Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 216283, 10 pages
http://dx.doi.org/10.1155/2014/216283

Research Article

Unified Finite Horizon H_∞ Fusion Filtering for Networked Dynamical System

Chenglin Wen,[1,2] Xiaoliang Feng,[1] and Jingjing Yan[1]

[1] *College of Electrical Engineering, Henan University of Technology, Zhengzhou 450001, China*
[2] *College of Automatic, Hangzhou Dianzi University, Hangzhou 310018, China*

Correspondence should be addressed to Xiaoliang Feng; fengxl2002@163.com

Received 5 February 2014; Accepted 11 March 2014; Published 23 April 2014

Academic Editor: Guanghui Wen

This paper addresses the H_∞ fusion filtering problem for networked dynamical systems, where measurements may arrive at fusion center in four different scenes and the fusion center could receive none, one, or multiple measurements in a fusion period. A unified H_∞ performance criterion function, which is suitable for different measurement arrival scenes, is designed for the filtering process of networked dynamical systems. Then, the H_∞ performance criterion function is described as an indefinite quadratic inequality and solved by a novel noise projection method in Krein space. On this basis, a unified finite horizon H_∞ filtering method is proposed for networked dynamical systems. Simulation results are provided to illustrate the correctness and the effectiveness of the theoretical analysis.

1. Introduction

The filtering methods are widely utilized in the fields of signal processing and automatic control for dynamical systems. With the development of computer and information technology, researchers begin to pay more and more attention on networked dynamical systems, such as the open channel networks and networked control systems [1–3]. However, it is inevitable that the measurement data is transmitted in the networked dynamical systems with different time delay.

In networked dynamical systems, the targets of interest are (remotely) observed by various sensors. The sampled measurements may arrive at information processing centers (especially refer to fusion filters, as in this paper) in different scenes through the transport network. Scene 1: the measurement arrives at the fusion filter in time, which is abbreviated to "ITM" in this paper. Scene 2: the measurement which arrives at the fusion filter with some time delay, but still in the sampled sequence, is abbreviated to "ISDM." Scene 3: the delay measurement arrives at the fusion filter out of the sampled sequence, which is abbreviated to "OOSM." Scene 4: the sampled measurement is missing in the transmitting process, which is also named as "packet dropout" (abbreviated to "PD"). The traditional filters are mainly proposed for systems with all measurements in Scene 1, such as Kalman filter, H_∞ filter. For the system with measurements arriving at fusion filter in other scenes, several effective filtering methods have also been proposed, recently.

(1) For systems with measurement in Scene 2, some novel filtering methods are proposed based on Kalman filter [4, 5]. And the developed H_∞ filtering approaches are also deduced for this kind of systems with bounded energy noises [6, 7].

(2) For systems with measurement in Scene 3, several OOSM filtering problems are investigated with the help of such technologies as nonstandard smoothing [8], Kalman filter with measurement weighted summation [9, 10], and reorganized innovation [11, 12].

(3) For systems with measurement in Scene 4, several filtering approaches are developed based on the traditional Kalman filter [13–15] or H_∞ filter [16], based on different descriptions of the packet dropout phenomenon, such as the Markovian jump approach [13] and the binary Bernoulli distribution approach [14–16].

Although some recent approaches have considered the systems in multiple measurement arrival scenes [17, 18], most results of them are deduced for the system with single sensor. Few papers address the filtering problems for the networked dynamical system with multiple sensors. In [19, 20], the fusion filtering methods for networked multisensor systems are deduced based on Kalman filter, which requires the system noise to satisfy zero mean Gaussian distribution with known variance, which, however, is usually not available. To the best of the author's knowledge, the filtering problem for the networked multisensor system with unknown statistic noises has not been fully investigated and still remains challenging.

