors repeated some well-known notions and proved the following new results.

Let $(X,\preceq)$ be a partially ordered set and $F:X\times X\to X$ and $g:X\to X$ be two mappings. Then

(1) an element $(x,y)\in X\times X$ is called a coupled coincidence point of F and g if

$$F(x,y)=gx,\qquad F(y,x)=gy.$$

(2) F and g are commutative if for all $x,y\in X$,

$$F(gx,gy)=g\bigl(F(x,y)\bigr).$$

(3) F is said to have the mixed g-monotone property if F is nondecreasing g-monotone in its first argument and is nonincreasing g-monotone in its second argument, that is, for any $x,y\in X$,

$$x_1,x_2\in X,\quad gx_1\preceq gx_2\quad\Rightarrow\quad F(x_1,y)\preceq F(x_2,y)$$

and

$$y_1,y_2\in X,\quad gy_1\preceq gy_2\quad\Rightarrow\quad F(x,y_1)\succeq F(x,y_2).$$

In particular, if g is an identity mapping, then F is said to have the mixed monotone property.

Let $(X,\preceq,d)$ be a partially ordered b-metric space with $s>1$, and let $T:X\times X\to X$ and $g:X\to X$ be two mappings. Set

$$\begin{aligned} &M_{s,T,g}(x,y,u,v)\\ &\quad=\max\biggl\{d(gx,gu),d(gy,gv),d\bigl(gx,T(x,y)\bigr),\\ &\qquad\frac{1}{2s}d\bigl(gu,T(u,v)\bigr),d\bigl(gy,T(y,x)\bigr),\frac{1}{2s}d\bigl(gv,T(v,u)\bigr),\\ &\qquad\frac{d(gx,T(u,v))+d(gu,T(x,y))}{2s},\frac{d(gy,T(v,u))+d(gv,T(y,x))}{2s}\biggr\} \end{aligned} \tag{1.7}$$

and

$$N_{T,g}(x,y,u,v) \\ = \min\bigl\{d\bigl(gx,T(x,y)\bigr),d\bigl(gu,T(u,v)\bigr),d\bigl(gu,T(x,y)\bigr),d\bigl(gx,T(u,v)\bigr)\bigr\}. \tag{1.8}$$

Let $\psi \in \Psi$, $\phi \in \Phi$ and $\theta \in \Theta$. The mapping T is called an almost generalized (ψ,ϕ,θ)-contractive mapping with respect to g if there exists $L \geq 0$ such that

$$\psi\bigl(s^3 d\bigl(T(x,y),T(u,v)\bigr)\bigr) \\ \leq \psi\bigl(M_{s,T,g}(x,y,u,v)\bigr) - \phi\bigl(M_{s,T,g}(x,y,u,v)\bigr) + L\theta\bigl(N_{T,g}(x,y,u,v)\bigr) \tag{1.9}$$

for all $x,y,u,v \in X$ with $gx \preceq gu$ and $gy \succeq gv$.

Theorem 1.7 ([34]) *Suppose that $(X,\preceq,d)$ is a partially ordered complete b-metric space with $s > 1$. Let $T : X \times X \to X$ be an almost generalized (ψ,ϕ,θ)-contractive mapping with respect to $g : X \to X$, and T and g be continuous such that T has the mixed g-monotone property and commutes with g. Also, suppose that $T(X \times X) \subseteq g(X)$. If there exists $(x_0,y_0) \in X \times X$ such that $gx_0 \preceq T(x_0,y_0)$ and $gy_0 \succeq T(y_0,x_0)$, then T and g have a coupled coincidence point in X.*

It needs emphasizing that the following crucial lemma is utilized again and again in proving of all main results from [29, 34] and [35].

Lemma 1.8 ([33]) *Let (X,d) be a b-metric space with $s \geq 1$ and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y, respectively. Then*

$$\frac{1}{s^2}d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Moreover, for each $z \in X$, we have

$$\frac{1}{s}d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq sd(x,z).$$

2 Main results

In this section, we improve and generalize coincidence and coupled coincidence point results of Theorems 1.3-1.7 in several directions without utilizing Lemma 1.8 in the proofs.

Theorem 2.1 *Let $(X,\preceq,d)$ be a partially ordered complete b-metric space with $s > 1$. Let $f,g,R,S : X \to X$ be four mappings such that $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$. Suppose that for every two comparable elements $Sx,Ry \in X$, we have*

$$s^\varepsilon d(fx,gy) \leq M_s(x,y), \tag{2.1}$$

where $\varepsilon > 1$ is a constant and $M_s(x,y)$ is given by (1.1). Let f, g, R and S be continuous, the pairs (f,S) and (g,R) be compatible and the pairs (f,g) and (g,f) be partially weakly *increasing with respect to R and S, respectively. Then the pairs* (f,S) *and* (g,R) *have a coincidence point z in X. Moreover, if Rz and Sz are comparable, then z is a coincidence point of* f, g, R *and S.*

Proof Let x_0 be an arbitrary point of X. Similar to [29], we construct a sequence $\{z_n\}$ in X such that $z_{2n+1} = fx_{2n} = Rx_{2n+1}$ and $z_{2n+2} = gx_{2n+1} = Sx_{2n+2}$ for all $n \geq 0$. Since the pairs (f,g) and (g,f) are partially weakly increasing with respect to R and S, respectively, it follows that $z_n \preceq z_{n+1}$ for all $n \geq 1$. We complete the proof only in two steps.

