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Solvability of anti-periodic boundary value problem for coupled system of fractional p-Laplacian equation

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Abstract

This paper studies the existence of solutions for anti-periodic boundary value problem for a coupled system of the fractional p-Laplacian equation. Under certain nonlinear growth conditions of the nonlinearity, a new existence result is obtained by using the Schaefer fixed point theorem. As an application, an example to illustrate our result is given.

MSC: 34A08; 34B15

Keywords: coupled system; p-Laplacian equation; anti-periodic boundary value conditions; Schaefer fixed point theorem

1 Introduction

The subject of fractional calculus has gained considerable popularity and importance due to its frequent appearance in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, electromagnetic, *etc.* (see [1–4]). Recently, fractional differential equations have been of great interest due to the intensive development of theory of itself and its applications (see [5–10]). Moreover, the existence of solutions to some coupled systems of fractional differential equations have been studied by many authors (see [11–16]). For instance, Ahmad and Nieto (see [11]) considered a three-point boundary value problem for a coupled system of nonlinear fractional differential equations given by

$$\begin{cases} D^{\alpha}u(t) = f(t, v(t), D^{p}v(t)), & t \in (0,1), \\ D^{\beta}v(t) = g(t, u(t), D^{q}u(t)), & t \in (0,1), \\ u(0) = 0, \quad u(1) = \gamma u(\eta), \quad v(0) = 0, \quad v(1) = \gamma v(\eta), \end{cases}$$

where $1 < \alpha, \beta < 2$, $p, q, \gamma > 0$, $0 < \eta < 1$, $\alpha - q, \beta - p \geq 1$, $\gamma\eta^{\alpha-1}, \gamma\eta^{\beta-1} < 1$, and D^{α} is the standard Riemann-Liouville fractional derivative. Under certain growth conditions on f and g, an existence result was obtained by using the Schauder fixed point theorem. In addition, Bai and Fang (see [12]) discussed the existence of a positive solution to the singular coupled system of the form

$$\begin{cases} D^{s}u = f(t, v), & 0 < t < 1, \\ D^{p}v = g(t, u), & 0 < t < 1, \end{cases}$$

where $0 < s,p < 1$, D^s is the standard Riemann-Liouville fractional derivative, $f,g : (0,1] \times [0,+\infty) \to [0,+\infty)$ are two given continuous functions, and $\lim_{t\to 0^+} f(t,\cdot) = \lim_{t\to 0^+} g(t,\cdot) = +\infty$. A nonlinear alternative of Leray-Schauder type and the Krasnoselskii fixed point theorem in a cone were applied to establish the existence results on a positive solution.

The anti-periodic boundary value problems occur in the mathematical modeling of a variety of physical processes (see [17, 18]) and recently received considerable attention. For an example and details of the anti-periodic boundary value problems, see [19, 20] and the references therein.

The turbulent flow in a porous medium is a fundamental mechanics problem. For studying this type of problems, Leibenson (see [21]) introduced the p-Laplacian equation as follows:

$$\bigl(\phi_p\bigl(x'(t)\bigr)\bigr)' = f\bigl(t,x(t),x'(t)\bigr), \tag{1.1}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. Obviously, ϕ_p is invertible and its inverse operator is ϕ_q, where $q > 1$ is a constant such that $1/p + 1/q = 1$. In the past few decades, many important results as regards (1.1) with certain boundary value conditions have been obtained. We refer the readers to [22–25] and the references cited therein. However, as far as we know, there are relatively few results on the anti-periodic boundary value problems (ABVPs for short) for coupled systems of the fractional p-Laplacian equations.

