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Lyapunov inequalities for a class of nonlinear dynamic systems on time scales

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Abstract

The purpose of this work is to obtain several Lyapunov inequalities for the nonlinear dynamic systems

$$\begin{cases} x^{\Delta}(t) = -A(t)x(\sigma(t)) - B(t)y(t)|\sqrt{B(t)}y(t)|^{p-2}, \\ y^{\Delta}(t) = C(t)x(\sigma(t))|x(\sigma(t))|^{q-2} + A^{T}(t)y(t), \end{cases}$$

on a given time scale interval $[a,b]_{\mathbb{T}}$ ($a,b \in \mathbb{T}$ with $\sigma(a) < b$), where $p,q \in (1,+\infty)$ satisfy $1/p + 1/q = 1$, $A(t)$ is a real $n \times n$ matrix-valued function on $[a,b]_{\mathbb{T}}$ such that $I + \mu(t)A(t)$ is invertible, $B(t)$ and $C(t)$ are two real $n \times n$ symmetric matrix-valued functions on $[a,b]_{\mathbb{T}}$, $B(t)$ is positive definite, and $x(t)$, $y(t)$ are two real n-dimensional vector-valued functions on $[a,b]_{\mathbb{T}}$.

MSC: 34K11; 39A10; 39A99

Keywords: Lyapunov inequality; nonlinear dynamic system; time scale

1 Introduction

The theory of dynamic equations on time scales, which follows Hilger's landmark paper [1], is a new study area of mathematics that has received a lot of attention. For example, we refer the reader to monographs [2, 3] and the references therein. During the last few years, some Lyapunov inequalities for dynamic equations on time scales have been obtained by many authors [4–7].

In 2002, Bohner *et al.* [8] investigated the second-order Sturm-Liouville dynamic equation

$$x^{\Delta^2}(t) + q(t)x^{\sigma}(t) = 0 \tag{1.1}$$

on time scale $\mathbb{T}$ under the conditions $x(a) = x(b) = 0$ ($a,b \in \mathbb{T}$ with $a < b$) and $q \in C_{\mathrm{rd}}(\mathbb{T},(0,\infty))$ and showed that if $x(t)$ is a solution of (1.1) with $\max_{t\in[a,b]_{\mathbb{T}}} |x(t)| > 0$, then

$$\int_a^b q(t)\Delta t \geq \frac{b-a}{C},$$

where $[a,b]_{\mathbb{T}} \equiv \{t \in \mathbb{T} : a \leq t \leq b\}$ and $C = \max\{(t-a)(b-t) : t \in [a,b]_{\mathbb{T}}\}$.

When $\mathbb{T} = \mathbb{R}$, (1.1) reduces to the Hills equation

$$x''(t) + u(t)x(t) = 0. \tag{1.2}$$

In 1907, Lyapunov [9] showed that if $u \in C([a,b],\mathbb{R})$ and $x(t)$ is a solution of (1.2) satisfying $x(a) = x(b) = 0$ and $\max_{t\in[a,b]} |x(t)| > 0$, then the following classical Lyapunov inequality holds:

$$\int_a^b |u(t)|\,dt > \frac{4}{b-a}.$$

This was later strengthened with $|u(t)|$ replaced by $u^+(t) = \max\{u(t), 0\}$ by Wintner [10] and thereafter by some other authors:

$$\int_a^b u^+(t)\,dt > \frac{4}{b-a}.$$

Moreover, the last inequality is optimal.

When $\mathbb{T}$ is the set $\mathbb{Z}$ of the integers, (1.1) reduces to the linear difference equation

$$\Delta^2 x(n) + u(n)x(n+1) = 0. \tag{1.3}$$

In 1983, Cheng [11] showed that if $a, b \in \mathbb{Z}$ with $0 < a < b$ and $x(n)$ is a solution of (1.3) satisfying $x(a) = x(b) = 0$ and $\max_{n\in\{a,a+1,\dots,b\}} |x(n)| > 0$, then

$$\sum_{n=a}^{b-2} |u(n)| \geq \begin{cases} \frac{4(b-a)}{(b-a)^2-1} & \text{if } b-a-1 \text{ is even}, \\ \frac{4}{b-a} & \text{if } b-a-1 \text{ is odd}. \end{cases}$$

The purpose of this paper is to establish several Lyapunov inequalities for the nonlinear dynamic system

$$\begin{cases} x^\Delta(t) = -A(t)x(\sigma(t)) - B(t)y(t)|\sqrt{B(t)}y(t)|^{p-2}, \\ y^\Delta(t) = C(t)x(\sigma(t))|x(\sigma(t))|^{q-2} + A^T(t)y(t), \end{cases} \tag{1.4}$$

on a given time scale interval $[a,b]_{\mathbb{T}}$ ($a, b \in \mathbb{T}$ with $\sigma(a) < b$), where $p, q \in (1, +\infty)$ satisfy $1/p + 1/q = 1$, $A(t)$ is a real $n \times n$ matrix-valued function on $[a,b]_{\mathbb{T}}$ such that $I + \mu(t)A(t)$ is invertible, $B(t)$ and $C(t)$ are two real $n \times n$ symmetric matrix-valued functions on $[a,b]_{\mathbb{T}}$, $B(t)$ being positive definite, $A^T(t)$ is the transpose of $A(t)$, and $x(t)$, $y(t)$ are two real n-dimensional vector-valued functions on $[a,b]_{\mathbb{T}}$.

