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Research Article

The Gerber-Shiu Discounted Penalty Function of Sparre Andersen Risk Model with a Constant Dividend Barrier

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This paper constructs a Sparre Andersen risk model with a constant dividend barrier in which the claim interarrival distribution is a mixture of an exponential distribution and an Erlang(n) distribution. We derive the integro-differential equation satisfied by the Gerber-Shiu discounted penalty function of this risk model. Finally, we provide a numerical example.

1. The Risk Model

Consider a Sparre Andersen risk model,

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i \quad \text{for } t \geq 0, \tag{1}$$

where $u \geq 0$ represents the initial capital, c is the insurer's rate of premium income per unit time, and $\{N(t), t \geq 0\}$ is the claim number process representing the number of claims up to time t. $\{X_i, i \geq 1\}$ is a sequence of i.i.d. random variables representing the individual claim amounts with distribution function $F(x)$ and density function $f(x)$ with mean μ. We assume that $\{N(t), t \geq 0\}$ and $\{X_i, i \geq 1\}$ are independent. Let $\{T_i, i \geq 1\}$ be sequence i.i.d. random variables, which represent the claim interarrival times, and T_i has a density function $K(t)$,

$$K(t) = \beta_1 \lambda e^{-\lambda t} + \beta_2 e^{-\lambda t} \frac{\lambda^n t^{n-1}}{(n-1)!}, \quad t \geq 0, \tag{2}$$

where $n \geq 1$ is a positive integer, $\lambda \geq 0$, $\beta_1, \beta_2 \geq 0$, and $\beta_1 + \beta_2 = 1$. We further assume that $cE[T_i] > E[X_i]$ for all i, which ensure that $\lim_{t\to\infty} U(t) = \infty$ almost surely. Throughout the paper we use the convention that $\sum_{i=1}^{0} X_i = 0$.

In recent years the Sparre Andersen model has been studied extensively. Ruin probabilities and many ruin related quantities such as the marginal and joint defective distributions of the time to ruin, the deficit at ruin, the surplus prior to ruin, and the claim size causing ruin have received considerable attention. Some related results can be found in Cai and Dickson [1], Sun and Yang [2], Gerber and Shiu [3], and Ko [4]. Li and Garrido [5] consider a compound renewal (Sparre Andersen) risk process in the presence of a constant dividend barrier in which the claim waiting times are generalized Erlang(n) distributed. The Sparre Andersen model with phase-type interclaim times has been studied by Ren [6]. Ng and Yang [7] study the ruin probability and the distribution of the severity of ruin in risk models with phase-type claims. Landriault and Willmot [8] study the Gerber-Shiu function in a Sparre Andersen model with general interclaim times. Yang and Zhang [9] study a Sparre Andersen model in which the interclaim times are generalized Erlang(n) distributed. They assume that the premium rate is a step function depending on the current surplus level. Landriault and Sendova [10] generalize the Sparre Andersen dual risk model with Erlang(n) interinnovation times by adding a budget-restriction strategy. Shi and Landriault [11] utilize the multivariate version of Lagrange expansion theorem to obtain a series expansion for the density of the time to ruin under a more general distribution assumption, namely, the combination of n exponentials. Yang and Sendova [12] study the Sparre Andersen dual risk model in which the times between positive gains are independently and identically distributed and have a generalized Erlang(n) distribution.

The barrier strategy was initially proposed by De Finetti [13] for a binomial model. From then on, barrier strategies have been studied in a number of papers and books, including Lin et al. [14], Dickson and Waters [15], Li and Lu [16], Yu [17–19], Yao et al. [20], Zhu [21], Tan et al. [22], and references therein for details. The purpose of this paper is to extend some results in Li and Garrido [5] and Yang and Zhang [9]. We study the Sparre Andersen risk model with a constant dividend barrier and the claim interarrival distribution is a mixture of an exponential distribution and an Erlang(n) distribution.