Motivated by the works mentioned previously, in this paper, we investigate the existence of solutions for ABVP for a coupled system of the fractional p-Laplacian equation of the form

$$\begin{cases} D_{0^+}^{\beta}\phi_p(D_{0^+}^{\alpha}u(t)) = f(t,v(t),D_{0^+}^{\gamma}v(t)), & t\in[0,1],\\ D_{0^+}^{\delta}\phi_p(D_{0^+}^{\gamma}v(t)) = g(t,u(t),D_{0^+}^{\alpha}u(t)), & t\in[0,1],\\ u(0) = -u(1), \qquad D_{0^+}^{\alpha}u(0) = -D_{0^+}^{\alpha}u(1), \\ v(0) = -v(1), \qquad D_{0^+}^{\gamma}v(0) = -D_{0^+}^{\gamma}v(1), \end{cases} \tag{1.2}$$

where $0 < \alpha,\beta,\gamma,\delta \le 1$, $D_{0^+}^{\alpha}$ is a Caputo fractional derivative of order α, and $f,g : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous. Note that the nonlinear operator $D_{0^+}^{\beta}\phi_p(D_{0^+}^{\alpha})$ reduces to the linear operator $D_{0^+}^{\beta}D_{0^+}^{\alpha}$ when $p = 2$ and the additive index law

$$D_{0^+}^{\beta}D_{0^+}^{\alpha}u(t) = D_{0^+}^{\alpha+\beta}u(t)$$

holds under some reasonable constraints on the function u (see [26]).

The rest of this paper is organized as follows. Section 2 contains some necessary notations, definitions and lemmas. In Section 3, based on the Schaefer fixed point theorem, we establish one theorem on the existence of solutions for ABVP (1.2) (Theorem 3.1). Finally, in Section 4, an explicit example is given to illustrate the main result.

2 Preliminaries

For convenience of the readers, we present here some necessary basic knowledge and definitions as regards the fractional calculus theory, which can be found, for instance, in [27, 28].

Definition 2.1 The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds,$$

provided that the right side integral is pointwise defined on $(0, +\infty)$.

Definition 2.2 The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$\begin{aligned} D_{0^+}^{\alpha} u(t) &= I_{0^+}^{n-\alpha} \frac{d^n u(t)}{dt^n} \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\, ds, \end{aligned}$$

where n is the smallest integer greater than or equal to α, provided that the right side integral is pointwise defined on $(0, +\infty)$.

Lemma 2.1 (see [28]) *Let $\alpha > 0$. Assume that $u, D_{0^+}^{\alpha} u \in L([0,1], \mathbb{R})$. Then the following equality holds*:

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$, and n is the smallest integer greater than or equal to α.

Next, we will give the Schaefer fixed point theorem (see for example [25]), which will be used in this paper.

Lemma 2.2 *Let X be a Banach space and $T : X \to X$ is a completely continuous operator. If the set $\Omega = \{u \in X | u = \lambda Tu, \lambda \in (0,1)\}$ is bounded, then T has at least one fixed point in X.*

In this paper, we take $Z = C([0,1], \mathbb{R})$ with the norm $\|z\|_0 = \max_{t \in [0,1]} |z(t)|$, $X = \{u | u, D_{0^+}^{\alpha} u \in Z\}$ with the norm $\|u\|_X = \max\{\|u\|_0, \|D_{0^+}^{\alpha} u\|_0\}$, and $Y = \{v | v, D_{0^+}^{\gamma} v \in Z\}$ with the norm $\|v\|_Y = \max\{\|v\|_0, \|D_{0^+}^{\gamma} v\|_0\}$. For $(u,v) \in X \times Y$, let $\|(u,v)\|_{X \times Y} = \max\{\|u\|_X, \|v\|_Y\}$. Obviously, $(X \times Y, \|\cdot\|_{X \times Y})$ is a Banach space.

3 Existence result

In this section, a theorem on the existence of solutions for ABVP (1.2) will be given under the nonlinear growth restrictions of f and g.

As a consequence of Lemma 2.1, we have the following result, which is useful in what follows.

Lemma 3.1 *Given $(h_1, h_2) \in Z \times Z$, the unique solution of*

$$\begin{cases} D_{0^+}^{\beta} \phi_p(D_{0^+}^{\alpha} u(t)) = h_1(t), & t \in [0,1], \\ D_{0^+}^{\delta} \phi_p(D_{0^+}^{\gamma} v(t)) = h_2(t), & t \in [0,1], \\ u(0) = -u(1), & D_{0^+}^{\alpha} u(0) = -D_{0^+}^{\alpha} u(1), \\ v(0) = -v(1), & D_{0^+}^{\gamma} v(0) = -D_{0^+}^{\gamma} v(1) \end{cases} \tag{3.1}$$