When $n = 1$ and $p = q = 2$, (1.4) reduces to

$$\begin{cases} x^\Delta(t) = u(t)x(\sigma(t)) + v(t)y(t), \\ y^\Delta(t) = -w(t)x(\sigma(t)) - u(t)y(t), \end{cases} \tag{1.5}$$

where $u(t)$, $v(t)$, and $w(t)$ are real-valued rd-continuous functions on $\mathbb{T}$ satisfying $v(t) \geq 0$ for any $t \in \mathbb{T}$.

In 2011, He *et al.* [12] obtained the following result.

Theorem 1.1 ([12]) *Let $1 - \mu(t)u(t) > 0$ for any $t \in \mathbb{T}$ and $a, b \in \mathbb{T}^k$ with $\sigma(a) \leq b$. If* (1.5) *has a real solution $(x(t), y(t))$ such that*

$$x(a) = 0 \quad or \quad x(a)x\bigl(\sigma(a)\bigr) < 0;$$
$$x(b) = 0 \quad or \quad x(b)x\bigl(\sigma(b)\bigr) < 0; \qquad \max_{t\in[a,b]_{\mathbb{T}}} \bigl|x(t)\bigr| > 0,$$

then we have the following inequality:

$$\int_a^b \bigl|u(t)\bigr|\Delta(t) + \left[\int_a^{\sigma(b)} v(t)\Delta(t)\int_a^b w^+(t)\Delta(t)\right]^{1/2} \geq 2,$$

where $w^+(t) = \max\{w(t), 0\}$.

In 2016, Liu *et al.* [13] obtained the following theorem.

Theorem 1.2 *Let $p = q = 2$ and $a, b \in \mathbb{T}$ with $\sigma(a) < b$. If* (1.4) *has a solution $(x(t), y(t))$ such that*

$$x(a) = x(b) = 0 \quad and \quad \max_{t\in[a,b]_{\mathbb{T}}} x^T(t)x(t) > 0, \tag{1.6}$$

then for any $n \times n$ symmetric matrix-valued function $C_1(t)$ with $C_1(t) - C(t) \geq 0$, we have the following inequalities:

(1)

$$\int_a^b \frac{[\int_a^{\sigma(t)} |B(s)||e_{\ominus A}(\sigma(t),s)|^2\Delta s][\int_{\sigma(t)}^b |B(s)||e_{\ominus A}(\sigma(t),s)|^2\Delta s]}{\int_a^b |B(s)||e_{\ominus A}(\sigma(t),s)|^2\Delta s}\bigl|C_1(t)\bigr|\Delta t \geq 1,$$

(2)

$$\int_a^b \bigl|C_1(t)\bigr|\left\{\int_a^b \bigl|B(s)\bigr|\bigl|e_{\ominus A}\bigl(\sigma(t),s\bigr)\bigr|^2\Delta s\right\}\Delta t \geq 4,$$

(3)

$$\int_a^b \bigl|A(t)\bigr|\Delta t + \left(\int_a^b \bigl|\sqrt{B(t)}\bigr|^2\Delta t\right)^{1/2}\left(\int_a^b \bigl|C_1(t)\bigr|\Delta t\right)^{1/2} \geq 2.$$

For some other related results on Lyapunov-type inequalities, see, for example, [14–23].

2 Preliminaries and some lemmas

Throughout this paper, we adopt basic definitions and notation of monograph [2]. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$. On a time scale $\mathbb{T}$, the forward jump operator, the backward jump operator, and the graininess function are defined as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \text{and} \quad \mu(t) = \sigma(t) - t,$$

respectively.

The point $t \in \mathbb{T}$ is said to be left-dense (resp. left-scattered) if $\rho(t) = t$ (resp. $\rho(t) < t$). The point $t \in \mathbb{T}$ is said to be right-dense (resp. right-scattered) if $\sigma(t) = t$ (resp. $\sigma(t) > t$). If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if f is continuous at right-dense points and has finite left-sided limits at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions from $\mathbb{T}$ to $\mathbb{R}$ is denoted by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. For a function $f : \mathbb{T} \to \mathbb{R}$, the notation f^σ means the composition $f \circ \sigma$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the (delta) derivative $f^\Delta(t)$ at $t \in \mathbb{T}$ is defined as the number (if it exists) such that for given any $\varepsilon > 0$, there is a neighborhood U of t with

$$\left|f\big(\sigma(t)\big) - f(s) - f^\Delta(t)\big(\sigma(t) - s\big)\right| \le \varepsilon\left|\sigma(t) - s\right|$$

for all $s \in U$. If the (delta) derivative $f^\Delta(t)$ exists for every $t \in \mathbb{T}^k$, then we say that f is Δ-differentiable on $\mathbb{T}$.