Step I. We prove that

$$d(z_{n+1}, z_{n+2}) \leq \lambda d(z_n, z_{n+1}) \tag{2.2}$$

for all $n \geq 1$, where $\lambda \in [0, \frac{1}{s})$.

We first assume that $z_n \neq z_{n+1}$ for all $n \geq 1$. Since $Sx_{2n} = z_{2n}$ and $Rx_{2n-1} = z_{2n-1}$ are comparable, then (2.1) means that

$$\begin{aligned}
s^{\varepsilon} d(z_{2n}, z_{2n+1}) &= s^{\varepsilon} d(gx_{2n-1}, fx_{2n}) \\
&= s^{\varepsilon} d(fx_{2n}, gx_{2n-1}) \\
&\leq \max\biggl\{ d(Sx_{2n}, Rx_{2n-1}), d(Sx_{2n}, fx_{2n}), d(Rx_{2n-1}, gx_{2n-1}), \\
&\quad \frac{d(Sx_{2n}, gx_{2n-1}) + d(Rx_{2n-1}, fx_{2n})}{2s} \biggr\} \\
&= \max\biggl\{ d(z_{2n}, z_{2n-1}), d(z_{2n}, z_{2n+1}), d(z_{2n-1}, z_{2n}), \\
&\quad \frac{d(z_{2n}, z_{2n}) + d(z_{2n-1}, z_{2n+1})}{2s} \biggr\} \\
&\leq \max\biggl\{ d(z_{2n-1}, z_{2n}), d(z_{2n}, z_{2n+1}), \frac{d(z_{2n-1}, z_{2n}) + d(z_{2n}, z_{2n+1})}{2} \biggr\} \\
&\leq \max\bigl\{ d(z_{2n-1}, z_{2n}), d(z_{2n}, z_{2n+1}) \bigr\}.
\end{aligned} \tag{2.3}$$

If $d(z_{2n-1}, z_{2n}) \leq d(z_{2n}, z_{2n+1})$, then (2.3) becomes

$$s^{\varepsilon} d(z_{2n}, z_{2n+1}) \leq d(z_{2n}, z_{2n+1}),$$

which gives a contradiction (because $s^{\varepsilon} > 1$). Thus

$$s^{\varepsilon} d(z_{2n}, z_{2n+1}) \leq d(z_{2n-1}, z_{2n}). \tag{2.4}$$

Again, since $Sx_{2n} = z_{2n}$ and $Rx_{2n+1} = z_{2n+1}$ are comparable, then (2.1) implies that

$$\begin{aligned}
s^{\varepsilon} d(z_{2n+1}, z_{2n+2}) &= s^{\varepsilon} d(fx_{2n}, gx_{2n+1}) \\
&\leq \max\biggl\{ d(Sx_{2n}, Rx_{2n+1}), d(Sx_{2n}, fx_{2n}), d(Rx_{2n+1}, gx_{2n+1}), \\
&\quad \frac{d(Sx_{2n}, gx_{2n+1}) + d(Rx_{2n+1}, fx_{2n})}{2s} \biggr\}
\end{aligned}$$

$$
\begin{aligned}
&= \max\left\{d(z_{2n},z_{2n+1}),d(z_{2n},z_{2n+1}),d(z_{2n+1},z_{2n+2}),\right.\\
&\quad \left.\frac{d(z_{2n},z_{2n+2})+d(z_{2n+1},z_{2n+1})}{2s}\right\}\\
&\le \max\left\{d(z_{2n},z_{2n+1}),d(z_{2n+1},z_{2n+2}),\frac{d(z_{2n},z_{2n+1})+d(z_{2n+1},z_{2n+2})}{2}\right\}\\
&\le \max\bigl\{d(z_{2n},z_{2n+1}),d(z_{2n+1},z_{2n+2})\bigr\}. \qquad (2.5)
\end{aligned}
$$

If $d(z_{2n},z_{2n+1}) \le d(z_{2n+1},z_{2n+2})$, then (2.5) becomes

$$s^{\varepsilon} d(z_{2n+1},z_{2n+2}) \le d(z_{2n+1},z_{2n+2}),$$

which gives a contradiction (because $s^{\varepsilon} > 1$). So

$$s^{\varepsilon} d(z_{2n+1},z_{2n+2}) \le d(z_{2n},z_{2n+1}). \qquad (2.6)$$

Now, combining (2.4) and (2.6), we get that (2.2), where $\lambda = \frac{1}{s^{\varepsilon}} \in [0,\frac{1}{s})$.