is

$$\begin{aligned}\big(u(t),v(t)\big) &= \big(B_1(h_1)+I_{0^+}^{\alpha}\phi_q\big(A_1(h_1)+I_{0^+}^{\beta}h_1\big)(t),\\ &\quad B_2(h_2)+I_{0^+}^{\gamma}\phi_q\big(A_2(h_2)+I_{0^+}^{\delta}h_2\big)(t)\big),\end{aligned}$$

where

$$\begin{aligned}A_1(h_1) &= -\frac{1}{2}I_{0^+}^{\beta}h_1(1)\\ &= -\frac{1}{2\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}h_1(s)\,ds,\\ B_1(h_1) &= -\frac{1}{2}I_{0^+}^{\alpha}\phi_q\big(A_1(h_1)+I_{0^+}^{\beta}h_1\big)(1)\\ &= -\frac{1}{2\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\phi_q\bigg(A_1(h_1)\\ &\quad+\frac{1}{\Gamma(\beta)}\int_0^s(s-\tau)^{\beta-1}h_1(\tau)\,d\tau\bigg)\,ds,\\ A_2(h_2) &= -\frac{1}{2}I_{0^+}^{\delta}h_2(1)\\ &= -\frac{1}{2\Gamma(\delta)}\int_0^1(1-s)^{\delta-1}h_2(s)\,ds,\\ B_2(h_2) &= -\frac{1}{2}I_{0^+}^{\gamma}\phi_q\big(A_2(h_2)+I_{0^+}^{\delta}h_2\big)(1)\\ &= -\frac{1}{2\Gamma(\gamma)}\int_0^1(1-s)^{\gamma-1}\phi_q\bigg(A_2(h_2)\\ &\quad+\frac{1}{\Gamma(\delta)}\int_0^s(s-\tau)^{\delta-1}h_2(\tau)\,d\tau\bigg)\,ds,\end{aligned}$$

and ϕ_q is understood as the operator $\phi_q : Z \to Z$ defined by $\phi_q(z)(t) = \phi_q(z(t))$.

Proof Assume that (u,v) satisfies the equations of ABVP (3.1), then Lemma 2.1 implies that

$$\phi_p\big(D_{0^+}^{\alpha}u(t)\big) = c_1 + I_{0^+}^{\beta}h_1(t), \quad \forall c_1 \in \mathbb{R}.$$

From the boundary value condition $D_{0^+}^{\alpha}u(0) = -D_{0^+}^{\alpha}u(1)$, one has

$$c_1 = -\frac{1}{2}I_{0^+}^{\beta}h_1(1) = A_1(h_1).$$

Thus we have

$$u(t) = c_2 + I_{0^+}^{\alpha}\phi_q\big(A_1(h_1)+I_{0^+}^{\beta}h_1\big)(t), \quad \forall c_2 \in \mathbb{R}.$$

By the condition $u(0) = -u(1)$, we get

$$c_2 = -\frac{1}{2}I_{0^+}^{\alpha}\phi_q\big(A_1(h_1)+I_{0^+}^{\beta}h_1\big)(1) = B_1(h_1).$$

A similar proof can show that

$$v(t) = c_4 + I_{0^+}^{\gamma}\phi_q\big(c_3 + I_{0^+}^{\delta}h_2\big)(t),$$

where

$$\begin{aligned} c_3 &= -\frac{1}{2}I_{0^+}^{\delta}h_2(1) = A_2(h_2), \\ c_4 &= -\frac{1}{2}I_{0^+}^{\gamma}\phi_q\big(A_2(h_2) + I_{0^+}^{\delta}h_2\big)(1) = B_2(h_2). \end{aligned}$$