Let $F, f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ satisfy $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^k$. Then, for any $c, d \in \mathbb{T}$, the Cauchy integral of f is defined as

$$\int_c^d f(t)\Delta t = F(d) - F(c).$$

For any $z \in \mathbb{R}^n$ and any $S \in \mathbb{R}^{n\times n}$ (the space of real $n \times n$ matrices), write

$$|z| = \sqrt{z^T z} \quad \text{and} \quad |S| = \max_{z\in\mathbb{R}^n, z\neq 0} \frac{|Sz|}{|z|},$$

which are called the Euclidean norm of z and the matrix norm of S, respectively. It is obvious that, for any $z \in \mathbb{R}^n$ and $U, V \in \mathbb{R}^{n\times n}$,

$$|Uz| \le |U||z| \quad \text{and} \quad |UV| \le |U||V|.$$

Let $\mathbb{R}_s^{n\times n}$ be the set of all symmetric real $n \times n$ matrices. We can show that, for any $U \in \mathbb{R}_s^{n\times n}$,

$$|U| = \max_{|\lambda I - U| = 0} |\lambda| \quad \text{and} \quad |U^2| = |U|^2.$$

A matrix $S \in \mathbb{R}_s^{n\times n}$ is said to be positive definite (resp. semipositive definite), written as $S > 0$ (resp. $S \ge 0$), if $y^T S y > 0$ (resp. $y^T S y \ge 0$) for any $y \in \mathbb{R}^n$ with $y \neq 0$. If S is positive definite (resp. semipositive definite), then there exists a unique positive definite matrix (resp. semipositive definite matrix), written as $\sqrt{S}$, satisfying $[\sqrt{S}]^2 = S$.

In this paper, we establish Lyapunov inequalities for (1.4) that has a solution $(x(t), y(t))$ satisfying

$$x(a) = x(b) = 0 \quad \text{and} \quad \max_{t\in[a,b]_{\mathbb{T}}} x^T(t)x(t) > 0. \tag{2.1}$$

We first introduce the following lemmas.

Lemma 2.1 ([2]) *Let* $1/p + 1/q = 1$ $(p, q \in (1, +\infty))$ *and* $a, b \in \mathbb{T}$ $(a < b)$. *Then, for any* $f, g \in C_{\mathrm{rd}}([a, b]_{\mathbb{T}}, \mathbb{R})$,

$$\int_a^b \left|f(t)g(t)\right| \Delta t \le \left(\int_a^b \left|f(t)\right|^p \Delta t\right)^{\frac{1}{p}} \left(\int_a^b \left|g(t)\right|^q \Delta t\right)^{\frac{1}{q}}.$$

Lemma 2.2 *Let* $a, b \in \mathbb{T}$ *with* $a < b$. *Suppose that* $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ *and* $p, q \in (1, +\infty)$ *with* $\alpha/p + \beta/q = \gamma/p + \delta/q = 1/p + 1/q = 1$. *Then, for any* $f, g \in C_{\mathrm{rd}}([a, b]_{\mathbb{T}}, (-\infty, 0) \cup (0, \infty))$,

$$\int_a^b \left|f(t)g(t)\right| \Delta t \le \left(\int_a^b \left|f(t)\right|^\alpha \left|g(t)\right|^\gamma \Delta t\right)^{\frac{1}{p}} \left(\int_a^b \left|f(t)\right|^\beta \left|g(t)\right|^\delta \Delta t\right)^{\frac{1}{q}}.$$

Proof Let $M(t) = (|f(t)|^\alpha |g(t)|^\gamma)^{\frac{1}{p}}$ and $N(t) = (|f(t)|^\beta |g(t)|^\delta)^{\frac{1}{q}}$. Then by Lemma 2.1 we have

$$\begin{aligned} \int_a^b \left|f(t)g(t)\right| \Delta t &= \int_a^b M(t)N(t)\Delta t \\ &\le \left(\int_a^b M^p(t)\Delta t\right)^{\frac{1}{p}} \left(\int_a^b N^q(t)\Delta t\right)^{\frac{1}{q}} \\ &= \left(\int_a^b \left|f(t)\right|^\alpha \left|g(t)\right|^\gamma \Delta t\right)^{\frac{1}{p}} \left(\int_a^b \left|f(t)\right|^\beta \left|g(t)\right|^\delta \Delta t\right)^{\frac{1}{q}}. \end{aligned}$$

This completes the proof of Lemma 2.2. □

Remark 2.3 Let $\gamma = 0$ in Lemma 2.2. Then we obtain that, for any $f, g \in C_{\mathrm{rd}}([a, b]_{\mathbb{T}}, (-\infty, 0) \cup (0, \infty))$,

$$\int_a^b \left|f(t)g(t)\right| \Delta t \le \left\{\max_{t \in [a,b]_{\mathbb{T}}} \left|f(t)\right|^\beta\right\}^{\frac{1}{q}} \left(\int_a^b \left|f(t)\right|^\alpha \Delta t\right)^{\frac{1}{p}} \left(\int_a^b \left|g(t)\right|^q \Delta t\right)^{\frac{1}{q}}.$$