Motivated by the above discussion, a unified finite horizon H_∞ filtering method is proposed for the networked dynamical system in this paper in which four different kinds of measurement arrival scenes are dealt with in a unified manner. Because of the complex arrival scenes of networked measurements, the fusion filter for the networked dynamical system could receive none, one, or multiple measurements in a fusion period. The H_∞ filtering algorithm should be deduced to achieve a H_∞ performance criterion function. In the traditional H_∞ performance criterion function, an ideal assumption condition is that the measurement sampling time is the same as the measurement arrival time and the filtering time. However, in networked dynamical systems, the arrival time of a networked measurement mostly is not equal to its sampling time. And the fusion filter deals with the sampled measurement at its arrival time, rather than its sampling time. It means that the traditional H_∞ performance criterion function cannot be applied to networked dynamical systems. In this paper, a novel unified H_∞ performance criterion function is built for the different measurement arrival scenes in networked dynamical systems, firstly. Secondly, the H_∞ performance criterion function is described as an indefinite quadratic inequality. The stationary point of indefinite quadratic form in Hilbert space corresponds to a projection in Krein space. In this paper, the stationary point of the indefinite quadratic inequality is obtained by solving a projection in Krein space. However, because of the random delay of networked measurements, the process of solving the projection with the delay measurement becomes more complex. Thirdly, a noise projection approach in Krein space is proposed to solve the projection corresponding to the stationary point. Then, a unified H_∞ filtering method is proposed for the networked dynamical system. Finally, the validity and effectiveness of the proposed method are verified in the final simulation.

The remainder of this paper is organized as follows. The problem of the fusion filtering for the networked dynamical system is formulated in Section 2. In Section 3, a unified finite horizon H_∞ filter is deduced based on a novel performance criterion function for the networked dynamical system in various measurement arrival scenes. An example for illustration is given in Section 4, and we conclude this paper in Section 5.

Notation. The elements in Hilbert space are denoted by bold face letters, such as "$\mathbf{x}, \hat{\mathbf{x}}$," and the elements in Krein space are denoted by the bold face letters with bar, such as "$\bar{\mathbf{x}}, \hat{\bar{\mathbf{x}}}$." The superscripts "^{T}" and "$^{-1}$" mean the transposed matrix and inverse matrix, respectively. $\theta(k) \in l_2[1, N)$ is the Euclidean norm; that is, $\sum_{k=1}^{N} \theta^T(k)\theta(k) < \infty$.

2. Problem Formulation

Consider the following discrete networked dynamical system, which is observed by N sensors

$$\begin{aligned}\mathbf{x}(k) &= \mathbf{F}(k, k-1)\,\mathbf{x}(k-1) + \mathbf{w}(k, k-1),\\ \mathbf{y}_l(k) &= \mathbf{H}_l(k)\,\mathbf{x}(k) + \mathbf{v}_l(k), \quad l = 1, 2, \dots, N,\\ \mathbf{z}(k) &= \mathbf{L}(k)\,\mathbf{x}(k),\end{aligned} \tag{1}$$

where $\mathbf{x}(k) \in \mathbf{R}^n$ is the state vector. $\mathbf{y}_l(k) \in \mathbf{R}^q$ is the measurement output of sensor l. $\mathbf{z}(k)$ is the signal to be estimated. $\mathbf{F}(k, k-1)$, $\mathbf{H}_l(k)$, and $\mathbf{L}(k)$ are the system matrices with compatible dimensions. $\mathbf{w}(k, k-1) \in l_2[1, N)$ is the process noise and $\mathbf{v}_l(k) \in l_2[1, N)$ is the corresponding measurement noise of sensor l.

According to the traditional finite horizon H_∞ filter for single-sensor system, for a given scalar $\gamma > 0$, $\hat{\mathbf{z}}(k \mid k)$ can be obtained as an approximation of $\mathbf{z}(k)$ based on the received measurements $\{\mathbf{y}(i) \mid i = 1, 2, \dots k\}$ to guarantee the following H_∞ performance criterion function:

$$\sup_{\mathbf{w},\mathbf{v}\in l_2[1,N)} \left(\left(\sum_{i=1}^{k} \mathbf{e}^T(i)\,\mathbf{e}(i) \right) \times \left(\sum_{i=1}^{k} \mathbf{v}^T(i)\,\mathbf{v}(i) + \sum_{i=1}^{k} \mathbf{w}^T(i, i-1)\,\mathbf{w}(i, i-1) + \tilde{\mathbf{x}}_0^T \mathbf{P}_0^{-1} \tilde{\mathbf{x}}_0 \right)^{-1} \right) < \gamma^2, \tag{2}$$

where $\mathbf{e}_z(k) = \mathbf{z}(k) - \hat{\mathbf{z}}(k \mid k)$, $\tilde{\mathbf{x}}_0 = \mathbf{x}(0) - \hat{\mathbf{x}}_0$, and $\hat{\mathbf{x}}_0$ is an initial estimate of $\mathbf{x}(0)$. $\mathbf{P}_0$ is a given positive definite matrix with compatible dimension.

