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Research Article

Existence Results and the Monotone Iterative Technique for Nonlinear Fractional Differential Systems with Coupled Four-Point Boundary Value Problems

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By establishing a comparison result and using the monotone iterative technique combined with the method of upper and lower solutions, we investigate the existence of solutions for nonlinear fractional differential systems with coupled four-point boundary value problems.

1. Introduction

This paper discusses the coupled four-point boundary value problems

$$\begin{gathered} D^p x(t) + f(t, x(t), y(t)) = 0, \quad t \in (0,1), \ 1 < p \le 2, \\ D^q y(t) + g(t, x(t), y(t)) = 0, \quad t \in (0,1), \ 1 < q \le 2, \\ x(0) = y(0) = 0, \qquad x(1) = a y(\xi), \qquad y(1) = b x(\eta), \end{gathered} \tag{1}$$

where f and $g : (0,1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, $\xi, \eta \in (0,1)$, $a, b > 0$ with $ab < 1$, and $D^p x$ denotes the Caputo fractional derivative of x with $1 < p \le 2$ defined by

$$D^p x(t) = I^{2-p} x''(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{1-p} x''(s)\, ds. \tag{2}$$

I^{2-p} is the Riemann-Liouville fractional integral of order $2-p$; see [1–4].

It is well known that

$$\begin{gathered} I^{2-p}\left(D^{2-p} x(t)\right) = x(t) - \sum_{k=0}^{1} \frac{x^{(k)}(0^+)}{k!} t^k, \\ D^{2-p}\left(I^{2-p} x(t)\right) = x(t). \end{gathered} \tag{3}$$

Fractional differential equation's modeling capabilities in physics, chemistry, economics, and other fields, over the last few decades, have resulted in the rapid development of the theory of fractional differential equations; we refer the reader to the books [1–4]. On the other hand, the study of systems involving coupled boundary value problems is also important as such systems occur in the study of reaction-diffusion equations and Sturm-Liouville problems, for example, [5–16]. In [17–25], using the upper and lower solutions method and the monotone iterative method, the authors considered the existence of solutions of initial value problems and boundary value problems for fractional differential equations. But, as far as we know, there have been few papers which have considered the existence of solutions of (1) by means of the monotone iterative method.

Motivated by the above papers, in this paper, we will investigate the existence of a solution of problem (1) by means of the upper and lower solutions method and the monotone iterative method. The novelty of this paper is that Caputo-type fractional differential systems involve two different fractional derivatives D^p and D^q and that the nonlinear terms f, g in the systems (1) involve unknown functions $x(t)$ and $y(t)$.

In the following, we denote

$$E = C^2([0,1], \mathbb{R}), \qquad E_1 = C([0,1], (0, +\infty)). \tag{4}$$

2. Preliminaries and Lemmas

In this section, we introduce the definition of the lower and upper solutions and present some existence and uniqueness results for linear problems together with comparison results for differential systems (1) which will be needed in the next section.

Throughout this paper, we always assume that the following condition is satisfied:

(H_1) $0 < ab < 1$.

Definition 1. $(u_0, v_0) \in E \times E$ is called a lower system of solutions of differential system (1) if

$$\begin{aligned} &D^p u_0(t) + f(t, u_0(t), v_0(t)) \geq 0, \quad t \in (0,1), \\ &D^q v_0(t) + g(t, u_0(t), v_0(t)) \geq 0, \quad t \in (0,1), \\ &u_0(0) \leq 0, \qquad v_0(0) \leq 0, \\ &u_0(1) \leq a v_0(\xi), \qquad v_0(1) \leq b u_0(\eta). \end{aligned} \tag{5}$$

Analogously, $(\alpha_0, \beta_0) \in E \times E$ is called an upper system of solutions if it satisfies the reversed inequalities.

If $u_0(t) \leq \alpha_0(t)$ and $v_0(t) \leq \beta_0(t)$, $t \in [0,1]$, we say that (u_0, v_0) and (α_0, β_0) are ordered lower and upper system of solutions of (1). In what follows, we assume that (u_0, v_0) and (α_0, β_0) are ordered lower and upper system of solutions of (1) and define the sector

$$\begin{aligned} \Omega = \{(x, y) \in E \times E : &(u_0(t), v_0(t)) \\ &\leq (x(t), y(t)) \\ &\leq (\alpha_0(t), \beta_0(t)), t \in [0,1]\}, \end{aligned} \tag{6}$$

where the vectorial inequalities mean that the same inequalities hold between their corresponding components.