The proof is complete. □

Define the operator $\mathcal{T} : X \times Y \to X \times Y$ by

$$\begin{aligned} \mathcal{T}(u,v)(t) &= \big(B_1(N_1v) + I_{0^+}^{\alpha}\phi_q\big(A_1(N_1v) + I_{0^+}^{\beta}N_1v\big)(t), \\ &\quad B_2(N_2u) + I_{0^+}^{\gamma}\phi_q\big(A_2(N_2u) + I_{0^+}^{\delta}N_2u\big)(t)\big) \\ &:= \big(T_1v(t), T_2u(t)\big), \quad \forall t \in [0,1], \end{aligned}$$

where

$$\begin{aligned} T_1v(t) &= -\frac{1}{2\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\bigg(-\frac{1}{2\Gamma(\beta)} \\ &\quad \cdot \int_0^1 (1-\tau)^{\beta-1}f\big(\tau, v(\tau), D_{0^+}^{\gamma}v(\tau)\big)\,d\tau \\ &\quad + \frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}f\big(\tau, v(\tau), D_{0^+}^{\gamma}v(\tau)\big)\,d\tau\bigg)\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\bigg(-\frac{1}{2\Gamma(\beta)} \\ &\quad \cdot \int_0^1 (1-\tau)^{\beta-1}f\big(\tau, v(\tau), D_{0^+}^{\gamma}v(\tau)\big)\,d\tau \\ &\quad + \frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}f\big(\tau, v(\tau), D_{0^+}^{\gamma}v(\tau)\big)\,d\tau\bigg)\,ds, \quad \forall t \in [0,1], \\ T_2u(t) &= -\frac{1}{2\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1}\phi_q\bigg(-\frac{1}{2\Gamma(\delta)} \\ &\quad \cdot \int_0^1 (1-\tau)^{\delta-1}g\big(\tau, u(\tau), D_{0^+}^{\alpha}u(\tau)\big)\,d\tau \\ &\quad + \frac{1}{\Gamma(\delta)}\int_0^s (s-\tau)^{\delta-1}g\big(\tau, u(\tau), D_{0^+}^{\alpha}u(\tau)\big)\,d\tau\bigg)\,ds \\ &\quad + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\phi_q\bigg(-\frac{1}{2\Gamma(\delta)} \\ &\quad \cdot \int_0^1 (1-\tau)^{\delta-1}g\big(\tau, u(\tau), D_{0^+}^{\alpha}u(\tau)\big)\,d\tau \\ &\quad + \frac{1}{\Gamma(\delta)}\int_0^s (s-\tau)^{\delta-1}g\big(\tau, u(\tau), D_{0^+}^{\alpha}u(\tau)\big)\,d\tau\bigg)\,ds, \quad \forall t \in [0,1], \end{aligned}$$

and $N_1 : Y \to Z$, $N_2 : X \to Z$ are Nemytskii operators defined by

$$N_1 v(t) = f\bigl(t, v(t), D_{0^+}^{\gamma} v(t)\bigr), \quad \forall t \in [0,1],$$
$$N_2 u(t) = g\bigl(t, u(t), D_{0^+}^{\alpha} u(t)\bigr), \quad \forall t \in [0,1].$$

Clearly, the fixed points of $\mathcal{T}$ are the solutions of ABVP (1.2).

Our main result, based on the Schaefer fixed point theorem and Lemma 3.1, is stated as follows.

Theorem 3.1 *Let $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Assume that*

(H) *for $\forall (u, v) \in \mathbb{R}^2$, $t \in [0,1]$, there exist nonnegative functions $a_1, b_1, c_1, a_2, b_2, c_2 \in Z$ such that*

$$\bigl|f(t, u, v)\bigr| \le a_1(t) + b_1(t)|u|^{p-1} + c_1(t)|v|^{p-1},$$
$$\bigl|g(t, u, v)\bigr| \le a_2(t) + b_2(t)|u|^{p-1} + c_2(t)|v|^{p-1}.$$

Then ABVP (1.2) *has at least one solution, provided that*

$$L := \frac{3\omega_1}{2\Gamma(\beta+1)} \cdot \frac{3\omega_2}{2\Gamma(\delta+1)} < 1, \tag{3.2}$$

where

$$\omega_1 = \frac{3^{p-1}\|b_1\|_0}{2^{p-1}(\Gamma(\gamma+1))^{p-1}} + \|c_1\|_0,$$
$$\omega_2 = \frac{3^{p-1}\|b_2\|_0}{2^{p-1}(\Gamma(\alpha+1))^{p-1}} + \|c_2\|_0.$$

Proof The proof will be given in the following two steps.