Lemma 2.4 ([2]) *If* $A \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ *with invertible* $I + \mu(t)A(t)$, $f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$, $t_0 \in \mathbb{T}$, *and* $a \in \mathbb{R}^n$, *then*

$$x(t) = e_{\ominus A}(t, t_0)a + \int_{t_0}^t e_{\ominus A}(t, \tau)f(\tau)\Delta\tau$$

is the unique solution of the initial value problem

$$\begin{cases} x^\Delta(t) = -A(t)x(\sigma(t)) + f(t), \\ x(t_0) = a, \end{cases}$$

where $(\ominus A)(t) = -[I + \mu(t)A(t)]^{-1}A(t)$ *for any* $t \in \mathbb{T}^k$, *and* $e_{\ominus A}(t, t_0)$ *is the unique matrix-valued solution of the initial value problem*

$$\begin{cases} Y^\Delta(t) = (\ominus A)(t)Y(t), \\ Y(t_0) = I. \end{cases}$$

Lemma 2.5 ([2]) *Let* $A, B \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$ *be* Δ*-differentiable. Then*

$$\left(A(t)B(t)\right)^\Delta = A^\sigma(t)B^\Delta(t) + A^\Delta(t)B(t) = A^\Delta(t)B^\sigma(t) + A(t)B^\Delta(t).$$

Lemma 2.6 ([13]) *If* $f_1(t), f_2(t), \dots, f_n(t)$ *are* Δ*-integrable on* $[a,b]_{\mathbb{T}}$ *and* $x(t) = (f_1(t), f_2(t), \dots, f_n(t))$, *then*

$$\left|\int_a^b x(t)\Delta t\right| = \left\{\sum_{i=1}^n \left(\int_a^b f_i(t)\Delta t\right)^2\right\}^{\frac{1}{2}} \leq \int_a^b \left\{\sum_{i=1}^n f_i^2(t)\right\}^{\frac{1}{2}} \Delta t = \int_a^b \left|x(t)\right|\Delta t.$$

Lemma 2.7 ([13]) *If* $A_1, A_2 \in \mathbb{R}_s^{n\times n}$ *and* $A_1 - A_2 \geq 0$, *then, for any* $x \in \mathbb{R}^n$,

$$\left(x^\sigma\right)^T A_2 x^\sigma \leq |A_1|\left|x^\sigma\right|^2.$$

3 Main results and proofs

In this section, we assume that $\alpha, \beta \in \mathbb{R}$ and $p, q \in (1, +\infty)$ satisfy

$$\alpha/p + \beta/q = 1/p + 1/q = 1.$$

For any $t, \tau \in [a,b]_{\mathbb{T}}$, write

$$F(t,\tau) = \left|e_{\ominus A}\left(\sigma(t),\tau\right)\right|\left|\sqrt{B(\tau)}\right|,$$

$$G(t) = \left|\sqrt{B(t)}y(t)\right|^{p-2} y^T(t)B(t)y(t) = \left|\sqrt{B(t)}y(t)\right|^p,$$

$$\Phi\left(\sigma(t)\right) = \left(\int_a^{\sigma(t)} F^\alpha(t,s)\Delta s\right)^{\frac{q}{p}},$$

$$\Psi\left(\sigma(t)\right) = \left(\int_{\sigma(t)}^b F^\alpha(t,s)\Delta s\right)^{\frac{q}{p}},$$

$$P(t) = \Phi\left(\sigma(t)\right)\Psi\left(\sigma(t)\right) \max_{a\leq\tau\leq\sigma(t)} F^\beta(t,\tau) \max_{\sigma(t)\leq\tau\leq b} F^\beta(t,\tau),$$

$$Q(t) = \Phi\left(\sigma(t)\right) \max_{a\leq\tau\leq\sigma(t)} F^\beta(t,\tau) + \Psi\left(\sigma(t)\right) \max_{\sigma(t)\leq\tau\leq b} F^\beta(t,\tau).$$

Theorem 3.1 *Let* $a, b \in \mathbb{T}$ *with* $\sigma(a) < b$ *and* $C_1 \in \mathbb{R}_s^{n\times n}$ *with* $C_1(t) - C(t) \geq 0$. *If* (1.4) *has a solution* $(x(t), y(t))$ *with* $x(t), y(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$ *satisfying* (2.1) *on the interval* $[a,b]_{\mathbb{T}}$, *then*

$$\int_a^b \frac{P(t)}{Q(t)}\left|C_1(t)\right|\Delta t \geq 1. \tag{3.1}$$

Proof Since $(x(t), y(t))$ is a solution of (1.4), we have

$$\left(y^T(t)x(t)\right)^\Delta = \left(x^\sigma(t)\right)^T C(t)x^\sigma(t)\left|x^\sigma(t)\right|^{q-2} - G(t). \tag{3.2}$$

Integrating (3.2) from a to b and noting that $x(a) = x(b) = 0$, we obtain

$$\int_a^b G(t)\Delta t = \int_a^b \left|x^\sigma(t)\right|^{q-2}\left(x^\sigma(t)\right)^T C(t)x^\sigma(t)\Delta t.$$

Noting that $B(t) > 0$, we know that $y^T(t)B(t)y(t) \geq 0$, $t \in [a,b]_{\mathbb{T}}$.