Lemma 2 (see [17]). *Let $z(t) \in E$ and $r(t) \in E_1$. If $z(t)$ satisfies the inequality*

$$\begin{aligned} &-D^p z(t) \leq -r(t) z(t), \quad p \in (1,2], \ t \in (0,1), \\ &z(0) \leq 0, \qquad z(1) \leq 0, \end{aligned} \tag{7}$$

then $z(t) \leq 0$, $\forall t \in [0,1]$.

We have the following important result.

Lemma 3 (comparison theorem). *Let $M(t), N(t) \in E_1$ be given. Assume that $x(t)$, $y(t)$ satisfy*

$$\begin{aligned} &-D^p x(t) \leq -M(t) x(t), \quad t \in (0,1), \\ &-D^q y(t) \leq -N(t) y(t), \quad t \in (0,1), \\ &x(0) \leq 0, \qquad y(0) \leq 0, \\ &x(1) \leq a y(\xi), \qquad y(1) \leq b x(\eta). \end{aligned} \tag{8}$$

Then $x(t) \leq 0$, $y(t) \leq 0$, $\forall t \in [0,1]$.

Proof. Suppose the contrary. By Lemma 2, We consider the following three possible cases.

Case 1. Consider $x(1) \leq 0$ and $y(1) > 0$. By Lemma 2, $x(t) \leq 0$, $\forall t \in [0,1]$. Then $y(1) \leq bx(\eta) \leq 0$ which contradicts $y(1) > 0$.

Case 2. Consider $y(1) \leq 0$ and $x(1) > 0$. By Lemma 2, $y(t) \leq 0$, $\forall t \in [0,1]$. Then $x(1) \leq ay(\xi) \leq 0$ which contradicts $x(1) > 0$.

Case 3. Consider $x(1) > 0$ and $y(1) > 0$. By Lemma 2, we have $x(1) = \max_{t \in [0,1]} x(t) > 0$ and $y(1) = \max_{t \in [0,1]} y(t) > 0$. We only prove that $x(1) = \max_{t \in [0,1]} x(t) > 0$. If not, $x(t)$ has a local positive maximum at some $t_0 \in (0,1)$ such that $x(t_0) = \max_{t \in [0,1]} x(t) > 0$. Then, by Theorem 2.1 in [21], we have the fact that the Caputo derivative of the function x is nonpositive at the point t_0. Thus,

$$0 \leq -D^p x(t_0) \leq -M(t_0) x(t_0) < 0, \tag{9}$$

which is a contradiction. Furthermore, considering the boundary condition $y(1) \leq bx(\eta)$, there exists $t_1 \in [0, \eta)$ such that

$$x(t) \leq 0, \quad t \in [0, t_1]; \qquad x(t) \geq 0, \quad t \in [t_1, 1]. \tag{10}$$

A similar proof, for $y(t)$, gives us that there exists $t_2 \in [0, \xi)$ such that

$$y(t) \leq 0, \quad t \in [0, t_2]; \qquad y(t) \geq 0, \quad t \in [t_2, 1]. \tag{11}$$

It follows from (10) and (11) that

$$x(1) \leq ay(\xi) \leq ay(1) \leq abx(\eta) \leq abx(1), \tag{12}$$

which implies that $ab \geq 1$, a contradiction. Hence $x(t) \leq 0$, $y(t) \leq 0$, $\forall t \in [0,1]$. □

Corollary 4. *Let $M(t), N(t) \in E_1$ be given. Assume that $x(t)$, $y(t)$ satisfy*

$$\begin{aligned} &-D^p x(t) = -M(t) x(t), \quad t \in (0,1), \\ &-D^q y(t) = -N(t) y(t), \quad t \in (0,1), \\ &x(0) = 0, \qquad y(0) = 0, \\ &x(1) = a y(\xi), \qquad y(1) = b x(\eta). \end{aligned} \tag{13}$$

Then $x(t) = y(t) = 0$, $\forall t \in [0,1]$.