Step 1: *$\mathcal{T} : X \times Y \to X \times Y$ is completely continuous.*

By the definitions of T_1 and T_2, we obtain

$$D_{0^+}^{\alpha} T_1 v(t) = \phi_q\bigl(A_1(N_1 v) + I_{0^+}^{\beta} N_1 v\bigr)(t),$$
$$D_{0^+}^{\gamma} T_2 u(t) = \phi_q\bigl(A_2(N_2 u) + I_{0^+}^{\delta} N_2 u\bigr)(t).$$

Obviously, the operators T_1, $D_{0^+}^{\alpha} T_1$, T_2, $D_{0^+}^{\gamma} T_2$ are compositions of the continuous operators. So T_1, $D_{0^+}^{\alpha} T_1$, T_2, $D_{0^+}^{\gamma} T_2$ are continuous in Z. Hence, $\mathcal{T}$ is a continuous operator in $X \times Y$.

Let $\Omega := \Omega_1 \times \Omega_2 \subset X \times Y$ be an open bounded set, then $T_1(\overline{\Omega_2})$, $T_2(\overline{\Omega_1})$, and $D_{0^+}^{\alpha} T_1(\overline{\Omega_2})$, $D_{0^+}^{\gamma} T_2(\overline{\Omega_1})$ are bounded. Moreover, for $\forall (u, v) \in \overline{\Omega}$, $t \in [0,1]$, there exist constants $L_1, L_2, L_3 > 0$ such that

$$\bigl|A_1(N_1 v) + I_{0^+}^{\beta} N_1 v(t)\bigr| \le L_1,$$
$$\bigl|A_2(N_2 u) + I_{0^+}^{\delta} N_2 u(t)\bigr| \le L_2,$$
$$\max\bigl\{\bigl|I_{0^+}^{\beta} N_1 v(t)\bigr|, \bigl|I_{0^+}^{\delta} N_2 u(t)\bigr|\bigr\} \le L_3.$$

Thus, in view of the Arzelà-Ascoli theorem, we need only to prove that $\mathcal{T}(\overline{\Omega}) \subset X \times Y$ is equicontinuous.

For $0 \leq t_1 < t_2 \leq 1$, $(u,v) \in \overline{\Omega}$, we have

$$\begin{aligned}
&\left|T_1 v(t_2) - T_1 v(t_1)\right| \\
&\quad = \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2 - s)^{\alpha-1} \phi_q\big(A_1(N_1 v) + I_{0^+}^{\beta} N_1 v(s)\big)\, ds \right. \\
&\qquad \left. - \int_0^{t_1} (t_1 - s)^{\alpha-1} \phi_q\big(A_1(N_1 v) + I_{0^+}^{\beta} N_1 v(s)\big)\, ds \right| \\
&\quad \leq \frac{L_1^{q-1}}{\Gamma(\alpha)} \left\{ \int_0^{t_1} \big[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}\big]\, ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\, ds \right\} \\
&\quad = \frac{L_1^{q-1}}{\Gamma(\alpha+1)} \big[t_1^{\alpha} - t_2^{\alpha} + 2(t_2 - t_1)^{\alpha}\big].
\end{aligned}$$

Similarly, one has

$$\left|T_2 u(t_2) - T_2 u(t_1)\right| \leq \frac{L_2^{q-1}}{\Gamma(\gamma+1)} \big[t_1^{\gamma} - t_2^{\gamma} + 2(t_2 - t_1)^{\gamma}\big].$$

Since t^{α} is uniformly continuous in $[0,1]$, we see that $(T_1(\overline{\Omega_2}), T_2(\overline{\Omega_1})) \subset Z \times Z$ is equicontinuous. A similar proof can show that $(I_{0^+}^{\beta} N_1(\overline{\Omega_2}), I_{0^+}^{\delta} N_2(\overline{\Omega_1})) \subset Z \times Z$ is equicontinuous. This, together with the uniformly continuity of $\phi_q(s)$ on $[-L_3, L_3]$, shows that $(D_{0^+}^{\alpha} T_1(\overline{\Omega_2}), D_{0^+}^{\gamma} T_2(\overline{\Omega_1})) \subset Z \times Z$ is also equicontinuous. Thus, we find that $\mathcal{T}: X \times Y \to X \times Y$ is compact.