We claim that $y^T(t)B(t)y(t) \not\equiv 0$ ($t \in [a,b]_{\mathbb{T}}$). Indeed, if $y^T(t)B(t)y(t) \equiv 0$ ($t \in [a,b]_{\mathbb{T}}$), then

$$\left|\sqrt{B(t)}y(t)\right|^2 = y^T(t)B(t)y(t) \equiv 0,$$

which implies $B(t)y(t) \equiv 0$ ($t \in [a,b]_{\mathbb{T}}$). Thus, the first equation of (1.4) reduces to

$$x^{\Delta}(t) = -A(t)x(\sigma(t)), \qquad x(a) = 0.$$

By Lemma 2.4 it follows

$$x(t) = e_{\ominus A}(t,a) \cdot 0 = 0,$$

which is a contradiction to (2.1). Hence, we obtain that

$$\int_a^b |x^{\sigma}(t)|^{q-2}(x^{\sigma}(t))^T C(t)x^{\sigma}(t)\Delta t = \int_a^b G(t)\Delta t > 0, \tag{3.3}$$

and it follows from Lemma 2.4 that, for $t \in [a,b]_{\mathbb{T}}$,

$$\begin{aligned} x(t) &= -\int_a^t e_{\ominus A}(t,\tau)B(\tau)y(\tau)\left|\sqrt{B(\tau)}y(\tau)\right|^{p-2}\Delta\tau \\ &= -\int_b^t e_{\ominus A}(t,\tau)B(\tau)y(\tau)\left|\sqrt{B(\tau)}y(\tau)\right|^{p-2}\Delta\tau, \end{aligned}$$

which implies that, for $t \in [a,b)_{\mathbb{T}}$,

$$\begin{aligned} x^{\sigma}(t) &= -\int_a^{\sigma(t)} e_{\ominus A}(\sigma(t),\tau)B(\tau)y(\tau)\left|\sqrt{B(\tau)}y(\tau)\right|^{p-2}\Delta\tau \\ &= +\int_{\sigma(t)}^b e_{\ominus A}(\sigma(t),\tau)B(\tau)y(\tau)\left|\sqrt{B(\tau)}y(\tau)\right|^{p-2}\Delta\tau. \end{aligned}$$

Note that, for $a \le \sigma(t) \le b$,

$$\begin{aligned} &\left|e_{\ominus A}(\sigma(t),\tau)B(\tau)y(\tau)\left|\sqrt{B(\tau)}y(\tau)\right|^{p-2}\right| \\ &\quad\le \left|e_{\ominus A}(\sigma(t),\tau)\right|\left|B(\tau)y(\tau)\right|\left|\sqrt{B(\tau)}y(\tau)\right|^{p-2} \\ &\quad\le F(t,\tau)\left|\sqrt{B(\tau)}y(\tau)\right|\left|\sqrt{B(\tau)}y(\tau)\right|^{p-2} \\ &\quad= F(t,\tau)G^{\frac{1}{q}}(\tau). \end{aligned}$$

Then by Remark 2.3 and Lemma 2.6 we obtain

$$\begin{aligned} \left|x^{\sigma}(t)\right|^q &= \left|\int_a^{\sigma(t)} e_{\ominus A}(\sigma(t),\tau)B(\tau)y(\tau)\left|\sqrt{B(\tau)}y(\tau)\right|^{p-2}\Delta\tau\right|^q \\ &\le \left[\int_a^{\sigma(t)} \left|e_{\ominus A}(\sigma(t),\tau)B(\tau)y(\tau)\left|\sqrt{B(\tau)}y(\tau)\right|^{p-2}\right|\Delta\tau\right]^q \\ &\le \left[\int_a^{\sigma(t)} F(t,\tau)G^{\frac{1}{q}}(\tau)\Delta\tau\right]^q \\ &\le \left(\int_a^{\sigma(t)} F^{\alpha}(t,\tau)\Delta\tau\right)^{\frac{q}{p}} \int_a^{\sigma(t)} F^{\beta}(t,\tau)G(\tau)\Delta\tau \\ &\le \max_{a\le\tau\le\sigma(t)} F^{\beta}(t,\tau)\left(\int_a^{\sigma(t)} F^{\alpha}(t,\tau)\Delta\tau\right)^{\frac{q}{p}} \int_a^{\sigma(t)} G(\tau)\Delta\tau, \end{aligned}$$

that is,

$$\left|x^\sigma(t)\right|^q \le \max_{a\le\tau\le\sigma(t)} F^\beta(t,\tau)\Phi\bigl(\sigma(t)\bigr)\int_a^{\sigma(t)} G(\tau)\Delta\tau. \tag{3.4}$$

Similarly, for $a \le \sigma(t) \le b$, we have

$$\left|x^\sigma(t)\right|^q \le \max_{\sigma(t)\le\tau\le b} F^\beta(t,\tau)\Psi\bigl(\sigma(t)\bigr)\int_{\sigma(t)}^{b} G(\tau)\Delta\tau. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\left|x^\sigma(t)\right|^q \le \frac{P(t)}{Q(t)}\int_a^b G(\tau)\Delta\tau.$$