Remark 1. In the above performance criterion function, the filtering time of the fusion filter is the same as the sampling time of measurements, which, however, is not this case in networked dynamical systems.

In a fusion period, the networked measurement could arrive at the fusion filter in four different scenes considered in this paper; namely, the measurement could be ITM, ISDM, or OOSM. The unified filter for networked dynamical systems would deal with various measurement arrival scenes. The filtering time of networked measurement is its arrival time, rather than its sampling time. This means that the performance criterion function shown in (2) cannot be directly extended to the unified filtering process of networked dynamical systems. In the next section, a novel performance criterion function is built for various measurement arrival scenes in the networked dynamical system, firstly. On this basis, a unified finite horizon H_∞ fusion filtering method is deduced.

3. Unified Finite Horizon H_∞ Fusion Filtering for Networked Dynamical System

3.1. Performance Criterion Function. Let $\kappa(i)$ be a counter, which counts for the number of the measurements received by the fusion filter in the fusion period $[i, i+1)$. At the start of the period, $\kappa(i) = 0$. Whenever a measurement arrives at the fusion filter, $\kappa(i) = \kappa(i) + 1$. Denote the jth measurement received by the fusion filter in the period $[i, i+1)$ as $\mathbf{y}_{\alpha_j^i}(\beta_j^i)$, in which the notations j, α_j^i, β_j^i mean that $\mathbf{y}_{\alpha_j^i}(\beta_j^i)$ is sampled by sensor α_j^i at the sampled time β_j^i. Here $\beta_j^i \le i$, $\alpha_j^i \le N$, and $j \le iN$ are all positive integers.

According to the measurement arrival scenes, for a given $\gamma > 0$, a novel finite horizon H_∞ fusion filtering performance criterion function could be given as follows in the fusion period $[k, k+1)$ to obtain $\hat{\mathbf{z}}(k \mid k)$ based on the received measurement space $\{\mathbf{y}_{\alpha_j^i}(\beta_j^i) \mid 1 \le i \le k$ and $\kappa(i) \ne 0$, $j = 1, \ldots, \kappa(i)\}$. Consider

$$\sup_{\mathbf{w},\mathbf{v}\in l_2[1,N)} \left(\left(\sum_{\substack{i=1\\ \kappa(i)\ne 0}}^{k} \sum_{j=1}^{\kappa(k)} \mathbf{e}_{z,j}^T(i)\, \mathbf{e}_{z,j}(i) \right) \times \left(\sum_{\substack{i=1\\ \kappa(i)\ne 0}}^{k} \sum_{j=1}^{\kappa(k)} \mathbf{v}_{\alpha_j^i}^T\left(\beta_j^i\right) \mathbf{v}_{\alpha_j^i}\left(\beta_j^i\right) + \sum_{i=1}^{k} \mathbf{w}^T(i, i-1)\, \mathbf{w}(i, i-1) + \tilde{\mathbf{x}}_0^T \mathbf{P}_0^{-1} \tilde{\mathbf{x}}_0 \right)^{-1} \right) < \gamma^2, \tag{3}$$

where $\mathbf{e}_{z,j}(i) = \hat{\mathbf{z}}_j(i \mid i) - \mathbf{z}(i)$. $\hat{\mathbf{z}}_j(i \mid i)$ is the jth estimate of $\mathbf{z}(i)$, updated with the measurement $\mathbf{y}_{\alpha_j^i}(\beta_j^i)$.

Remark 2. The performance criterion function shown in (3) can be utilized for the finite horizon H_∞ fusion filtering processes of the dynamical system with measurements which could arrive at fusion filter in the aforementioned four kinds of arrival scenes.

(1) When none measurement arrives at the fusion filter in the period $[k, k+1)$, $\kappa(k) = 0$.