The contents of this paper are organized as follows. Section 2 introduces the risk model. In Section 3, we derive the higher-order integro-differential equation for the Gerber-Shiu discounted penalty function. Finally, in the special case we provide the numerical example in Section 4.

2. The Risk Model

Let $U_b(t)$ be the surplus process with initial surplus $U_b(0) = u$ under the barrier strategy. Thus, it can be expressed as

$$dU_b(t) = \begin{cases} cdt - dS(t) & U_b(t) < b \\ -dS(t) & U_b(t) \geq b, \end{cases} \tag{3}$$

where $S(t) = \sum_{i=1}^{N(t)} X_i$. Define $T_b = \inf\{t : U_b(t) < 0\}$ to be the first time that the surplus becomes negative. The stopping time T_b is referred to as the time of ruin. Let $\psi_b(u) = \Pr(T_b < \infty)$ be the ruin probability.

In this paper, we will study the time of ruin T_b and its related functions such as the surplus before ruin $U_b(T_b-)$ and the deficit at ruin $|U_b(T_b)|$. By using a renewal equation approach, we will be able to get a number of analytic and probabilistic properties of these quantities. Our analysis will involve the Gerber-Shiu discounted penalty function that is defined below.

Let $\omega(x, y)$, $0 \leq x, y < \infty$, be a nonnegative function. For $\delta \geq 0$, define

$$\begin{aligned} m_b(u) = E\big[&e^{-\delta T_b}\omega(U(T_b-), |U(T_b)|) \\ &I(T_b < \infty) \mid U(0) = u\big], \end{aligned} \tag{4}$$

where $I(\cdot)$ is the indicator function, $I(T_b < \infty) = 1$ if $T_b < \infty$, and $I(T_b < \infty) = 0$ otherwise. The function $m_b(u)$ in (4) is useful for deriving results in connection with joint and marginal distributions of T_b, $U_b(T_b-)$ and $|U_b(T_b)|$. While δ may be interpreted as a force of interest, function (4) may also be viewed in terms of a Laplace transform with δ serving as the argument. In particular, if we let $\omega(x, y) = 1$, (4) is the Laplace transform of the time of ruin T_b. If we let $\delta = 0$ and $\omega(x, y) = 1$, then $m_b(u)$ becomes the ruin probability $\psi(u)$. If we let $\delta = 0$ and $\omega(x, y) = I(x \leq z_1)I(y \leq z_2)$, (4) becomes the joint df of the surplus before ruin and the deficit at ruin. Furthermore, if $\delta = 0$ and $\omega(x, y) = x_1^n$, we obtain the nth moment of the surplus before ruin. Likewise, if $\delta = 0$ and $\omega(x, y) = x_2^n$, we obtain the nth moment of the deficit at ruin. For other functions of interest, see Gerber and Shiu [23] and Lin and Willmot [24]. Let f^* denote the Laplace transform of the function f, that is, $f^*(s) = \int_0^\infty e^{-sx} f(x)dx$.

3. An Integro-Differential Equation

In this section, we show $m_b(u)$ satisfies a higher-order integro-differential equation.

Lemma 1. *Assume $s > u$; then $H(u, s) = K((s - u)/c)e^{-\delta((s-u)/c)}$ satisfies the following differential equation:*

$$\begin{aligned} &\sum_{j=0}^{n-1} C_{n-1}^j c^j (-\lambda - \delta)^{n-1-j} \frac{\partial^j H(u, s)}{\partial u^j} \\ &\quad = (-1)^{n-1}\beta_2\lambda^n e^{-(\lambda+\delta)((s-u)/c)}, \end{aligned} \tag{5}$$

with the boundary conditions when $s = u$,

$$\begin{aligned} \frac{\partial^k H(u, s)}{\partial u^k} &= \beta_1\lambda\left(\frac{\lambda+\delta}{c}\right)^k, \quad k = 0, 1, 2, \ldots, n-2, \\ \frac{\partial^{n-1} H(u, s)}{\partial u^{n-1}} &= \beta_1\lambda\left(\frac{\lambda+\delta}{c}\right)^{n-1} + \beta_2\lambda^n\left(-\frac{1}{c}\right)^{n-1}. \end{aligned} \tag{6}$$