Assume now that $z_{n_0} = z_{n_0+1}$ for some n_0. If $n_0 = 2k-1$, then $z_{2k-1} = z_{2k}$ gives that $z_{2k} = z_{2k+1}$. Indeed, since $Sx_{2k} = z_{2k}$ and $Rx_{2k-1} = z_{2k-1}$ are comparable, then by (2.3) we have that

$$
\begin{aligned}
s^{\varepsilon} d(z_{2k},z_{2k+1}) &\le \max\bigl\{d(z_{2k-1},z_{2k}),d(z_{2k},z_{2k+1})\bigr\}\\
&= \max\bigl\{0,d(z_{2k},z_{2k+1})\bigr\} = d(z_{2k},z_{2k+1}),
\end{aligned}
$$

which establishes that $d(z_{2k},z_{2k+1}) = 0$, that is, $z_{2k} = z_{2k+1}$. If $n_0 = 2k$, then $z_{2k} = z_{2k+1}$ gives that $z_{2k+1} = z_{2k+2}$. Actually, since $Sx_{2k} = z_{2k}$ and $Rx_{2k+1} = z_{2k+1}$ are comparable, then by (2.5) we have that

$$
\begin{aligned}
s^{\varepsilon} d(z_{2k+1},z_{2k+2}) &\le \max\bigl\{d(z_{2k},z_{2k+1}),d(z_{2k+1},z_{2k+2})\bigr\}\\
&= \max\bigl\{0,d(z_{2k+1},z_{2k+2})\bigr\} = d(z_{2k+1},z_{2k+2}),
\end{aligned}
$$

which implies that $d(z_{2k+1},z_{2k+2}) = 0$, that is, $z_{2k+1} = z_{2k+2}$. Consequently, the sequence $\{z_n\}$ in both cases becomes constant for $n \ge n_0$ and hence (2.2) holds.

Step II. We show that f, g, R and S have a coincidence point.

Making the most of (2.2) and Lemma 3.1 of [18], we obtain that $\{z_n\}$ is a b-Cauchy sequence. Since (X,d) is b-complete, then there exists $z \in X$ such that z_n b-converges to z. Accordingly,

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} z_{2n} = z, \qquad \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} z_{2n+1} = z.$$

Note that (f,S) is compatible, that is, $\lim_{n\to\infty} d(Sfx_{2n},fSx_{2n}) = 0$. Otherwise, by the continuity of f and S, it is valid that $Sfx_{2n} \to Sz$ and $fSx_{2n} \to fz$, as $n \to \infty$. Now, we have that

$$
\begin{aligned}
\frac{1}{s} d(Sz,fz) &\le d(Sz,Sfx_{2n}) + d(Sfx_{2n},fz)\\
&\le d(Sz,Sfx_{2n}) + s\bigl[d(Sfx_{2n},fSx_{2n}) + d(fSx_{2n},fz)\bigr]. \qquad (2.7)
\end{aligned}
$$

Taking the limit as $n \to \infty$ in (2.7), we get $\frac{1}{s} d(Sz,fz) \le 0$, *i.e.*, $fz = Sz$.

Following an argument similar to that mentioned above, we obtain $gz = Rz$. Now that Sz and Rz are comparable, hence by (2.1) it is obvious that

$$s^{\varepsilon} d(fz, gz) \leq M_s(z, z) = d(Sz, Rz) = d(fz, gz),$$

which establishes that $fz = gz$ (because $s^{\varepsilon} > 1$). Therefore, $fz = gz = Sz = Rz$. □

Theorem 2.2 *Let $(X, \preceq, d)$ be a regular partially ordered complete b-metric space with $s > 1$, $f, g, R, S : X \to X$ be four mappings such that $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$ and $R(X)$ and $S(X)$ are b-closed subsets of X. Suppose that for every two comparable elements $Sx, Ry \in X$, we have*

$$s^{\varepsilon} d(fx, gy) \leq M_s(x, y), \tag{2.8}$$

where $\varepsilon > 1$ is a constant and $M_s(x, y)$ is given by (1.1). *Then the pairs (f, S) and (g, R) have a coincidence point z in X provided that the pairs (f, S) and (g, R) are weakly compatible and the pairs (f, g) and (g, f) are partially weakly increasing with respect to R and S, respectively. Moreover, if Rz and Sz are comparable, then z is a coincidence point of f, g, R and S.*

Proof Similar to the proof of Theorem 2.1, we can construct the sequence $\{z_n\}$ and obtain that there exists $z \in X$ such that $z_n \to z$ as $n \to \infty$. Since $R(X)$ and $S(X)$ are b-closed, $\{z_{2n+1}\} \subseteq R(X)$ and $\{z_{2n+2}\} \subseteq S(X)$, then there exist $u, v \in X$ such that $z = Ru$ and $z = Sv$. That is,

$$\lim_{n\to\infty} Rx_{2n+1} = \lim_{n\to\infty} z_{2n+1} = z = Sv,$$
$$\lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} z_{2n+2} = z = Sv.$$

We now prove that z is a coincidence point of f and S.