Step 2: *A priori bounds.*

Set

$$\Omega = \big\{(u,v) \in X \times Y \,|\, (u,v) = \lambda^{q-1} \mathcal{T}(u,v), \lambda \in (0,1)\big\}.$$

Now it remains to show that the set Ω is bounded.

Since $0 < \alpha \leq 1$, by Lemma 2.1, we have

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_0.$$

So we get

$$c_0 = -u(0) = I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(1) - u(1).$$

Hence, from the anti-periodic boundary value condition $u(0) = -u(1)$, one has

$$c_0 = \frac{1}{2} I_{0+}^{\alpha} D_{0^+}^{\alpha} u(1).$$

Thus we obtain

$$u(t) = -\frac{1}{2} I_{0+}^{\alpha} D_{0^+}^{\alpha} u(1) + I_{0+}^{\alpha} D_{0^+}^{\alpha} u(t),$$

which together with

$$\begin{aligned}\left|I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t)\right| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^t (t-s)^{\alpha-1}D_{0^+}^{\alpha}u(s)\,ds\right| \\ &\le \frac{1}{\Gamma(\alpha)}\left\|D_{0^+}^{\alpha}u\right\|_0\cdot\frac{1}{\alpha}t^{\alpha} \\ &\le \frac{1}{\Gamma(\alpha+1)}\left\|D_{0^+}^{\alpha}u\right\|_0, \quad \forall t\in[0,1],\end{aligned}$$

yields

$$\|u\|_0 \le \frac{3}{2\Gamma(\alpha+1)}\left\|D_{0^+}^{\alpha}u\right\|_0. \tag{3.3}$$

Similarly, we can get

$$\|v\|_0 \le \frac{3}{2\Gamma(\gamma+1)}\left\|D_{0^+}^{\gamma}v\right\|_0. \tag{3.4}$$

For $(u,v)\in\Omega$, we have

$$\begin{aligned}&u(t) = \lambda^{q-1}\bigl(B_1(N_1v) + I_{0^+}^{\alpha}\phi_q\bigl(A_1(N_1v) + I_{0^+}^{\beta}N_1v\bigr)(t)\bigr), \\ &v(t) = \lambda^{q-1}\bigl(B_2(N_2u) + I_{0^+}^{\gamma}\phi_q\bigl(A_2(N_2u) + I_{0^+}^{\delta}N_2u\bigr)(t)\bigr).\end{aligned}$$

Thus we get

$$\begin{aligned}&D_{0^+}^{\alpha}u(t) = \lambda^{q-1}\phi_q\bigl(A_1(N_1v) + I_{0^+}^{\beta}N_1v(t)\bigr), \\ &D_{0^+}^{\gamma}v(t) = \lambda^{q-1}\phi_q\bigl(A_2(N_2u) + I_{0^+}^{\delta}N_2u(t)\bigr),\end{aligned}$$

which together with $\phi_q(\lambda) = \lambda^{q-1}$ ($\lambda\in(0,1)$) yields

$$\begin{aligned}&\phi_p\bigl(D_{0^+}^{\alpha}u(t)\bigr) = \lambda\bigl(A_1(N_1v) + I_{0^+}^{\beta}N_1v(t)\bigr), \\ &\phi_p\bigl(D_{0^+}^{\gamma}v(t)\bigr) = \lambda\bigl(A_2(N_2u) + I_{0^+}^{\delta}N_2u(t)\bigr).\end{aligned}$$