Proof. Note that $H(u, s) = [\beta_1\lambda + (\lambda^n\beta_2/(n-1)!)((\lambda+\delta)/c)^{n-1}]e^{-(\lambda+\delta)((s-u)/c)}$. Taking derivative with respect to variable u for k times and by induction, we can obtain

$$\begin{aligned} &\frac{\partial^k H(u, s)}{\partial u^k} \\ &\quad = -\sum_{j=0}^{k-1} C_k^j \left(-\frac{\lambda+\delta}{c}\right)^{k-j} \frac{\partial^j H(u, s)}{\partial u^j} \\ &\qquad + \frac{\beta_2\lambda^n}{(n-1-k)!}\left(\frac{s-u}{c}\right)^{n-1-k}\left(-\frac{1}{c}\right)^k e^{-(\lambda+\delta)((s-u)/c)}, \\ &\qquad\qquad 0 \leq k \leq n-1. \end{aligned} \tag{7}$$

When $k = n - 1$, one gets (5). On substituting $s = u$ in (7), we get the boundary conditions. □

Theorem 2. *The Gerber-Shiu discounted penalty function $m_b(u)$ satisfies the higher-order integro-differential equation*

$$\begin{aligned} &\sum_{k=0}^{n} C_n^k c^k (-\lambda - \delta)^{n-k} \frac{d^k m_b(u)}{du^k} \\ &\quad = \left[\beta_2(-\lambda)^n + \beta_1(-\lambda)(-\lambda-\delta)^{n-1}\right] \\ &\qquad \times \int_0^\infty m_b(u - x)\, dF(x) \\ &\qquad - \beta_1\lambda\sum_{k=1}^{n-1} C_{n-1}^k c^k (-\lambda-\delta)^{n-k-1}\frac{d^k}{du^k} \\ &\qquad \times \left(\int_0^\infty m_b(u - x)\, dF(x)\right). \end{aligned} \tag{8}$$

Proof. Let t be the time of the first claim and let x be the amount of the claim. There are two possibilities. First, $t < (b-u)/c$ and the surplus has not yet reached the barrier. In this case, the surplus immediately before time t is $u + ct$. Second, $t \geq (b - u)/c$ and the surplus immediately before time t is b. And since the "probability" that the claim occurs at time t is $K(t)dt$ and the "probability" of the claim amount being x is $dF(x)$, we have, for $0 \leq u \leq b$,

$$\begin{aligned} m_b(u) = & \int_0^{(b-u)/c} K(t) e^{-\delta t} \\ & \times \left[\int_0^{u+ct} m_b(u + ct - x) \, dF(x) \right. \\ & \left. + \int_{u+ct}^{\infty} w(u + ct, x - u - ct) \, dF(x) \right] dt \\ & + \int_{((b-u)/c)}^{\infty} K(t) e^{-\delta t} \\ & \times \left[\int_0^b m_b(b - x) \, dF(x) \right. \\ & \left. + \int_b^{\infty} w(b, x - b) \, dF(x) \right] dt. \end{aligned} \tag{9}$$

Using the substitution $s = u + ct$, we have

$$\begin{aligned} m_b(u) = & \int_u^b K\left(\frac{s-u}{c}\right) e^{-\delta((s-u)/c)} \\ & \times \left[\int_0^s m_b(s - x) \, dF(x) \right. \\ & \left. + \int_s^{\infty} w(s, x - s) \, dF(x) \right] \frac{1}{c} ds \\ & + \int_b^{\infty} K\left(\frac{s-u}{c}\right) e^{-\delta((s-u)/c)} \\ & \times \left[\int_0^b m_b(b - x) \, dF(x) \right. \\ & \left. + \int_b^{\infty} w(b, x - b) \, dF(x) \right] \frac{1}{c} ds \end{aligned} \tag{10}$$