By using $Rx_{2n+1} \to Sv$ $(n \to \infty)$ and the regularity of $(X, \preceq, d)$, it follows that $Rx_{2n+1} \preceq Sv$. As a consequence, by (2.8) we have that

$$s^{\varepsilon} d(fv, gx_{2n+1}) \leq M_s(v, x_{2n+1}), \tag{2.9}$$

where

$$\begin{aligned} M_s(v, x_{2n+1}) &= \max\biggl\{ d(Sv, Rx_{2n+1}), d(Sv, fv), d(Rx_{2n+1}, gx_{2n+1}), \\ &\quad \frac{d(Sv, gx_{2n+1}) + d(Rx_{2n+1}, fv)}{2s} \biggr\} \\ &\leq \max\biggl\{ d(Sv, Rx_{2n+1}), d(Sv, fv), d(Rx_{2n+1}, gx_{2n+1}), \\ &\quad \frac{d(Sv, gx_{2n+1})}{2s} + \frac{d(Rx_{2n+1}, Sv) + d(Sv, fv)}{2} \biggr\} \\ &\to \max\biggl\{ 0, d(Sv, fv), 0, \frac{d(Sv, fv)}{2} \biggr\} = d(Sv, fv) \quad (n \to \infty). \end{aligned} \tag{2.10}$$

By virtue of the triangle inequality and (2.9), it may be verified that

$$\begin{aligned}\frac{1}{s}d(Sv,fv) &\le d(Sv,gx_{2n+1}) + d(gx_{2n+1},fv) \\ &\le d(Sv,gx_{2n+1}) + \frac{1}{s^{\varepsilon}}M_s(v,x_{2n+1}).\end{aligned} \tag{2.11}$$

Taking the limit as $n \to \infty$ in (2.11) together with (2.10), we arrive at

$$\frac{1}{s}d(Sv,fv) \le \frac{1}{s^{\varepsilon}}d(Sv,fv).$$

As a result, $Sv = fv$ (because $s^{\varepsilon} > s > 1$). Hence, $z = Sv = fv$. Next by the compatibility of f and S, we claim that $fz = fSv = Sfv = Sz$. That is to say, z is a coincidence point of f and S.

Similarly, it can be shown that z is a coincidence point of g and R. The remainder is the same as the proof of Theorem 2.1 and therefore we omit it. □

Corollary 2.3 *Let $(X,\preceq,d)$ be a partially ordered complete b-metric space with $s > 1$. Let $f,g : X \to X$ be two mappings. Suppose that for every comparable elements $x,y \in X$,*

$$s^{\varepsilon}d(fx,gy) \le M_s(x,y),$$

where

$$M_s(x,y) = \max\left\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy) + d(y,fx)}{2s}\right\}.$$

Then the pair (f,g) has a common fixed point z in X provided that the pair (f,g) is weakly increasing and either,

(a) *f and g are continuous, or*

(b) *$(X,d,\preceq)$ is regular.*

Proof Taking $R = S = I_X$ (an identity mapping on X) in Theorems 2.1 and 2.2, the desired result holds. □

Remark 2.4 Compared with Theorem 2.1, Theorem 2.2 omits the assumption of continuity of f, g, R and S, and replaces the compatibility of the pairs (f,S) and (g,R) by the weak compatibility of the pairs.

Remark 2.5 Theorem 2.1 and Theorem 2.2 greatly generalize Theorem 1.3 and Theorem 1.4, respectively. In fact, condition (2.1) or (2.8) is much wider than condition (1.2) or (1.3). On the one hand, we delete the functions ψ and φ. On the other hand, our condition is much more general because $\varepsilon > 1$ is arbitrary. In addition, the proofs of Theorem 2.1 and Theorem 2.2 are shorter than those of Theorem 1.3 and Theorem 1.4 because we never use Lemma 1.8, but Theorem 1.3 and Theorem 1.4 are strongly dependent on this lemma.

Definition 2.6 Let $(X,\preceq,d)$ be a partially ordered b-metric space with $s > 1$. The mapping $T : X \to X$ is called an almost generalized (ψ,L)-contractive mapping with respect to $g : X \to X$ for some $\psi \in \Psi$, $\varepsilon > 1$ and $L \ge 0$ if

$$\psi\left(s^{\varepsilon}d(Tx,Ty)\right) \le \psi\left(M(x,y)\right) + L\psi\left(N(x,y)\right) \tag{2.12}$$

for all $x, y \in X$ with $gx \preceq gy$, where $M(x,y)$ and $N(x,y)$ are given by (1.4) and (1.5), respectively.