Then by (3.3) and Lemma 2.7 we have

$$\begin{aligned}
&\int_a^b \left|C_1(t)\right|\left|x^\sigma(t)\right|^q\Delta t \\
&\quad\le \int_a^b \left|C_1(t)\right|\frac{P(t)}{Q(t)}\Delta t\int_a^b G(t)\Delta t \\
&\quad= \int_a^b \left|C_1(t)\right|\frac{P(t)}{Q(t)}\Delta t\int_a^b \left|x^\sigma(t)\right|^{q-2}\bigl(x^\sigma(t)\bigr)^T C(t)x^\sigma(t)\Delta t \\
&\quad\le \int_a^b \left|C_1(t)\right|\frac{P(t)}{Q(t)}\Delta t\int_a^b \left|C_1(t)\right|\left|x^\sigma(t)\right|^q\Delta t.
\end{aligned}$$

Since

$$\int_a^b \left|C_1(t)\right|\left|x^\sigma(t)\right|^q\Delta t \ge \int_a^b \left|x^\sigma(t)\right|^{q-2}\bigl(x^\sigma(t)\bigr)^T C(t)x^\sigma(t)\Delta t = \int_a^b G(t)\Delta t > 0,$$

we get

$$\int_a^b \frac{P(t)}{Q(t)}\left|C_1(t)\right|\Delta t \ge 1.$$

This completes the proof of Theorem 3.1. □

Corollary 3.2 *Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $C_1 \in \mathbb{R}_s^{n\times n}$ with $C_1(t) - C(t) \ge 0$. If* (1.4) *has a solution $(x(t), y(t))$ with $x(t), y(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$ satisfying* (2.1) *on the interval $[a,b]_{\mathbb{T}}$, then*

$$\int_a^b Q(t)\left|C_1(t)\right|\Delta t \ge 4. \tag{3.6}$$

Proof Note that

$$\frac{P(t)}{Q(t)} \le \frac{Q(t)}{4}.$$

It follows from (3.1) that

$$\int_a^b \frac{Q(t)}{4}\left|C_1(t)\right|\Delta t \ge 1,$$

that is,

$$\int_a^b Q(t)\big|C_1(t)\big|\Delta t \geq 4.$$

This completes the proof of Corollary 3.2. □

Corollary 3.3 *Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $C_1 \in \mathbb{R}_s^{n\times n}$ with $C_1(t) - C(t) \geq 0$. If* (1.4) *has a solution $(x(t), y(t))$ with $x(t), y(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$ satisfying* (2.1) *on the interval $[a,b]_{\mathbb{T}}$, then*

$$\int_a^b \sqrt{P(t)}\big|C_1(t)\big|\Delta t \geq 2. \tag{3.7}$$

Proof Note that

$$Q(t) \geq 2\sqrt{P(t)}.$$

It follows from (3.1) that

$$\int_a^b \sqrt{P(t)}\big|C_1(t)\big|\Delta t \geq \int_a^b 2\frac{P(t)}{Q(t)}\big|C_1(t)\big|\Delta t \geq 2.$$

This completes the proof of Corollary 3.3. □

Theorem 3.4 *Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $C_1 \in \mathbb{R}_s^{n\times n}$ with $C_1(t) - C(t) \geq 0$. If* (1.4) *has a solution $(x(t), y(t))$ with $x(t), y(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$ satisfying* (2.1) *on the interval $[a,b]_{\mathbb{T}}$, then there exists $c \in (a,b)$ such that*

$$\begin{cases} \int_a^{\sigma(c)} \Phi(\sigma(t)) \max_{a\leq\tau\leq\sigma(t)} F^{\beta}(t,\tau)|C_1(t)|\Delta t \geq 1, \\ \int_c^b \Psi(\sigma(t)) \max_{\sigma(t)\leq\tau\leq b} F^{\beta}(t,\tau)|C_1(t)|\Delta t \geq 1. \end{cases} \tag{3.8}$$

Proof Set $U(t) = \Phi(\sigma(t)) \max_{a\leq\tau\leq\sigma(t)} F^{\beta}(t,\tau)$ and $V(t) = \Psi(\sigma(t)) \max_{\sigma(t)\leq\tau\leq b} F^{\beta}(t,\tau)$. Let

$$f(t) = \int_a^t U(s)\big|C_1(s)\big|\Delta s - \int_t^b V(s)\big|C_1(s)\big|\Delta s.$$

Then we have $f(a) < 0$ and $f(b) > 0$. Hence, we can choose $c \in (a,b)$ such that $f(c) \leq 0$ and $f(\sigma(c)) \geq 0$, that is,

$$\int_a^c U(s)\big|C_1(s)\big|\Delta s \leq \int_c^b V(s)\big|C_1(s)\big|\Delta s \tag{3.9}$$

and

$$\int_a^{\sigma(c)} U(s)\big|C_1(s)\big|\Delta s \geq \int_{\sigma(c)}^b V(s)\big|C_1(s)\big|\Delta s. \tag{3.10}$$

By (3.4) we have that

$$\big|C_1(t)\big|\big|x^{\sigma}(t)\big|^q \leq U(t)\big|C_1(t)\big| \int_a^{\sigma(t)} G(\tau)\Delta\tau. \tag{3.11}$$