Lemma 5. *Let ρ, $\sigma \in C[0,1]$, then the linear differential system with coupled four-point boundary value problem*

$$\begin{gathered} D^{p} x(t)+\rho(t)=0, \quad t \in(0,1), \; 1<p \leq 2, \\ D^{q} y(t)+\sigma(t)=0, \quad t \in(0,1), \; 1<q \leq 2, \\ x(0)=y(0)=0, \qquad x(1)=a y(\xi), \qquad y(1)=b x(\eta) \end{gathered} \tag{14}$$

has integral representation

$$\begin{aligned} x(t) &= \int_{0}^{1} G_{1}(t, s) \rho(s) d s+\int_{0}^{1} H_{1}(t, s) \sigma(s) d s, \\ y(t) &= \int_{0}^{1} G_{2}(t, s) \sigma(s) d s+\int_{0}^{1} H_{2}(t, s) \rho(s) d s, \end{aligned} \tag{15}$$

where

$$\begin{gathered} G_{1}(t, s)=G_{p}(t, s)+\frac{a b \xi t}{1-a b \xi \eta} G_{p}(\eta, s), \\ H_{1}(t, s)=\frac{a t}{1-a b \xi \eta} G_{q}(\xi, s), \\ G_{2}(t, s)=G_{q}(t, s)+\frac{a b \eta t}{1-a b \xi \eta} G_{q}(\xi, s), \\ H_{2}(t, s)=\frac{b t}{1-a b \xi \eta} G_{p}(\eta, s), \\ G_{p}(t, s)= \begin{cases}\dfrac{t(1-s)^{p-1}-(t-s)^{p-1}}{\Gamma(p)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{t(1-s)^{p-1}}{\Gamma(p)}, & 0 \leq t \leq s \leq 1.\end{cases} \end{gathered} \tag{16}$$

Proof. It follows from [21] that (14) is equivalent to the system of integral equations

$$\begin{aligned} x(t) &= x(1) t+\int_{0}^{1} G_{p}(t, s) \rho(s) d s, \quad t \in[0,1], \\ y(t) &= y(1) t+\int_{0}^{1} G_{q}(t, s) \sigma(s) d s, \quad t \in[0,1]. \end{aligned} \tag{17}$$

By coupled four-point boundary value conditions of problem (14), we have

$$y(1)=b x(\eta)=b \eta x(1)+b \int_{0}^{1} G_{p}(\eta, s) \rho(s) d s, \tag{18}$$

$$x(1)=a y(\xi)=a \xi y(1)+a \int_{0}^{1} G_{q}(\xi, s) \sigma(s) d s. \tag{19}$$

After simple computation, we get

$$\begin{aligned} x(1)= & \frac{a b \xi}{1-a b \xi \eta} \int_{0}^{1} G_{p}(\eta, s) \rho(s) d s \\ & +\frac{a}{1-a b \xi \eta} \int_{0}^{1} G_{q}(\xi, s) \sigma(s) d s, \end{aligned} \tag{20}$$

$$\begin{aligned} y(1)= & \frac{a b \eta}{1-a b \xi \eta} \int_{0}^{1} G_{q}(\xi, s) \sigma(s) d s \\ & +\frac{b}{1-a b \xi \eta} \int_{0}^{1} G_{p}(\eta, s) \rho(s) d s. \end{aligned} \tag{21}$$

Substituting (20) into (18) and (21) into (19), respectively, we obtain the desired results. □

Now we enunciate the following existence and uniqueness results for differential system:

$$\begin{gathered} D^{p} x(t)-M(t) x(t)+\rho(t)=0, \quad t \in(0,1), \; 1<p \leq 2, \\ D^{q} y(t)-N(t) y(t)+\sigma(t)=0, \quad t \in(0,1), \; 1<q \leq 2, \\ x(0)=y(0)=0, \qquad x(1)=a y(\xi), \qquad y(1)=b x(\eta), \end{gathered} \tag{22}$$

where $M, N \in E_1$.