Theorem 2.7 *Suppose that $(X, \preceq, d)$ is a partially ordered complete b-metric space with $s > 1$. Let $T : X \to X$ be an almost generalized (ψ, L)-contractive mapping with respect to $g : X \to X$, and T and g be continuous such that T is a monotone g-nondecreasing mapping, compatible with g and $T(X) \subseteq g(X)$. If there exists $x_0 \in X$ such that $gx_0 \preceq Tx_0$, then T and g have a coincidence point in X.*

Proof Similar to [35], we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_n = Tx_n = gx_{n+1} \quad (n \geq 0) \tag{2.13}$$

for which

$$gx_0 \preceq gx_1 \preceq \cdots \preceq gx_n \preceq gx_{n+1} \preceq \cdots. \tag{2.14}$$

The same as in [35] we can assume that $y_n \neq y_{n+1}$ for all $n \geq 0$. For this case we shall show that

$$d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n) \tag{2.15}$$

for all $n \geq 1$, where $\lambda \in [0, \frac{1}{s})$.

Indeed, by (2.12)-(2.14), we have that

$$\begin{aligned}\psi\left(s^{\varepsilon} d(y_n, y_{n+1})\right) &= \psi\left(s^{\varepsilon} d(Tx_n, Tx_{n+1})\right) \\ &\leq \psi\left(M(x_n, x_{n+1})\right) + L\psi\left(N(x_n, x_{n+1})\right),\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}M(x_n, x_{n+1}) &= \max\left\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{d(y_{n-1}, y_{n+1})}{2s}\right\} \\ &\leq \max\left\{d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{d(y_{n-1}, y_n) + d(y_n, y_{n+1})}{2}\right\} \\ &= \max\left\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\right\}\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}N(x_n, x_{n+1}) &= \min\left\{d(y_{n-1}, y_n), d(y_n, y_{n+1}), d(y_{n-1}, y_{n+1}), d(y_n, y_n)\right\} \\ &= 0.\end{aligned} \tag{2.18}$$

Hence, by (2.16)-(2.18), we arrive at

$$\psi\left(s^{\varepsilon} d(y_n, y_{n+1})\right) \leq \psi\left(\max\left\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\right\}\right). \tag{2.19}$$

If $d(y_n, y_{n+1}) \geq d(y_{n-1}, y_n) > 0$ for some $n \in \mathbb{N}$, then by (2.19) we get that

$$\psi\bigl(s^\varepsilon d(y_n, y_{n+1})\bigr) \leq \psi\bigl(d(y_n, y_{n+1})\bigr),$$

or equivalently,

$$s^\varepsilon d(y_n, y_{n+1}) \leq d(y_n, y_{n+1}).$$

This is a contradiction. Thus from (2.19) it follows that

$$s^\varepsilon d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n),$$

that is, (2.15) holds, where $\lambda = \frac{1}{s^\varepsilon} \in [0, \frac{1}{s})$.

Now combining (2.15) and Lemma 3.1 of [18], we claim that $\{y_n\} = \{Tx_n\} = \{gx_{n+1}\}$ is a b-Cauchy sequence. Since (X, d) is b-complete, then there exists $x \in X$ such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_{n+1} = x.$$

Thus by the compatibility of T and g, we get that

$$\lim_{n\to\infty} d(Tgx_n, gTx_n) = 0. \tag{2.20}$$

By the continuity of T and g, it may be verified that

$$\lim_{n\to\infty} Tgx_n = Tx, \qquad \lim_{n\to\infty} gTx_n = gx. \tag{2.21}$$

Hence by the triangle inequality together with (2.20) and (2.21), it ensures us that

$$\begin{aligned}\frac{1}{s} d(Tx, gx) &\leq d(Tx, Tgx_n) + d(Tgx_n, gx)\\ &\leq d(Tx, Tgx_n) + s\bigl[d(Tgx_n, gTx_n) + d(gTx_n, gx)\bigr] \to 0 \quad (n \to \infty).\end{aligned}$$

Therefore, we obtain that $Tx = gx$, that is, x is a coincidence point of T and g. □

In the following theorem we omit the assumption of continuity of T and g.

Theorem 2.8 *Suppose that $(X, \preceq, d)$ is a partially ordered complete b-metric space with $s > 1$. Let $T : X \to X$ be an almost generalized (ψ, L)-contractive mapping with respect to $g : X \to X$, T be a monotone g-nondecreasing mapping and $T(X) \subseteq g(X)$. Also suppose that if $\{gx_n\} \subset X$ is a nondecreasing sequence with $gx_n \to gz$ in gX, then $gx_n \preceq gz$, $gz \preceq g(gz)$ for all n hold. Also suppose that gX is b-closed. If there exists $x_0 \in X$ such that $gx_0 \preceq Tx_0$, then T and g have a coincidence point. Further, if T and g commute at their coincidence points, then T and g have a common fixed point.*

Proof By the proof of Theorem 2.7, we can show that $\{y_n\} = \{Tx_n\} = \{gx_{n+1}\}$ is a b-Cauchy sequence. Since gX is b-closed, then there exists $x \in X$ such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} gx_{n+1} = gx.$$