(2) When a measurement firstly arrives at the fusion filter in $[k, k+1)$, $\kappa(k) = 1$. If $\beta_1^k = k$, the measurement is an ITM $\mathbf{y}_{\alpha_1^k}(k)$. If $\beta_1^k < k$, this measurement is a delay measurement (an ISDM or a OOSM) sampled at β_1^k by sensor α_1^k.

(3) When a second (third,. . ., etc.) measurement arrives in $[k, k+1)$, then let $\kappa(k) = \kappa(k) + 1$. The new measurement would also be an ITM or a delay measurement.

3.2. Unified Finite Horizon H_∞ Filter Design. The performance criterion function shown in (3) can also be described as the following indefinite quadratic inequality:

$$J_{\kappa(k)} = \sum_{\substack{i=1\\ \kappa(i)\ne 0}}^{k} \sum_{j=1}^{\kappa(k)} \left(\mathbf{v}_{\alpha_j^i}^T\left(\beta_j^i\right) \mathbf{v}_{\alpha_j^i}\left(\beta_j^i\right) - \gamma^{-2} \mathbf{e}_{z,j}^T(i)\, \mathbf{e}_{z,j}(i) \right) + \sum_{i=1}^{k} \mathbf{w}^T(i, i-1)\, \mathbf{w}(i, i-1) + \tilde{\mathbf{x}}_0^T \mathbf{P}_0^{-1} \tilde{\mathbf{x}}_0 > 0. \tag{4}$$

Remark 3. The quadratic form $J_{\kappa(k)}$ satisfies the indefinite quadratic inequality above if and only if (1) $J_{\kappa(k)}$ has a stationary point, (2) the value of $J_{\kappa(k)}$ at the stationary point is a minimum, and (3) the minimum is positive.

The stationary point of an indefinite quadratic form in Hilbert space corresponds to a projection in Krein space which is solved to obtain the stationary point of $J_{\kappa(k)}$ in this paper. A Krein space state-space model associated with the system shown in (1) is introduced as

$$\begin{gathered} \bar{\mathbf{x}}(i) = \mathbf{F}(i, i-1)\, \bar{\mathbf{x}}(i-1) + \bar{\mathbf{w}}(i, i-1), \quad i = 1, \ldots, k, \\ \bar{\mathbf{y}}_{\alpha_j^i}\left(\beta_j^i\right) = \mathbf{H}_{\alpha_j^i}\left(\beta_j^i\right) \bar{\mathbf{x}}\left(\beta_j^i\right) + \bar{\mathbf{v}}_{y,\alpha_j^i}\left(\beta_j^i\right), \\ 1 \le i \le k, \quad \kappa(i) \ne 0, \quad 0 < j \le \kappa(i), \\ \breve{\bar{\mathbf{z}}}(i) = \mathbf{L}(i)\, \bar{\mathbf{x}}(i) + \bar{\mathbf{v}}_{z,j}(i), \\ 1 \le i \le k, \quad \kappa(i) \ne 0, \quad 0 < j \le \kappa(i), \end{gathered} \tag{5}$$

with

$$\left\langle \begin{bmatrix} \bar{\mathbf{x}}(0) - \hat{\bar{\mathbf{x}}}_0 \\ \bar{\mathbf{w}}(j_1, j_1 - 1) \\ \bar{\mathbf{v}}_{y,j_3}\left(\beta_{j_3}^{l_1}\right) \\ \bar{\mathbf{v}}_{z,j_3}(l_1) \end{bmatrix}, \begin{bmatrix} \bar{\mathbf{x}}(0) - \hat{\bar{\mathbf{x}}}_0 \\ \bar{\mathbf{w}}(j_2, j_2 - 1) \\ \bar{\mathbf{v}}_{y,j_4}\left(\beta_{j_4}^{l_2}\right) \\ \bar{\mathbf{v}}_{z,j_4}(l_2) \end{bmatrix} \right\rangle = \begin{bmatrix} \mathbf{P}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}\delta_{j_1,j_2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}\delta_{j_3,j_4}\delta_{l_1,l_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\gamma^2 \mathbf{I}\delta_{j_3,j_4}\delta_{l_1,l_2} \end{bmatrix}, \tag{6}$$

$$1 \le j_1, \quad j_2 \le k,$$
$$1 \le j_3, \quad j_4 \le k, \quad \kappa(j_3) > 1,$$
$$0 < l_1 \le \kappa(j_3), \quad 0 < l_2 \le \kappa(j_4).$$