Lemma 6. *Let $M, N \in E_1$. Then differential system* (22) *has a unique solution.*

Proof. Indeed, by Lemma 5, differential system (22) is equivalent to the operator equation

$$(x, y)=T(x, y)+(\tilde{\rho}, \tilde{\sigma}), \tag{23}$$

where

$$\begin{aligned} & T(x, y)(t) \\ & =\left(-\int_{0}^{1} G_{1}(t, s) M(s) x(s) d s-\int_{0}^{1} H_{1}(t, s) N(s) y(s) d s,\right. \\ & \quad -\int_{0}^{1} G_{2}(t, s) N(s) y(s)(s) d s \\ & \quad \left.-\int_{0}^{1} H_{2}(t, s) M(s) x(s) d s\right), \\ & \tilde{\rho}(t)=\int_{0}^{1} G_{1}(t, s) \rho(s) d s+\int_{0}^{1} H_{1}(t, s) \sigma(s) d s, \\ & \tilde{\sigma}(t)=\int_{0}^{1} G_{2}(t, s) \sigma(s) d s+\int_{0}^{1} H_{2}(t, s) \rho(s) d s. \end{aligned} \tag{24}$$

We apply the Fredholm theorem to find the unique solution of differential system (22). By using standard arguments, we can easily show that $T : C[0,1] \times C[0,1] \rightarrow C[0,1] \times C[0,1]$ is linear completely continuous Also, by Corollary 4, the operator equation $(x, y) = T(x, y)$ has only the zero solution. Thus, for given $(\tilde{\rho}, \tilde{\sigma}) \in C[0,1] \times C[0,1]$, operator equation (23) has a unique solution in $C[0,1]\times C[0,1]$, by the Fredholm theorem. This ends the proof. □

3. Main Results

In this section, we prove the existence of extremal solutions of differential system (1).

Theorem 7. *Assume that* $f \in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $g \in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$. *Let* (u_0, v_0) *and* (α_0, β_0) *be ordered lower and upper system of solutions of* (1). *In addition, we assume that*

(H_2) $f(t,x,y)$ *is nondecreasing in* y *and there exists* $M(t) \in E_1$ *such that*

$$f(t,x_1,y) - f(t,x_2,y) \geq -M(t)(x_1 - x_2), \tag{25}$$

where $u_0(t) \leq x_2 \leq x_1 \leq \alpha_0(t)$, $v_0(t) \leq y \leq \beta_0(t)$;

(H_3) $g(t,x,y)$ *is nondecreasing in* x *and there exists* $N(t) \in E_1$ *such that*

$$g(t,x,y_1) - g(t,x,y_2) \geq -N(t)(y_1 - y_2), \tag{26}$$

where $v_0(t) \leq y_2 \leq y_1 \leq \beta_0(t)$, $u_0(t) \leq x \leq \alpha_0(t)$.

Then differential system (1) has extremal solutions in the section Ω.

Proof. Let us define two sequences $\{(u_n, v_n), (\alpha_n, \beta_n)\}$ by relations

$$\begin{gathered}
D^p u_{n+1}(t) - M(t)u_{n+1}(t) + f(t,u_n(t),v_n(t)) + M(t)u_n(t) = 0, \quad t \in (0,1), \\
D^q v_{n+1}(t) - N(t)v_{n+1}(t) + g(t,u_n(t),v_n(t)) + N(t)v_n(t) = 0, \quad t \in (0,1), \\
u_{n+1}(0) = v_{n+1}(0) = 0, \quad u_{n+1}(1) = a v_{n+1}(\xi), \\
v_{n+1}(1) = b u_{n+1}(\eta), \\
D^p \alpha_{n+1}(t) - M(t)\alpha_{n+1}(t) + f(t,\alpha_n(t),\beta_n(t)) + M(t)\alpha_n(t) = 0, \quad t \in (0,1), \\
D^q \beta_{n+1}(t) - N(t)\beta_{n+1}(t) + g(t,\alpha_n(t),\beta_n(t)) + N(t)\beta_n(t) = 0, \quad t \in (0,1), \\
\alpha_{n+1}(0) = \beta_{n+1}(0) = 0, \quad \alpha_{n+1}(1) = a\beta_{n+1}(\xi), \\
\beta_{n+1}(1) = b\alpha_{n+1}(\eta),
\end{gathered} \tag{27}$$

for $n = 1, 2, \ldots$. Note that $\{(u_1, v_1), (\alpha_1, \beta_1)\}$ are well defined, by Lemma 6. First, we show that

$$(u_0, v_0) \leq (u_1, v_1) \leq (\alpha_1, \beta_1) \leq (\alpha_0, \beta_0). \tag{28}$$