We shall show that x is a coincidence point of T and g. As a matter of fact, owing to $gx_n \preceq gx$ for all n, by (2.12) it should be noticed that

$$\psi\big(s^{\varepsilon} d(Tx_n, Tx)\big) \le \psi\big(M(x_n, x)\big) + L\psi\big(N(x_n, x)\big), \tag{2.22}$$

where

$$\begin{aligned} M(x_n, x) &= \max\left\{d(gx_n, gx), d(gx_n, Tx_n), d(gx, Tx), \frac{d(gx_n, Tx) + d(gx, Tx_n)}{2s}\right\} \\ &\le \max\left\{d(gx_n, gx), d(gx_n, Tx_n), d(gx, Tx), \frac{d(gx_n, gx) + d(gx, Tx)}{2} + \frac{d(gx, Tx_n)}{2s}\right\} \\ &\to \max\left\{0, 0, d(gx, Tx), \frac{d(gx, Tx)}{2}\right\} \\ &= d(gx, Tx) \quad (n \to \infty), \end{aligned} \tag{2.23}$$

and

$$N(x_n, x) = \min\big\{d(gx_n, Tx_n), d(gx, Tx), d(gx_n, Tx), d(gx, Tx_n)\big\} \to 0 \quad (n \to \infty). \tag{2.24}$$

Taking the limit as $n \to \infty$ from (2.22) together with (2.23) and (2.24), we deduce that

$$\psi\Big(s^{\varepsilon} \lim_{n\to\infty} d(Tx_n, Tx)\Big) \le \psi\big(d(gx, Tx)\big),$$

or equivalently,

$$\lim_{n\to\infty} d(Tx_n, Tx) \le \frac{1}{s^{\varepsilon}} d(gx, Tx). \tag{2.25}$$

However, note that

$$\frac{1}{s} d(gx, Tx) \le d(gx, gx_{n+1}) + d(Tx_n, Tx), \tag{2.26}$$

then (2.25) and (2.26) lead to a contradiction if $gx \neq Tx$. In other words, $gx = Tx$.

Set $y = gx = Tx$. Now that T and g commute at x, it follows that $Ty = T(gx) = g(Tx) = gy$. Since $gx \preceq g(gx) = gy$, then by (2.12) and $gx = Tx$ and $gy = Ty$, we demonstrate that

$$\begin{aligned} \psi\big(s^{\varepsilon} d(Tx, Ty)\big) &\le \psi\big(M(x, y)\big) + L\psi\big(N(x, y)\big) \\ &= \psi\big(d(Tx, Ty)\big) + 0 = \psi\big(d(Tx, Ty)\big), \end{aligned}$$

or equivalently,

$$s^{\varepsilon} d(Tx, Ty) \le d(Tx, Ty).$$

This is a contradiction if $Tx \neq Ty$. Hence, we claim that $Tx = Ty = y$. Therefore, $Ty = gy = y$. That is to say, y is a common fixed point of T and g. □

Corollary 2.9 *Let $(X, \preceq, d)$ be a partially ordered complete b-metric space with $s > 1$ and $T : X \to X$ be a nondecreasing mapping. Suppose that there exist $\psi \in \Psi$ and $L \geq 0$ such that*

$$\psi\left(s^{\varepsilon} d(Tx, Ty)\right) \leq \psi\left(M(x,y)\right) + L\psi\left(N(x,y)\right),$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2s}\right\}$$

and

$$N(x,y) = \min\left\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right\}$$

for all $x, y \in X$ with $x \preceq y$. Also suppose that either

(a) *$(X, d, \preceq)$ is regular, or*

(b) *T is continuous.*

If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point in X.

Remark 2.10 Theorems 2.7 and 2.8 improve and generalize Theorems 1.5 and 1.6 in many ways. First, Theorems 2.7 and 2.8 delete the function φ in (1.6), since due to the proofs of Theorem 1.5 and Theorem 1.6, it is superfluous based on the fact that $\varphi(t) \leq t$ for each $t \in [0, \infty)$. Second, condition (2.12) is wider than (1.6) because the constant $\varepsilon > 1$ is optional. Third, the compatible condition of Theorem 2.7 is weaker than the commutative condition of Theorem 1.5. This is because if T and g are commutative, then $Tgx_n = gTx_n$. This is natural that $\lim_{n\to\infty} d(Tgx_n, gTx_n) = 0$. That is to say, the pair (T, g) is compatible. However, the converse is not true. Otherwise, the proofs of Theorems 2.7 and 2.8 are shorter than the ones of Theorems 1.5 and 1.6 since they do not utilize Lemma 1.8, but Theorems 1.5 and 1.6 rely on this lemma entirely.

Definition 2.11 Let $(X, \preceq, d)$ be a partially ordered b-metric space with $s > 1$, $\psi \in \Psi$ and $\theta \in \Theta$. The mapping $T : X \times X \to X$ is called an almost generalized (ψ, θ)-contractive mapping with respect to $g : X \to X$ if there exists $L \geq 0$ such that

$$\psi\left(s^{\varepsilon} d\left(T(x,y), T(u,v)\right)\right) \leq \psi\left(M_{s,T,g}(x,y,u,v)\right) + L\theta\left(N_{T,g}(x,y,u,v)\right) \tag{2.27}$$

for all $x, y, u, v \in X$ with $gx \preceq gu$ and $gy \succeq gv$, where $M_{s,T,g}(x,y,u,v)$ and $N_{T,g}(x,y,u,v)$ are given by (1.7) and (1.8), respectively.