which implies that

$$\begin{aligned} c m_b(u) = & \int_u^b H(u, s) \int_0^{\infty} m_b(s - x) \, dF(x) \, ds \\ & + \int_b^{\infty} H(u, s) \int_0^{\infty} m_b(b - x) \, dF(x) \, ds, \end{aligned} \tag{11}$$

where $H(u, s)$ is defined in Lemma 1. Differentiating the above equation k times and using condition (6) yield

$$\begin{aligned} & c \frac{d^k m_b(u)}{du^k} \\ & = -\beta_1 \lambda \sum_{i=0}^{k-1} \left(\frac{\lambda + \delta}{c}\right)^{k-1-i} \frac{d^i}{du^i} \int_0^{\infty} m_b(u - x) \, dF(x) \\ & + \int_u^b \frac{\partial^k H(u, s)}{\partial u^k} \int_0^{\infty} m_b(s - x) \, dF(x) \, ds \\ & + \int_b^{\infty} \frac{\partial^k H(u, s)}{\partial u^k} \int_0^{\infty} m_b(b - x) \, dF(x) \, ds. \end{aligned} \tag{12}$$

Multiplying (12) by $c^k(-\lambda-\delta)^{n-1-k} C_{n-1}^k$ for $k = 0, 1, 2, \ldots, n-1$, then adding up these equations, and using (5), we obtain

$$\begin{aligned} & \sum_{k=0}^{n-1} C_{n-1}^k (-\lambda - \delta)^{n-1-k} c^{k+1} \frac{d^k m_b(u)}{du^k} \\ & = \beta_2 \lambda^n (-1)^{n-1} \int_u^b e^{-(\lambda+\delta)((s-u)/c)} \\ & \quad \times \int_0^{\infty} m_b(s - x) \, dF(x) \, ds \\ & \quad + \beta_2 \lambda^n (-1)^{n-1} \int_b^{\infty} e^{-(\lambda+\delta)((s-u)/c)} \\ & \quad \times \int_0^{\infty} m_b(b - x) \, dF(x) \, ds \\ & \quad - \beta_1 \lambda \sum_{k=1}^{n-1} C_{n-1}^k (-\lambda - \delta)^{n-1-k} c^k \\ & \quad \times \left[\sum_{i=0}^{k-1} \left(\frac{\lambda + \delta}{c}\right)^{k-1-i} \frac{d^i}{du^i} \left(\int_0^{\infty} m_b(u - x) \, dF(x) \right) \right]. \end{aligned} \tag{13}$$

Differentiating (13) again, we have

$$\begin{aligned} & \sum_{k=0}^{n-1} C_{n-1}^k (-\lambda - \delta)^{n-1-k} c^{k+1} \frac{d^{k+1} m_b(u)}{du^{k+1}} \\ & = (-1)^n \beta_2 \lambda^n \int_0^{\infty} m_b(u - x) \, dF(x) \\ & \quad - \beta_1 \lambda \sum_{k=1}^{n-1} C_{n-1}^k (-\lambda - \delta)^{n-1-k} c^k \end{aligned}$$

$$\times\left[\sum_{i=0}^{k-1}\left(\frac{\lambda+\delta}{c}\right)^{k-1-i}\frac{d^{i+1}}{du^{i+1}}\left(\int_0^\infty m_b(u-x)\,dF(x)\right)\right]$$
$$+\frac{\lambda+\delta}{c}\beta_2\lambda^n(-1)^{n-1}$$
$$\times\left[\int_u^b e^{-(\lambda+\delta)((s-u)/c)}\int_0^\infty m_b(s-x)\,dF(x)\,ds\right.$$
$$\left.+\int_b^\infty e^{-(\lambda+\delta)((s-u)/c)}\int_0^\infty m_b(b-x)\,dF(x)\,ds\right] \tag{14}$$