Let $z = u_0 - u_1$, $w = v_0 - v_1$. This and the assumption that (u_0, v_0) is a lower system of solutions of (1) yield

$$\begin{gathered}
-D^p z(t) \leq -M(t)z(t), \quad t \in (0,1), \\
-D^q w(t) \leq -N(t)w(t), \quad t \in (0,1), \\
z(0) \leq 0, \quad w(0) \leq 0, \quad z(1) \leq a w(\xi), \\
w(1) \leq b z(\eta).
\end{gathered} \tag{29}$$

Hence, $u_0(t) \leq u_1(t)$ and $v_0(t) \leq v_1(t)$, $t \in [0,1]$, by Lemma 3. By a similar way, we can show that $\alpha_1(t) \leq \alpha_0(t)$ and $\beta_1(t) \leq \beta_0(t)$, $t \in [0,1]$. Now we put $z = u_1 - \alpha_1$, $w = v_1 - \beta_1$. Hence, in view of assumptions (H_2), (H_3), we have

$$\begin{aligned}
-D^p z(t) &= -M(t)z(t) - f(t,\alpha_0(t),\beta_0(t)) \\
&\quad - M(t)\alpha_0(t) + f(t,u_0(t),v_0(t)) + M(t)u_0(t) \\
&\leq -M(t)z(t) - f(t,\alpha_0(t),\beta_0(t)) \\
&\quad - M(t)\alpha_0(t) + f(t,u_0(t),\beta_0(t)) + M(t)u_0(t) \\
&\leq -M(t)z(t), \quad t \in (0,1), \\
-D^q w(t) &\leq -N(t)w(t), \quad t \in (0,1), \\
z(0) = 0, &\quad w(0) = 0, \quad z(1) = a w(\xi), \\
w(1) &= b z(\eta).
\end{aligned} \tag{30}$$

This and Lemma 3 prove that $(u_1, v_1) \leq (\alpha_1, \beta_1)$, $t \in [0,1]$, so, relation (28) holds.

Now we show that (u_1, v_1) is a lower system of solution of problem (1). Note that

$$\begin{aligned}
D^p u_1(t) &+ f(t,u_1(t),v_1(t)) \\
&\geq M(t)u_1(t) - f(t,u_0(t),v_0(t)) - M(t)u_0(t) \\
&\quad + f(t,u_1(t),v_0(t)) \geq 0, \quad t \in (0,1), \\
D^q v_1(t) &+ g(t,u_1(t),v_1(t)) \geq 0 \quad t \in (0,1), \\
u_1(0) &= v_1(0) = 0, \quad u_1(1) = a v_1(\xi), \\
v_1(1) &= b u_1(\eta),
\end{aligned} \tag{31}$$

by assumptions (H_2), (H_3). It proves that (u_1, v_1) is a lower system of solution of (1). Similarly, we can prove that (α_1, β_1) is an upper system of solution of problem (1).

By mathematical induction we can show that

$$\begin{gathered}
(u_0, v_0) \leq (u_1, v_1) \leq \cdots \leq (u_n, v_n) \leq (\alpha_n, \beta_n) \\
\leq \cdots \leq (\alpha_1, \beta_1) \leq (\alpha_0, \beta_0)
\end{gathered} \tag{32}$$

for $t \in [0,1]$ and $n = 1, 2, \ldots$. Employing standard arguments we see that the sequences $\{(u_n, v_n), (\alpha_n, \beta_n)\}$ converge to their limit functions (u_*, v_*), (α^*, β^*), respectively. Indeed, (u_*, v_*) and (α^*, β^*) are solutions of problem (1) and $(u_0(t), v_0(t)) \leq (u_*, v_*) \leq (\alpha^*, \beta^*) \leq (\alpha_0, \beta_0)$ on $[0,1]$.