Theorem 2.12 *Suppose that $(X, \preceq, d)$ is a partially ordered complete b-metric space with $s > 1$. Let $T : X \times X \to X$ be an almost generalized (ψ, θ)-contractive mapping with respect to $g : X \to X$, and T and g be continuous such that T has the mixed g-monotone property and commutes with g. Also, suppose that $T(X \times X) \subseteq g(X)$. If there exists $(x_0, y_0) \in X \times X$ such that $gx_0 \preceq T(x_0, y_0)$ and $gy_0 \succeq T(y_0, x_0)$, then T and g have a coupled coincidence point in X.*

Proof By the given assumption and the proof of [34], Theorem 2.2 (also see Theorem 1.7), we construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = T(x_n, y_n), \qquad gy_{n+1} = T(y_n, x_n) \quad (n \geq 0)$$

for which $\{gx_n\}_{n=0}^{\infty}$ is nondecreasing and $\{gy_n\}_0^{\infty}$ is nonincreasing. Putting $x = x_n$, $y = y_n$, $u = x_{n+1}$ and $v = y_{n+1}$ in (2.27), we obtain that

$$\begin{aligned} \psi\bigl(s^{\varepsilon} d(gx_{n+1}, gx_{n+2})\bigr) \leq{}& \psi\bigl(M_{s,T,g}(x_n, y_n, x_{n+1}, y_{n+1})\bigr) \\ &+ L\theta\bigl(N_{T,g}(x_n, y_n, x_{n+1}, y_{n+1})\bigr). \end{aligned} \tag{2.28}$$

According to the proof of [34], Theorem 2.2, we get that

$$\begin{aligned} M_{s,T,g}(x_n, y_n, x_{n+1}, y_{n+1}) \leq{}& \max\bigl\{d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}), \\ & d(gy_n, gy_{n+1}), d(gy_{n+1}, gy_{n+2})\bigr\} \end{aligned} \tag{2.29}$$

and

$$N_{T,g}(x_n, y_n, x_{n+1}, y_{n+1}) = 0. \tag{2.30}$$

Since ψ is nondecreasing, then by (2.28)-(2.30) it is not hard to verify that

$$\begin{aligned} s^{\varepsilon} d(gx_{n+1}, gx_{n+2}) \leq{}& \max\bigl\{d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}), \\ & d(gy_n, gy_{n+1}), d(gy_{n+1}, gy_{n+2})\bigr\}. \end{aligned} \tag{2.31}$$

Similarly, putting $x = y_{n+1}$, $y = x_{n+1}$, $u = y_n$ and $v = x_n$ in (2.27), we acquire that

$$\begin{aligned} s^{\varepsilon} d(gy_{n+1}, gy_{n+2}) \leq{}& \max\bigl\{d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}), \\ & d(gy_n, gy_{n+1}), d(gy_{n+1}, gy_{n+2})\bigr\}. \end{aligned} \tag{2.32}$$

Further, denote

$$\delta_n = \max\bigl\{d(gx_{n+1}, gx_{n+2}), d(gy_{n+1}, gy_{n+2})\bigr\}, \tag{2.33}$$

$$\xi_n = \max\bigl\{d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}), d(gy_n, gy_{n+1}), d(gy_{n+1}, gy_{n+2})\bigr\}. \tag{2.34}$$

It follows immediately from (2.31)-(2.34) that

$$s^{\varepsilon} \delta_n \leq \xi_n. \tag{2.35}$$

Now, we shall prove that

$$\delta_n \leq \lambda \delta_{n-1} \tag{2.36}$$

for all $n \geq 1$, where $\lambda = \frac{1}{s^{\varepsilon}} \in [0, \frac{1}{s})$.

Indeed, if $\xi_n = \delta_n$, then (2.35) means $s^\varepsilon \delta_n \leq \delta_n$. This leads to $\delta_n = 0$ (because $s^\varepsilon > 1$) and (2.36) holds trivially. If $\xi_n = \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}$, *i.e.*, $\xi_n = \delta_{n-1}$, then (2.35) follows (2.36).

Now by (2.36) we get $\delta_n \leq \lambda^n \delta_0$. Therefore,

$$d(gx_{n+1}, gx_{n+2}) \leq \lambda^n \delta_0, \qquad d(gy_{n+1}, gy_{n+2}) \leq \lambda^n \delta_0.$$

Finally, according to [18], Lemma 3.1, the sequences $\{gx_n\}$ and $\{gy_n\}$ are b-Cauchy sequences. The rest of the proof is the same as in [34], Theorem 2.2. □

Remark 2.13 Theorem 2.8 is more superior in several aspects as compared to Theorem 1.7. Indeed, (2.27) dismisses the condition $-\phi(M_{s,T,g}(x,y,u,v))$ of (1.9). This indicates that (2.27) is much broader than (1.9). Further, the constant $\varepsilon > 1$ is much more general in (2.27) because it is not only restricted to $\varepsilon = 3$ in (1.9). In addition, the proof of Theorem 2.8 is simpler than the one of Theorem 1.7 because it ignores Lemma 1.8, but Theorem 1.7 depends on this lemma utterly.