which, together with (13), implies

$$\sum_{k=0}^{n-1}C_{n-1}^k c^{k+1}(-\lambda-\delta)^{n-1-k}\frac{d^{k+1}m_b(u)}{du^{k+1}}$$
$$+\sum_{k=0}^{n-1}C_{n-1}^k c^k(-\lambda-\delta)^{n-k}\frac{d^k m_b(u)}{du^k}$$
$$=-\beta_1\lambda\sum_{k=1}^{n-1}C_{n-1}^k(-\lambda-\delta)^{n-1-k}c^k$$
$$\times\left[\sum_{i=0}^{k-1}\left(\frac{\lambda+\delta}{c}\right)^{k-1-i}\frac{d^{i+1}}{du^{i+1}}\left(\int_0^\infty m_b(u-x)\,dF(x)\right)\right]$$
$$-\beta_1\lambda\sum_{k=1}^{n-1}C_{n-1}^k(-\lambda-\delta)^{n-k}c^{k-1}$$
$$\times\left[\sum_{i=0}^{k-1}\left(\frac{\lambda+\delta}{c}\right)^{k-1-i}\frac{d^{i}}{du^{i}}\left(\int_0^\infty m_b(u-x)\,dF(x)\right)\right]$$
$$+(-1)^n\beta_2\lambda^n\int_0^\infty m_b(u-x)\,dF(x). \tag{15}$$

Moreover, note that

$$\sum_{k=0}^{n-1}C_{n-1}^k c^{k+1}(-\lambda-\delta)^{n-1-k}\frac{d^{k+1}m_b(u)}{du^{k+1}}$$
$$=c^n\frac{d^n m_b(u)}{du^n}+\sum_{k=1}^{n-1}C_{n-1}^{k-1}c^k(-\lambda-\delta)^{n-k}\frac{d^k m_b(u)}{du^k},$$
$$\sum_{k=0}^{n-1}C_{n-1}^k c^k(-\lambda-\delta)^{n-k}\frac{d^k m_b(u)}{du^k}$$
$$=(-\lambda-\delta)^n m_b(u)+\sum_{k=1}^{n-1}C_{n-1}^k c^k(-\lambda-\delta)^{n-k}\frac{d^k m_b(u)}{du^k}. \tag{16}$$

So, it follows from (16) that

$$\sum_{k=0}^{n-1}C_{n-1}^k c^{k+1}(-\lambda-\delta)^{n-1-k}\frac{d^{k+1}m_b(u)}{du^{k+1}}$$
$$+\sum_{k=0}^{n-1}C_{n-1}^k c^k(-\lambda-\delta)^{n-k}\frac{d^k m_b(u)}{du^k} \tag{17}$$
$$=\sum_{k=0}^{n}C_n^k c^k(-\lambda-\delta)^{n-k}\frac{d^k m_b(u)}{du^k}$$

and thus the result follows from (15) and (17). □

Remark 3. Letting $\beta_1=0$, $\beta_2=1$, $n=2$ in (8), we get the integro-differential equation for Erlang (2) risk model with a constant dividend barrier of Li and Garrido [5].

Remark 4. Letting $\beta_1=0$, $\beta_2=1$, $n=2$, $b=\infty$ in (8), we obtain the integro-differential equation for Erlang (2) risk model with no dividend barrier, which has been considered in Dickson and Hipp [25].

Remark 5. Letting $n=1$, $b=\infty$ in (8), we derive the integro-differential equation for classical risk model. For details, see Gerber and Shiu [23].

Remark 6. Letting $n=1$, the case has been studied in Lin et al. [14].

Remark 7. Letting $b=\infty$, the case has been studied in Zhao and Yin [26].