Integrating (3.11) from a to $\sigma(c)$, we obtain

$$\begin{aligned}\int_a^{\sigma(c)} \left|C_1(t)\right|\left|x^\sigma(t)\right|^q \Delta t &\le \int_a^{\sigma(c)} U(t)\left|C_1(t)\right|\left(\int_a^{\sigma(t)} G(\tau)\Delta\tau\right)\Delta t \\ &\le \int_a^{c} U(t)\left|C_1(t)\right|\Delta t \int_a^{\sigma(c)} G(\tau)\Delta\tau \\ &\quad + U(c)\left|C_1(c)\right|\left(\sigma(c)-c\right)\int_a^{\sigma(c)} G(\tau)\Delta\tau \\ &= \int_a^{\sigma(c)} U(t)\left|C_1(t)\right|\Delta t \int_a^{\sigma(c)} G(\tau)\Delta\tau.\end{aligned}$$

Similarly, we obtain from (3.4) and (3.10) that

$$\begin{aligned}\int_{\sigma(c)}^{b} \left|C_1(t)\right|\left|x^\sigma(t)\right|^q \Delta t &\le \int_{\sigma(c)}^{b} V(t)\left|C_1(t)\right|\Delta t \int_{\sigma(c)}^{b} G(\tau)\Delta\tau \\ &\le \int_a^{\sigma(c)} U(t)\left|C_1(t)\right|\Delta t \int_{\sigma(c)}^{b} G(\tau)\Delta\tau.\end{aligned}$$

This yields

$$\begin{aligned}\int_a^{b} \left|C_1(t)\right|\left|x^\sigma(t)\right|^q \Delta t &\le \int_a^{\sigma(c)} U(t)\left|C_1(t)\right|\Delta t \int_a^{b} G(t)\Delta t \\ &= \int_a^{\sigma(c)} U(t)\left|C_1(t)\right|\Delta t \int_a^{b} \left|x^\sigma(t)\right|^{q-2}\left(x^\sigma(t)\right)^T C(t)x^\sigma(t)\Delta t \\ &\le \int_a^{\sigma(c)} U(t)\left|C_1(t)\right|\Delta t \int_a^{b} \left|C_1(t)\right|\left|x^\sigma(t)\right|^q \Delta t.\end{aligned}$$

Since

$$\begin{aligned}\int_a^{b} \left|C_1(t)\right|\left|x^\sigma(t)\right|^q \Delta t &\ge \int_a^{b} \left|x^\sigma(t)\right|^{q-2}\left(x^\sigma(t)\right)^T C(t)x^\sigma(t)\Delta t \\ &= \int_a^{b} G(t)\Delta t > 0,\end{aligned}$$

we have $\int_a^{\sigma(c)} U(t)|C_1(t)|\Delta t \ge 1$.

Next, we obtain from (3.5) that

$$\left|x^\sigma(t)\right|^q\left|C_1(t)\right| \le V(t)\left|C_1(t)\right|\int_{\sigma(t)}^{b} G(\tau)\Delta\tau. \tag{3.12}$$

Integrating (3.12) from c to b, we have

$$\begin{aligned}\int_c^{b} \left|C_1(t)\right|\left|x^\sigma(t)\right|^q \Delta t &\le \int_c^{b} V(t)\left|C_1(t)\right|\left(\int_{\sigma(t)}^{b} G(\tau)\Delta\tau\right)\Delta t \\ &\le \int_c^{b} V(t)\left|C_1(t)\right|\Delta t \int_{\sigma(c)}^{b} G(\tau)\Delta\tau.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\int_a^c |C_1(t)||x^\sigma(t)|^q \Delta t &\le \int_a^c U(t)|C_1(t)|\Delta t \int_a^{\sigma(c)} G(\tau)\Delta\tau \\ &\le \int_c^b V(t)|C_1(t)|\Delta t \int_a^{\sigma(c)} G(\tau)\Delta\tau.\end{aligned}$$

This yields

$$\begin{aligned}\int_a^b |C_1(t)||x^\sigma(t)|^q \Delta t &\le \int_c^b V(t)|C_1(t)|\Delta t \int_a^b G(t)\Delta t \\ &= \int_c^b V(t)|C_1(t)|\Delta t \int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T C(t)x^\sigma(t)\Delta t \\ &\le \int_c^b V(t)|C_1(t)|\Delta t \int_a^b |C_1(t)||x^\sigma(t)|^q \Delta t.\end{aligned}$$

Thus, we have $\int_c^b V(t)|C_1(t)|\Delta t \ge 1$. This completes the proof of Theorem 3.4. □

Theorem 3.5 *Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$ and $C_1 \in \mathbb{R}_s^{n\times n}$ with $C_1(t) - C(t) \ge 0$. If (1.4) has a solution $(x(t), y(t))$ with $x(t), y(t) \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$ satisfying (2.1) on the interval $[a, b]_{\mathbb{T}}$, then*

$$\int_a^b |A(t)|\Delta t + \left\{\max_{a\le t\le b} \left|\sqrt{B(t)}\right|^\beta\right\}^{\frac{1}{q}} \left(\int_a^b \left|\sqrt{B(t)}\right|^\alpha \Delta t\right)^{\frac{1}{p}} \left(\int_a^b |C_1(t)|\Delta t\right)^{\frac{1}{q}} \ge 2.$$