Denote $\overline{\mathbf{W}}(k) := [\bar{\mathbf{w}}^T(1,0), \ldots, \bar{\mathbf{w}}^T(k, k-1)]^T$, $\bar{\xi}(k) := [\bar{\mathbf{x}}^T(0), \overline{\mathbf{W}}^T(k)]^T$. The stationary point of the indefinite quadratic form shown in (4) corresponds to the projection of $\bar{\xi}(k)$ into the Krein subspace $\Gamma_{\kappa(k)}(k)$ spanned by $\{\bar{\mathbf{y}}_{\alpha_j^i}(\beta_j^i), \breve{\bar{\mathbf{z}}}_j(i) \mid 1 \le i \le k$ and $\kappa(i) \ne 0$, $j = 1, \ldots, \kappa(i)\}$.

According to the Krein space state equation shown in (5), we have $\bar{\mathbf{x}}(i-1) = \mathbf{F}(i-1,i)(\bar{\mathbf{x}}(i) - \bar{\mathbf{w}}(i,i-1))$, and thus

$$\begin{aligned}\bar{\mathbf{y}}_{\alpha_j^i}\left(\beta_j^i\right) &= \mathbf{H}_{\alpha_j^i}\left(\beta_j^i\right)\bar{\mathbf{x}}\left(\beta_j^i\right) + \bar{\mathbf{v}}_{y,\alpha_j^i}\left(\beta_j^i\right)\\ &= \mathbf{H}_{\alpha_j^i}\left(\beta_j^i\right)\mathbf{F}\left(\beta_j^i,i\right)\bar{\mathbf{x}}(i) + \bar{\mathbf{v}}_{y,\alpha_j^i}\left(\beta_j^i\right)\\ &\quad - \mathbf{H}_{\alpha_j^i}\left(\beta_j^i\right)\mathbf{F}\left(\beta_j^i,i\right)\bar{\mathbf{w}}\left(i,\beta_j^i\right).\end{aligned} \tag{7}$$

Let $\bar{\mathbf{y}}_j^*(i) = \bar{\mathbf{y}}_{\alpha_j^i}(\beta_j^i)$; $\mathbf{H}_j^*(i) = \mathbf{H}_{\alpha_j^i}(\beta_j^i)\mathbf{F}(\beta_j^i,i)$; $\bar{\mathbf{v}}_{y,j}^*(i) = \bar{\mathbf{v}}_{y,\alpha_j^i}(\beta_j^i) - \mathbf{H}_{\alpha_j^i}(\beta_j^i)\mathbf{F}(\beta_j^i,i)\bar{\mathbf{w}}(i,\beta_j^i)$; we have

$$\bar{\mathbf{y}}_j^*(i) = \mathbf{H}_j^*(i)\bar{\mathbf{x}}(i) + \bar{\mathbf{v}}_{y,j}^*(i), \tag{8}$$

with the Gramian matrixes

$$\begin{aligned}&\left\langle \bar{\mathbf{w}}(l,l-1), \bar{\mathbf{v}}_{y,j}^*(i)\right\rangle\\ &\quad = \mathbf{Q}(l,l-1)\mathbf{F}^T(i,l+1)\left(\mathbf{H}_j^*(i)\right)^T,\\ &\qquad\qquad \beta_j^i \le l \le i-1,\\ &\mathbf{P}_{vyy,j}^*(i) = \left\langle \bar{\mathbf{v}}_{y,j}^*(i), \bar{\mathbf{v}}_{y,j}^*(i)\right\rangle\\ &\quad = \mathbf{I} + \mathbf{H}_j^*(i)\left(\sum_{l=\beta_j^i+1}^{i}\mathbf{F}(i,l)\mathbf{F}^T(i,l)\right)\left(\mathbf{H}_j^*(i)\right)^T,\\ &\qquad\qquad \beta_j^i < i,\end{aligned} \tag{9}$$

where $\mathbf{F}(i,i) = \mathbf{I}$.