Theorem 8. *The Laplace transform of $m_b(u)$ is*

$$m_b^*(s)$$
$$=\frac{A\int_0^\infty e^{-su}\int_u^\infty\omega(u,x-u)\,dF(x)\,du+G(s)+D(s)}{(sc-\lambda-\delta)^n-\left[\beta_2(-\lambda)^n-\lambda\beta_1(sc-\lambda-\delta)^{n-1}\right]f^*(s)}, \tag{18}$$

where

$$A=\beta_2(-\lambda)^n+\beta_1(-\lambda)(-\lambda-\delta)^{n-1},$$
$$G(s)=\sum_{k=1}^{n-1}\sum_{j=0}^{k-1}C_n^k(-\lambda-\delta)^{n-k}c^k s^{k-1-j}m_b^{(j)}(0),$$
$$D(s)=\beta_1\lambda\sum_{k=2}^{n-1}C_{n-1}^k(-\lambda-\delta)^{n-1-k}c^k$$
$$\times\sum_{j=1}^{k-1}s^{k-1-j}\sum_{l=0}^{j-1}m_b^{(l)}(0)\,f^{(j-1-l)}(0) \tag{19}$$
$$-\beta_1\lambda\sum_{k=1}^{n-1}C_{n-1}^k(-\lambda-\delta)^{n-1-k}c^k$$
$$\times\int_0^\infty e^{-su}\left[\frac{d^k}{du^k}\int_u^\infty\omega(u,x-u)\,dF(x)\right]du.$$

Proof. It is easy to see that

$$\int_0^\infty e^{-su}\frac{d^k m_b(u)}{du^k}du = s^k m_b^*(s) - \sum_{j=0}^{k-1} s^{k-1-j} m_b^{(j)}(0), \quad (20)$$

$$\begin{aligned}
&\int_0^\infty e^{-su}\int_0^\infty m_b(u-x)\,dF(x)\,du \\
&\quad = \int_0^\infty e^{-su}\int_0^u m_b(u-x)\,dF(x)\,du \\
&\quad\quad + \int_0^\infty e^{-su}\int_u^\infty \omega(u,x-u)\,dF(x)\,du \\
&\quad = s m_b^*(s) f^*(s) + \int_0^\infty e^{-su}\int_u^\infty \omega(u,x-u)\,dF(x)\,du,
\end{aligned} \quad (21)$$

$$\begin{aligned}
&\int_0^\infty e^{-su}\frac{d^k}{du^k}\left(\int_0^\infty m_b(u-x)\,dF(x)\right)du \\
&\quad = \int_0^\infty e^{-su}\frac{d^k}{du^k}\left(\int_0^u m_b(u-x)\,dF(x)\right)du \\
&\quad\quad + \int_0^\infty e^{-su}\frac{d^k}{du^k}\left(\int_u^\infty \omega(u,x-u)\,dF(x)\right)du \\
&\quad = s^k m_b^*(s) f^*(s) - \sum_{j=1}^{k-1} s^{k-1-j}\sum_{l=0}^{j-1} m_b^{(l)}(0) f^{(j-1-l)}(0) \\
&\quad\quad + \int_0^\infty e^{-su}\frac{d^k}{du^k}\left(\int_u^\infty \omega(u,x-u)\,dF(x)\right)du.
\end{aligned} \quad (22)$$

Taking the Laplace transform on both sides of (8), and together with (20), (21), and (22), we have

$$\begin{aligned}
&\sum_{k=0}^{n} C_n^k c^k(-\lambda-\delta)^{n-k}\left(s^k m_b^*(s) - \sum_{j=0}^{k-1} s^{k-1-j} m_b^{(j)}(0)\right) \\
&\quad = \left[\beta_2(-\lambda)^n + \beta_1(-\lambda)(-\lambda-\delta)^{n-1}\right] \\
&\quad\quad \times\left[m_b^*(s) f^*(s) + \int_0^\infty e^{-su}\int_u^\infty \omega(u,x-u)\,dF(x)\,du\right] \\
&\quad\quad - \beta_1\lambda\left[(sc-\lambda-\delta)^{n-1} - (-\lambda-\delta)^{n-1}\right] m_b^*(s) f^*(s) \\
&\quad\quad + \beta_1\lambda\sum_{k=2}^{n-1} C_{n-1}^k c^k(-\lambda-\delta)^{n-1-k} \\
&\quad\quad \times \sum_{j=1}^{k-1} s^{k-1-j}\sum_{l=0}^{j-1} m_b^{(l)}(0) f^{(j-1-l)}(0) \\
&\quad\quad - \beta_1\lambda\sum_{k=1}^{n-1} C_{n-1}^k c^k(-\lambda-\delta)^{n-1-k} \\
&\quad\quad \times \int_0^\infty e^{-su}\left[\frac{d^k}{du^k}\int_u^\infty \omega(u,x-u)\,dF(x)\right]du
\end{aligned} \quad (23)$$