From the hypothesis (H), for $\forall t\in[0,1]$, we get

$$\begin{aligned}\left|I_{0^+}^{\beta}N_1v(t)\right| &\le \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\bigl|f\bigl(s,v(s),D_{0^+}^{\gamma}v(s)\bigr)\bigr|\,ds \\ &\le \frac{1}{\Gamma(\beta)}\bigl(\|a_1\|_0 + \|b_1\|_0\|v\|_0^{p-1} + \|c_1\|_0\left\|D_{0^+}^{\gamma}v\right\|_0^{p-1}\bigr)\cdot\frac{1}{\beta}t^{\beta} \\ &\le \frac{1}{\Gamma(\beta+1)}\bigl(\|a_1\|_0 + \|b_1\|_0\|v\|_0^{p-1} + \|c_1\|_0\left\|D_{0^+}^{\gamma}v\right\|_0^{p-1}\bigr),\end{aligned}$$

which together with $|\phi_p(D_{0^+}^{\alpha}u(t))| = |D_{0^+}^{\alpha}u(t)|^{p-1}$ yields

$$\left\|D_{0^+}^{\alpha}u\right\|_0^{p-1} \le \frac{3}{2\Gamma(\beta+1)}\bigl(\|a_1\|_0 + \|b_1\|_0\|v\|_0^{p-1} + \|c_1\|_0\left\|D_{0^+}^{\gamma}v\right\|_0^{p-1}\bigr). \tag{3.5}$$

Repeating arguments similar to the above we can arrive at

$$\left\|D_{0^+}^{\gamma}v\right\|_0^{p-1} \le \frac{3}{2\Gamma(\delta+1)}\left(\|a_2\|_0 + \|b_2\|_0\|u\|_0^{p-1} + \|c_2\|_0\left\|D_{0^+}^{\alpha}u\right\|_0^{p-1}\right). \tag{3.6}$$

From (3.3)-(3.6), we obtain

$$\begin{aligned}
\left\|D_{0^+}^{\alpha}u\right\|_0^{p-1} &\le \frac{3}{2\Gamma(\beta+1)}\biggl[\|a_1\|_0 + \biggl(\|c_1\|_0 \\
&\quad + \frac{3^{p-1}\|b_1\|_0}{2^{p-1}(\Gamma(\gamma+1))^{p-1}}\biggr)\left\|D_{0^+}^{\gamma}v\right\|_0^{p-1}\biggr] \\
&= \frac{3}{2\Gamma(\beta+1)}\left(\|a_1\|_0 + \omega_1\left\|D_{0^+}^{\gamma}v\right\|_0^{p-1}\right), \\
\left\|D_{0^+}^{\gamma}v\right\|_0^{p-1} &\le \frac{3}{2\Gamma(\delta+1)}\biggl[\|a_2\|_0 + \biggl(\|c_2\|_0 \\
&\quad + \frac{3^{p-1}\|b_2\|_0}{2^{p-1}(\Gamma(\alpha+1))^{p-1}}\biggr)\left\|D_{0^+}^{\alpha}u\right\|_0^{p-1}\biggr] \\
&= \frac{3}{2\Gamma(\delta+1)}\left(\|a_2\|_0 + \omega_2\left\|D_{0^+}^{\alpha}u\right\|_0^{p-1}\right).
\end{aligned}$$

So we have

$$\begin{aligned}
\left\|D_{0^+}^{\alpha}u\right\|_0^{p-1} &\le \frac{3}{2\Gamma(\beta+1)}\left(\|a_1\|_0 + \frac{3\omega_1}{2\Gamma(\delta+1)}\left(\|a_2\|_0 + \omega_2\left\|D_{0^+}^{\alpha}u\right\|_0^{p-1}\right)\right), \\
\left\|D_{0^+}^{\gamma}v\right\|_0^{p-1} &\le \frac{3}{2\Gamma(\delta+1)}\left(\|a_2\|_0 + \frac{3\omega_2}{2\Gamma(\beta+1)}\left(\|a_1\|_0 + \omega_1\left\|D_{0^+}^{\gamma}v\right\|_0^{p-1}\right)\right).
\end{aligned}$$

Hence, in view of (3.2), we can get

$$\left\|D_{0^+}^{\alpha}u\right\|_0 \le \left(\frac{M_1}{1-L}\right)^{q-1} := L_{11}, \tag{3.7}$$

$$\left\|D_{0^+}^{\gamma}v\right\|_0 \le \left(\frac{M_2}{1-L}\right)^{q-1} := L_{21}, \tag{3.8}$$

where

$$\begin{aligned}
M_1 &= \frac{3}{2\Gamma(\beta+1)}\left(\|a_1\|_0 + \frac{3\omega_1}{2\Gamma(\delta+1)}\|a_2\|_0\right), \\
M_2 &= \frac{3}{2\Gamma(\delta+1)}\left(\|a_2\|_0 + \frac{3\omega_2}{2\Gamma(\beta+1)}\|a_1\|_0\right).
\end{aligned}$$