Denote the Krein subspace spanned by $\{\bar{\mathbf{y}}_l(\beta_l^\tau), \breve{\bar{\mathbf{z}}}_l(\tau) \mid 1 \le \tau \le i-1 \text{ and } \kappa(\tau) \ne 0, l = 1,\dots,\kappa(\tau)\}$ and $\{\bar{\mathbf{y}}_l(\beta_l^i), \breve{\bar{\mathbf{z}}}_l(i) \mid l = 1,\dots,j-1\}$ by $\Gamma_{j-1}(i)$. The projection of $\bar{\mathbf{y}}_j^*(i)$ into $\Gamma_{j-1}(i)$ is given by

$$\hat{\bar{\mathbf{y}}}_j^*(i \mid i-1) = \mathbf{H}_j^*(i)\hat{\bar{\mathbf{x}}}_j(i \mid i-1) + \hat{\bar{\mathbf{v}}}_{y,j}^*(i \mid i-1), \tag{10}$$

in which

$$\begin{aligned}\hat{\bar{\mathbf{x}}}_j(i \mid i-1) &= \hat{\bar{\mathbf{x}}}_{j-1}(i \mid i), \quad j > 1,\\ \hat{\bar{\mathbf{x}}}_1(i \mid i-1) &= \mathbf{F}(i,i-1)\hat{\bar{\mathbf{x}}}_{\kappa(i-1)}(i-1 \mid i-1),\end{aligned} \tag{11}$$

$$\hat{\bar{\mathbf{v}}}_{y,j}^*(i \mid i-1) = -\mathbf{H}_j^*(i)\hat{\bar{\mathbf{w}}}\left(i,\beta_j^i \mid i-1\right), \tag{12}$$

$$\hat{\bar{\mathbf{z}}}_j(i \mid i-1) = \mathbf{L}(i)\hat{\bar{\mathbf{x}}}_j(i \mid i-1). \tag{13}$$

Let $\bar{\mathbf{e}}_{yz,j}^*(i) := \begin{bmatrix}\bar{\mathbf{y}}_j^*(i) - \hat{\bar{\mathbf{y}}}_j^*(i|i-1)\\ \breve{\bar{\mathbf{z}}}_j(i) - \hat{\bar{\mathbf{z}}}_j(i|i-1)\end{bmatrix} = \begin{bmatrix}\mathbf{H}_j^*(i)\\ \mathbf{L}(i)\end{bmatrix}\tilde{\bar{\mathbf{x}}}_j(i \mid i-1) + \begin{bmatrix}\tilde{\bar{\mathbf{v}}}_{y,j}^*(i|i-1)\\ \bar{\mathbf{v}}_{z,j}(i)\end{bmatrix}$, $\mathbf{R}_{wyz,l}(i,\beta_j^i,k) := \langle \bar{\mathbf{w}}(i,\beta_j^i), \bar{\mathbf{e}}_{yz,l}^*(k)\rangle$, $\mathbf{R}_{eyz,j}^*(i) := \langle \bar{\mathbf{e}}_{yz,j}^*(i), \bar{\mathbf{e}}_{yz,j}^*(i)\rangle$. It is obvious that $\bar{\mathbf{e}}_{yz,j-1}^*(i \mid i-1) \perp \Gamma_{j-1}(i)$, and $\{\bar{\mathbf{e}}_{yz,j}^*(i \mid i-1) \mid 1 \le i \le k \text{ and } \kappa(i) \ne 0, j = 1,\dots,\kappa(i)\}$ is an orthogonal basis of $\Gamma_{\kappa(k)}(k)$. The projection of $\bar{\xi}(k)$ into $\Gamma_{\kappa(k)}(k)$ is given by

$$\hat{\bar{\xi}}_{\kappa(k)}(k \mid k) = \sum_{\substack{i=1,\\ \kappa(i)\ne 0}}^{k}\sum_{j=1}^{\kappa(i)}\left\langle \bar{\xi}(k), \bar{\mathbf{e}}_{yz,j}^*(i)\right\rangle\left(\mathbf{R}_{eyz,j}^*(i)\right)^{-1}\bar{\mathbf{e}}_{yz,j}^*(i). \tag{14}$$