which implies (8). □

Lemma 9. *Let δ be strictly positive and n is a positive integer; then the equation*

$$\begin{aligned}
(sc-\lambda-\delta)^n &= f^*(s)\left[\beta_2(-\lambda)^n + \beta_1(-\lambda)(-\lambda-\delta)^{n-1}\right] \\
&\quad - f^*(s)\lambda\beta_1\left[(sc-\lambda-\delta)^{n-1} - (-\lambda-\delta)^{n-1}\right]
\end{aligned} \quad (24)$$

has exact n roots $s_l(\delta)$ with $\mathrm{Re}(s_l(\delta)) > 0$ $(l = 1,2,3,\ldots,n)$.

Proof. When $s = 0$,we have

$$\left|\left[\beta_2(-\lambda)^n + \beta_1(-\lambda)(-\lambda-\delta)^{n-1}\right] f^*(0)\right| < \left|(-\lambda-\delta)^n\right|. \quad (25)$$

So for $\rho > 0$ sufficiently big, the inequality

$$\begin{aligned}
&\left|(sc-\lambda-\delta)^n\right| \\
&> \left|\left[\beta_2(-\lambda)^n + \beta_1(-\lambda)(-\lambda-\delta)^{n-1}\right.\right. \\
&\quad \left.\left. -\lambda\beta_1(sc-\lambda-\delta)^{n-1} + \lambda\beta_1(-\lambda-\delta)^{n-1}\right] f^*(s)\right|
\end{aligned} \quad (26)$$

holds on the imaginary axis and on the semicircle $\{s \in £,\ \mathrm{Re}(s) > 0,\ |s| = \rho\}$. By Rouches theorem (20) has exact n roots on the right-half plane. □

4. Numerical Illustration for Ruin Probability

In this section, we give the numerical illustration for $m_b(u)$ when the claim number process has Erlang (2) process ($\beta_1 = 0$, $\beta_2 = 1$, $n = 2$), $\delta = 0$ and $w(x,y) = 1$. At this time, $m_b(u)$ turns to ruin probability $\psi_b(u)$. By conditioning on the time of the first claim we have, for $0 \le u \le b$,

$$\begin{aligned}
m_b(u) &= \int_0^{((b-u)/c)} K_1(t)\,\gamma_b(u+ct)\,dt \\
&\quad + \int_{((b-u)/c)}^\infty K_1(t)\,\gamma_b(b)\,dt,
\end{aligned} \quad (27)$$

where

$$\gamma_b(t) = \int_0^t m_b(t-x)\,dF(x) + 1 - F(t). \quad (28)$$

Substituting $K_1(t) = \lambda^2 t e^{-\lambda t}$ into (27), we obtain

$$\begin{aligned}
m_b(u) &= \left(\frac{\lambda}{c}\right)^2\int_u^b (t-u)\,e^{-(\lambda/c)(t-u)}\gamma_b(t)\,dt \\
&\quad + \gamma_b(b)\,e^{-(\lambda/c)(t-u)}\left[1 + \frac{\lambda}{c}(b-u)\right].
\end{aligned} \quad (29)$$