The following examples show the superiority of the obtained results.

Example 2.14 Let $X = [0, \infty)$ and d on X be given by $d(x,y) = |x-y|^2$ for all $x, y \in X$. Then d is a b-metric on X, with $s = 2$. Define an ordering '$\preceq$' on X as follows:

$$x \preceq y \quad \Leftrightarrow \quad x \leq y, \quad \forall x, y \in X.$$

Define self-maps f, g, S and R on X by

$$fx = \ln\bigl(\sqrt{x^2+1} + x\bigr) = \sinh^{-1} x, \qquad Rx = \sinh 3x,$$
$$gx = \sinh^{-1}\left(\frac{x}{2}\right), \qquad Sx = \sinh 6x.$$

Take $1 < \varepsilon < 2$. Note that

$$\begin{aligned} s^\varepsilon d(fx, gy) &= 2^\varepsilon |fx - gy|^2 \\ &= 2^\varepsilon \left| \sinh^{-1} x - \sinh^{-1}\left(\frac{y}{2}\right)\right|^2 \\ &\leq 2^\varepsilon \left| x - \frac{y}{2}\right|^2 \\ &= 2^\varepsilon \frac{|6x-3y|^2}{36} \\ &\leq \frac{2^\varepsilon}{36} |\sinh 6x - \sinh 3y|^2 \\ &\leq |Sx - Ry|^2 \\ &= d(Sx, Ry) \\ &\leq M_s(x,y). \end{aligned}$$

Make full use of [29], Example 2.6, it is easy to see that all conditions of Theorem 2.1 are satisfied and hence the corresponding conclusions hold. However, if $1 < \varepsilon < 2$, then condition (1.2) does not hold but our condition (2.1) holds. As a consequence, this example shows that our theorem is a genuine generalization of Theorem 1.3.

Example 2.15 Let X be the set of Lebesgue measurable functions on $[0,1]$ such that $\int_0^1 x(t)\,\mathrm{d}t < 1$. Define $d : X \times X \to [0,\infty)$ by

$$d(x,y) = \int_0^1 \bigl|x(t) - y(t)\bigr|^2\,\mathrm{d}t.$$

Then d is a b-metric on X, with $s = 2$. Also, this space can also be equipped with a partial order given by

$$x,y \in X, \quad x \preceq y \quad \Leftrightarrow \quad x(t) \le y(t), \quad \forall t \in [0,1].$$

The operator $T : X \to X$ is defined by

$$Tx(t) = \frac{\sqrt{2}}{4} \ln\bigl(1 + \bigl|x(t)\bigr|\bigr).$$

Take $1 < \varepsilon < 3$, then

$$\begin{aligned}
s^\varepsilon d(Tx, Ty) &= 2^\varepsilon \int_0^1 \bigl|Tx(t) - Ty(t)\bigr|^2\,\mathrm{d}t \\
&= 2^\varepsilon \int_0^1 \left|\frac{\sqrt{2}}{4} \ln\bigl(1 + \bigl|x(t)\bigr|\bigr) - \frac{\sqrt{2}}{4} \ln\bigl(1 + \bigl|y(t)\bigr|\bigr)\right|^2\,\mathrm{d}t \\
&= 2^{\varepsilon-3} \int_0^1 \left|\ln\left(\frac{1 + |x(t)|}{1 + |y(t)|}\right)\right|^2\,\mathrm{d}t \\
&= 2^{\varepsilon-3} \int_0^1 \left|\ln\left(1 + \frac{|x(t)| - |y(t)|}{1 + |y(t)|}\right)\right|^2\,\mathrm{d}t \\
&\le 2^{\varepsilon-3} \int_0^1 \left|\ln\left(1 + \frac{|x(t) - y(t)|}{1 + |y(t)|}\right)\right|^2\,\mathrm{d}t \\
&\le 2^{\varepsilon-3} \int_0^1 \bigl|x(t) - y(t)\bigr|^2\,\mathrm{d}t \\
&< \int_0^1 \bigl|x(t) - y(t)\bigr|^2\mathrm{d}t = d(x,y) \\
&\le M(x,y).
\end{aligned}$$

Let $x_0 = 0$, $L = 0$ and $g = I_X$ (an identity mapping on X). Simple circulations show that all conditions of Theorem 2.7 are satisfied for any $\psi \in \Psi$ and hence T and g have a coincidence point in X. However, $1 < \varepsilon < 3$ never includes $\varepsilon = 3$. That is to say, this example is not applicable for Theorem 1.5. Consequently, our theorem is more convenient in applications.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

The authors contribute equally and significantly in writing this paper. All authors read and approved the final manuscript.

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