Proof Since $x(a) = x(b) = 0$, we have

$$\int_a^b G(t)\Delta t = \int_a^b |x^\sigma(t)|^{q-2}(x^\sigma(t))^T C(t)x^\sigma(t)\Delta t.$$

It follows from the first equation of (1.4) that, for all $a \le t \le b$,

$$\begin{aligned}x(t) &= \int_a^t \left(-A(\tau)x^\sigma(\tau) - B(\tau)\left|\sqrt{B(\tau)}y(\tau)\right|^{p-2} y(\tau)\right)\Delta\tau \\ &= \int_t^b \left(A(\tau)x^\sigma(\tau) + B(\tau)\left|\sqrt{B(\tau)}y(\tau)\right|^{p-2} y(\tau)\right)\Delta\tau.\end{aligned}$$

Thus, we have

$$\begin{aligned}|x(t)| &= \left|\int_a^t \left(-A(\tau)x^\sigma(\tau) - B(\tau)y(\tau)\right)\left|\sqrt{B(\tau)}y(\tau)\right|^{p-2}\Delta\tau\right| \\ &\le \int_a^t \left|A(\tau)x^\sigma(\tau) + B(\tau)y(\tau)\left|\sqrt{B(\tau)}y(\tau)\right|^{p-2}\right|\Delta\tau \\ &\le \int_a^t |A(\tau)x^\sigma(\tau)|\Delta\tau + \int_a^t |B(\tau)y(\tau)|\left|\sqrt{B(\tau)}y(\tau)\right|^{p-2}\Delta\tau \\ &\le \int_a^t |A(\tau)||x^\sigma(\tau)|\Delta\tau + \int_a^t \left|\sqrt{B(\tau)}\right| G^{\frac{1}{q}}(\tau)\Delta\tau.\end{aligned}$$

Similarly, we have

$$|x(t)| \leq \int_t^b |A(\tau)||x^\sigma(\tau)|\Delta\tau + \int_t^b |\sqrt{B(\tau)}|G^{\frac{1}{q}}(\tau)\Delta\tau.$$

Then we obtain

$$\begin{aligned}
|x(t)| &\leq \frac{1}{2}\left[\int_a^b |A(t)||x^\sigma(t)|\Delta t + \int_a^b |\sqrt{B(t)}|G^{\frac{1}{q}}(t)\Delta t\right] \\
&\leq \frac{1}{2}\left[\int_a^b |A(t)||x^\sigma(t)|\Delta t + \left\{\max_{a\leq t\leq b}|\sqrt{B(t)}|^\beta\right\}^{\frac{1}{q}}\right. \\
&\quad \left.\times \left(\int_a^b |\sqrt{B(t)}|^\alpha \Delta t\right)^{\frac{1}{p}}\left(\int_a^b G(t)\Delta t\right)^{\frac{1}{q}}\right] \\
&= \frac{1}{2}\left[\int_a^b |A(t)||x^\sigma(t)|\Delta t + \left\{\max_{a\leq t\leq b}|\sqrt{B(t)}|^\beta\right\}^{\frac{1}{q}}\left(\int_a^b |\sqrt{B(t)}|^\alpha \Delta t\right)^{\frac{1}{p}}\right. \\
&\quad \left.\times \left(\int_a^b |x^\sigma(t)|^{q-2}\big(x^\sigma(t)\big)^T C(t)x^\sigma(t)\Delta t\right)^{\frac{1}{q}}\right] \\
&\leq \frac{1}{2}\left[\int_a^b |A(t)||x^\sigma(t)|\Delta t + \left\{\max_{a\leq t\leq b}|\sqrt{B(t)}|^\beta\right\}^{\frac{1}{q}}\right. \\
&\quad \left.\times \left(\int_a^b |\sqrt{B(t)}|^\alpha \Delta t\right)^{\frac{1}{p}}\left(\int_a^b |C_1(t)||x^\sigma(t)|^q\Delta t\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Denote $M = \max_{a\leq t\leq b}|x(t)| > 0$. Then

$$M \leq \frac{1}{2}\left[\int_a^b |A(t)|M\Delta t + \left\{\max_{a\leq t\leq b}|\sqrt{B(t)}|^\beta\right\}^{\frac{1}{q}}\left(\int_a^b |\sqrt{B(t)}|^\alpha \Delta t\right)^{\frac{1}{p}}\left(\int_a^b |C_1(t)|M^q\Delta t\right)^{\frac{1}{q}}\right].$$

Thus,

$$\int_a^b |A(t)|\Delta t + \left\{\max_{a\leq t\leq b}|\sqrt{B(t)}|^\beta\right\}^{\frac{1}{q}}\left(\int_a^b |\sqrt{B(t)}|^\alpha \Delta t\right)^{\frac{1}{p}}\left(\int_a^b |C_1(t)|\Delta t\right)^{\frac{1}{q}} \geq 2.$$

This completes the proof of Theorem 3.5. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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