We need to show now that (u_*, v_*) and (α^*, β^*) are extremal solutions of problem (1) in the segment Ω. To prove it, we assume that (x, y) is another solution of problem (1) and $(u_n, v_n) \leq (x(t), y(t)) \leq (\alpha_n(t), \beta_n(t))$, $t \in [0,1]$ for some positive integer n. Put $z = u_{n+1} - x$, $w = v_{n+1} - y$. Hence, in view of assumptions (H_2), (H_3), we have

$$\begin{aligned} -D^{p} z(t) &= -M(t) z(t) - f(t, x(t), y(t)) \\ &\quad - M(t) x(t) + f(t, u_n(t), v_n(t)) + M(t) u_n(t) \\ &\leq -M(t) z(t) - f(t, x(t), y(t)) - M(t) x(t) \\ &\quad + f(t, u_n(t), y(t)) + M(t) u_n(t) \\ &\leq -M(t) z(t), \quad t \in (0,1), \\ -D^{q} w(t) &\leq -N(t) w(t), \quad t \in (0,1), \\ z(0) = 0, \quad & w(0) = 0, \quad z(1) = a w(\xi), \\ & w(1) = b z(\eta). \end{aligned} \tag{33}$$

Hence, $(u_{n+1}(t), v_{n+1}(t)) \leq (x(t), y(t))$, $t \in [0,1]$, by Lemma 3. By a similar way, we can show that $(x(t), y(t)) \leq (\alpha_{n+1}(t), \beta_{n+1}(t))$, $t \in [0,1]$. So by induction, $(u_n(t), v_n(t)) \leq (x(t), y(t)) \leq (\alpha_n(t), \beta_n(t))$, $t \in [0,1]$ on $[0,1]$ for all n. Taking the limit as $n \to +\infty$, we conclude $(u_*(t), v_*(t)) \leq (x(t), y(t)) \leq (\alpha^*(t), \beta^*(t))$, $t \in [0,1]$. That is, $(u_*(t), v_*(t))$ and $(\alpha^*(t), \beta^*(t))$ are extremal systems of solutions of (1) in Ω. □

4. Example

Consider the following problems:

$$\begin{aligned} D^{5/4} x(t) + \sin t - 2x(t) + \frac{1}{8} y^3(t) t^3 = 0, \quad & t \in (0,1), \\ D^{7/4} y(t) - y^3(t) + x^2(t) + 1 = 0, \quad & t \in (0,1), \\ x(0) = y(0) = 0, \quad x(1) = \frac{1}{4} y\left(\frac{1}{2}\right), & \\ y(1) = 2x\left(\frac{3}{4}\right). & \end{aligned} \tag{34}$$

Obviously,

$$\begin{aligned} f(t, x, y) &= \sin t - 2x + \frac{1}{8} y^3 t^3, \\ g(t, x, y) &= -y^3 + x^2 + 1. \end{aligned} \tag{35}$$

Take $(u_0(t), v_0(t)) = (0,0)$, $(\alpha_0(t), \beta_0(t)) = (t, 2)$; then

$$\begin{aligned} & D^{5/4} u_0(t) + \sin t - 2u_0(t) + \frac{1}{8} v_0^3(t) t^3 = \sin t \geq 0, \\ & \qquad t \in (0,1), \\ & D^{7/4} v_0(t) - v_0^3(t) + u_0^2(t) + 1 = 1 \geq 0, \quad t \in (0,1), \\ & u_0(0) = v_0(0) = 0, \quad u_0(1) = \frac{1}{4} v_0\left(\frac{1}{2}\right), \\ & v_0(1) = 2u_0\left(\frac{3}{4}\right), \\ & D^{5/4} \alpha_0(t) + \sin t - 2\alpha_0(t) + \frac{1}{8} \beta_0^3(t) t^3 \\ & \quad = \sin t - 2t + t^3 \leq 0, \quad t \in (0,1), \\ & D^{7/4} \beta_0(t) - \beta_0^3(t) + \alpha_0^2(t) + 1 = -7 + t^2 \leq 0, \\ & \qquad t \in (0,1), \\ & \alpha_0(0) = 0, \quad \beta_0(0) \geq 0, \\ & \alpha_0(1) \geq \frac{1}{4} \beta_0\left(\frac{1}{2}\right), \quad \beta_0(1) \geq 2\alpha_0\left(\frac{3}{4}\right). \end{aligned} \tag{36}$$

It shows that $(u_0(t), v_0(t))$ and $(\alpha_0(t), \beta_0(t))$ are lower and upper systems of solutions of (34).