Differentiating (29) with respect to u, we have, for $0 \le u \le b$,

$$\begin{aligned}
m_b'(u) &= \frac{\lambda}{c}m_b(u) - \left(\frac{\lambda}{c}\right)^2\int_u^b e^{-(\lambda/c)(t-u)}\gamma_b(t)\,dt \\
&\quad - \frac{\lambda\gamma_b(b)}{c}e^{-(\lambda/c)(b-u)}.
\end{aligned} \quad (30)$$

Differentiating (30) again with respect to u, we have

$$\begin{aligned} m_b''(u) &= \frac{\lambda}{c} m_b'(u) \\ &- \frac{\lambda}{c}\left[\left(\frac{\lambda}{c}\right)^2 \int_u^b e^{-(\lambda/c)(t-u)} \gamma_b(t)\,dt \right. \\ &\left. + \frac{\lambda}{c}\gamma_b(b) e^{-(\lambda/c)(b-u)}\right] \\ &+ \left(\frac{\lambda}{c}\right)^2 \gamma_b(u). \end{aligned} \quad (31)$$

Suppose the claim size distribution is exponential. Let $F(x) = 1 - e^{-\alpha x}$, $\alpha > 0$; then substituting (30) into (31), we have

$$\begin{aligned} m_b''(u) &= \frac{2\lambda}{c} m_b'(u) - \left(\frac{\lambda}{c}\right)^2 m_b(u) \\ &+ \left(\frac{\lambda}{c}\right)^2 \alpha e^{-\alpha u} \int_0^u m_b(t) e^{-\alpha t} dt + \left(\frac{\lambda}{c}\right)^2 e^{-\alpha u}. \end{aligned} \quad (32)$$

Differentiating (32) with respect to u, we have

$$\begin{aligned} m_b'''(u) &= \frac{2\lambda}{c} m_b''(u) - \left(\frac{\lambda}{c}\right)^2 m_b'(u) \\ &- \left(\frac{\lambda}{c}\right)^2 \alpha^2 e^{-\alpha u} \int_0^u m_b(t) e^{-\alpha t} dt \\ &+ \left(\frac{\lambda}{c}\right)^2 \alpha m_b(u) - \left(\frac{\lambda}{c}\right)^2 \alpha e^{-\alpha u}. \end{aligned} \quad (33)$$

(32) $\times\, \alpha$ + (33) implies

$$m_b'''(u) + \left(\alpha - \frac{2\lambda}{c}\right) m_b''(u) + \frac{\lambda^2 - 2\alpha c\lambda}{c^2} m_b'(u) = 0. \quad (34)$$

This is a three-order differential equation with constant coefficients, so we can carry on the numerical solution. Suppose $\alpha = 10000$, $c = 200$, $\lambda = 0.0001$, $b = 20$; then by the Matlab, we obtain the curve of ruin probability (see Figure 1). As is known to all ruin must occur under the constant dividend barrier. From Figure 1, we know that ruin probability $\psi_b(u)$ is an increasing function of the initial surplus u (convex function) and the function value of 1 is its asymptote.

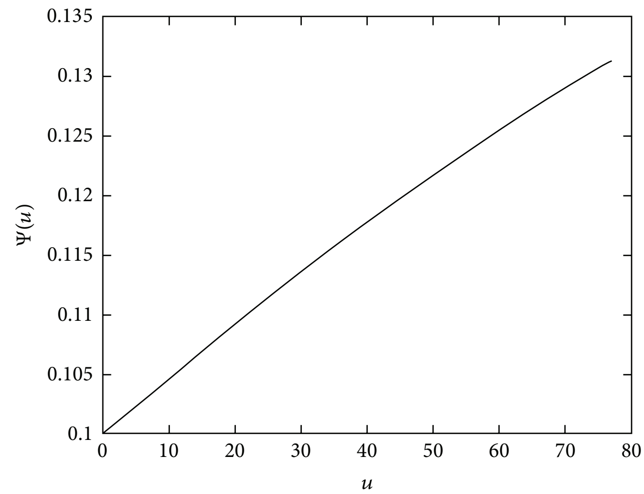

Figure 1: The curve of ruin probability.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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