Thus, from (3.3) and (3.4), one has

$$\|u\|_0 \le \frac{3L_{11}}{2\Gamma(\alpha+1)} := L_{12}, \tag{3.9}$$

$$\|v\|_0 \le \frac{3L_{21}}{2\Gamma(\gamma+1)} := L_{22}. \tag{3.10}$$

Therefore, combining (3.7) and (3.9) with (3.8) and (3.10), we have

$$\begin{aligned}\|(u,v)\|_{X\times Y} &= \max\{\|u\|_0, \|D_{0^+}^{\alpha}u\|_0, \|v\|_0, \|D_{0^+}^{\gamma}v\|_0\}\\ &\le \max\{L_{11}, L_{12}, L_{21}, L_{22}\}.\end{aligned}$$

As a consequence of the Schaefer fixed point theorem, we deduce that $\mathcal{T}$ has at least one fixed point which is the solution of ABVP (1.2). The proof is complete. □

4 An example

In this section, we will give an example to illustrate our main result.

Example 4.1 Consider the following ABVP for the coupled system of the fractional p-Laplacian equation:

$$\begin{cases} D_{0^+}^{\frac{3}{4}}\phi_3(D_{0^+}^{\frac{1}{2}}u(t)) = -\frac{25}{3} + \frac{1}{10}v^2(t) + te^{-|D_{0^+}^{\frac{3}{4}}v(t)|}, & t\in[0,1],\\ D_{0^+}^{\frac{1}{2}}\phi_3(D_{0^+}^{\frac{3}{4}}v(t)) = \cos t + \frac{1}{4}u^2(t) + t\cos(D_{0^+}^{\frac{1}{2}}u(t)), & t\in[0,1],\\ u(0) = -u(1), \qquad D_{0^+}^{\frac{1}{2}}u(0) = -D_{0^+}^{\frac{1}{2}}u(1),\\ v(0) = -v(1), \qquad D_{0^+}^{\frac{3}{4}}v(0) = -D_{0^+}^{\frac{3}{4}}v(1).\end{cases} \tag{4.1}$$

Corresponding to ABVP (1.2), we get $p = 3$, $\alpha = 1/2$, $\beta = 3/4$, $\gamma = 3/4$, $\delta = 1/2$, and

$$\begin{aligned} f(t,u,v) &= -\frac{25}{3} + \frac{1}{10}u^2 + te^{-|v|},\\ g(t,u,v) &= \cos t + \frac{1}{4}u^2 + t\cos v.\end{aligned}$$

Choose $a_1(t) = 10$, $b_1(t) = 1/10$, $c_1(t) = 0$, $a_2(t) = 2$, $b_2(t) = 1/4$, $c_2(t) = 0$. By a simple calculation, we obtain $\|b_1\|_0 = 1/10$, $\|c_1\|_0 = 0$, $\|b_2\|_0 = 1/4$, $\|c_2\|_0 = 0$, and

$$\begin{aligned}\omega_1 &= \frac{3^2}{2^2(\Gamma(\frac{3}{4}+1))^2} \times \frac{1}{10} + 0 \le 0.266374,\\ \omega_2 &= \frac{3^2}{2^2(\Gamma(\frac{1}{2}+1))^2} \times \frac{1}{4} + 0 \le 0.716197,\\ L &= \frac{3}{2}\frac{\omega_1}{\Gamma(\frac{3}{4}+1)}\frac{3}{2}\frac{\omega_2}{\Gamma(\frac{1}{2}+1)} < 1.\end{aligned}$$

Obviously, ABVP (4.1) satisfies all assumptions of Theorem 3.1. Hence, ABVP (4.1) has at least one solution.

Competing interests
The author declares that she has no competing interests.

Author's contributions
The author contributed to the manuscript. The author read and approved the final manuscript.

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