On the other hand, it is easy to verify that conditions (H_2), (H_3) hold for $M(t) = 2$ and $N(t) = 12$.

By Theorem 7, problem (34) has an extremal system of solutions $(u_*(t), v_*(t))$ and $(\alpha^*(t), \beta^*(t))$, which can be obtained by taking limits from some iterative sequences.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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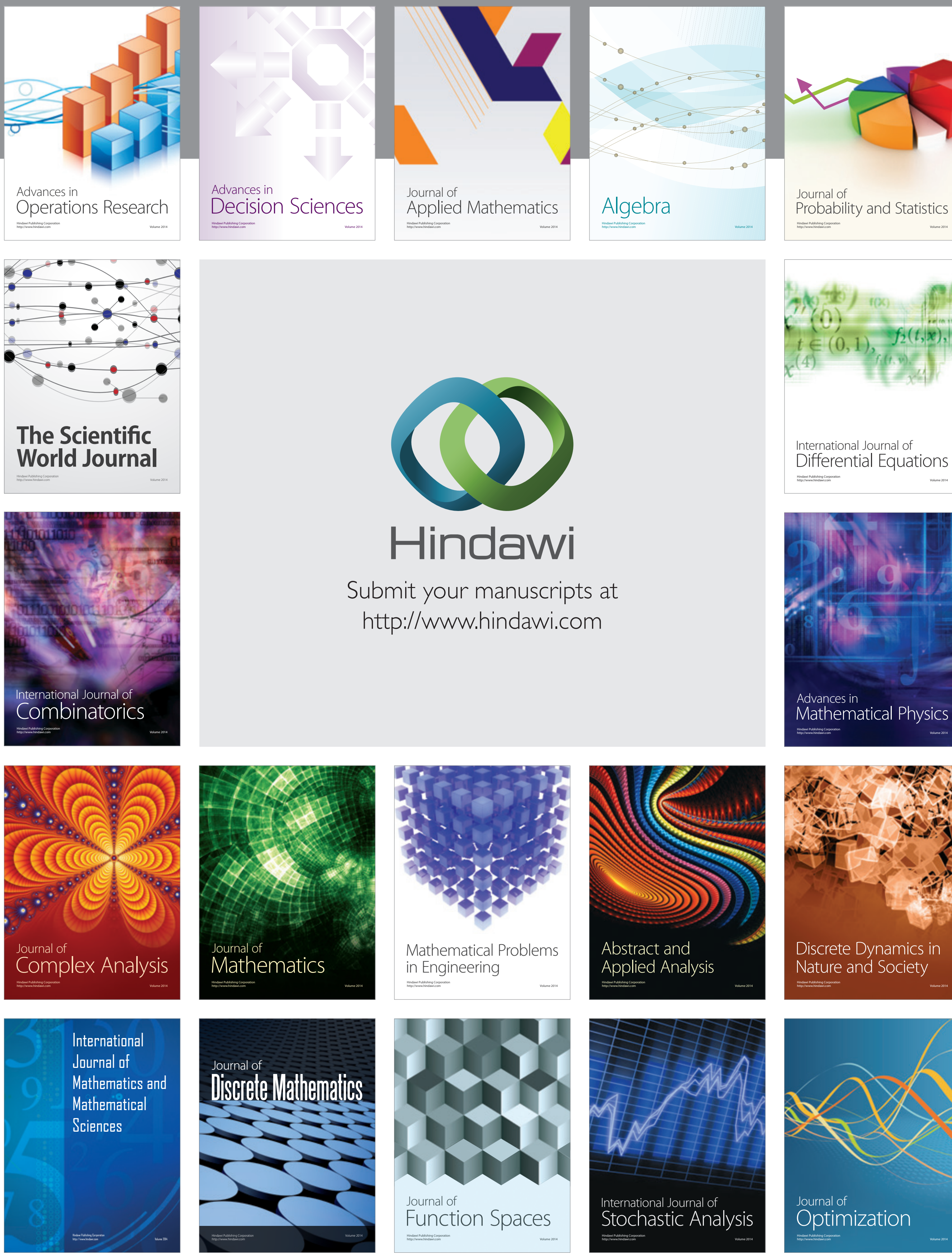
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