The projection of $\bar{\mathbf{x}}(k)$ into $\Gamma_{\kappa(k)}(k)$ is

$$\begin{aligned}\hat{\bar{\mathbf{x}}}_{\kappa(k)}(k \mid k) &= \sum_{\substack{i=1,\\ \kappa(i)\ne 0}}^{k}\sum_{j=1}^{\kappa(i)}\left\langle \bar{\mathbf{x}}(k), \bar{\mathbf{e}}_{yz,j}^*(i)\right\rangle\left(\mathbf{R}_{eyz,j}^*(i)\right)^{-1}\bar{\mathbf{e}}_{yz,j}^*(i)\\ &= \sum_{\substack{i=1,\\ \kappa(i)\ne 0}}^{k}\sum_{j=1}^{\kappa(i)}\left\langle \bar{\mathbf{x}}(k), \bar{\mathbf{e}}_{yz,j}^*(i)\right\rangle\left(\mathbf{R}_{eyz,j}^*(i)\right)^{-1}\bar{\mathbf{e}}_{yz,j}^*(i)\\ &= \hat{\bar{\mathbf{x}}}_{\kappa(k)-1}(k \mid k) + \left\langle \bar{\mathbf{x}}(k), \bar{\mathbf{e}}_{yz,\kappa(k)}^*(k)\right\rangle\\ &\quad \times \left(\mathbf{R}_{eyz,\kappa(k)}^*(k)\right)^{-1}\bar{\mathbf{e}}_{yz,\kappa(k)}^*(k)\\ &= \hat{\bar{\mathbf{x}}}_{\kappa(k)}(k \mid k-1) + \left\langle \bar{\mathbf{x}}(k), \bar{\mathbf{e}}_{yz,\kappa(k)}^*(k)\right\rangle\\ &\quad \times \left(\mathbf{R}_{eyz,\kappa(k)}^*(k)\right)^{-1}\bar{\mathbf{e}}_{yz,\kappa(k)}^*(k)\\ \hat{\bar{\mathbf{x}}}_0(k \mid k) &= \mathbf{F}(k,k-1)\hat{\bar{\mathbf{x}}}_{\kappa(k-1)}(k-1 \mid k-1),\end{aligned} \tag{15}$$

and the noise projection $\hat{\bar{\mathbf{w}}}(i,\beta_j^i \mid i-1)$ in (12) is given by

$$\begin{aligned}\hat{\bar{\mathbf{w}}}_j\left(i,\beta_j^i \mid i-1\right) &= \hat{\bar{\mathbf{w}}}_{j-1}\left(i,\beta_j^i, i-1\right)\\ &\quad + \mathbf{R}_{wyz,j-1}\left(i,\beta_j^i,i\right)\left(\mathbf{R}_{eyz,j-1}^*(i)\right)^{-1}\\ &\quad \times \bar{\mathbf{e}}_{yz,j-1}^*(i),\\ \hat{\bar{\mathbf{w}}}_1\left(i,\beta_j^i \mid i-1\right) &= \hat{\bar{\mathbf{w}}}_{\kappa(i-1)}\left(i,\beta_j^i \mid i-2\right)\\ &\quad + \mathbf{R}_{wyz,\kappa(i-1)}\left(i,\beta_j^i,i-1\right)\\ &\quad \times \left(\mathbf{R}_{eyz,\kappa(i-1)}^*(i-1)\right)^{-1}\\ &\quad \times \bar{\mathbf{e}}_{yz,\kappa(i-1)}^*(i-1),\\ \hat{\bar{\mathbf{w}}}_{\kappa(i-1)}\left(i,\beta_j^i \mid i-2\right) &= \mathbf{F}(i,i-1)\hat{\bar{\mathbf{w}}}_{\kappa(i-1)}\left(i-1,\beta_j^i \mid i-2\right),\end{aligned} \tag{16}$$

where

$$\mathbf{K}_{\kappa(k)}(k) := \left\langle \overline{\mathbf{x}}(k), \overline{\mathbf{e}}^{*}_{yz,\kappa(k)}(k) \right\rangle \left(\mathbf{R}^{*}_{eyz,\kappa(k)}(k)\right)^{-1} := \mathbf{R}_{xyz,\kappa(k)}(k)\left(\mathbf{R}^{*}_{eyz,\kappa(k)}(k)\right)^{-1}, \tag{17}$$

$$\begin{aligned}
&\mathbf{R}_{xyz,\kappa(k)}(k) \\
&= \left\langle \overline{\mathbf{x}}(k), \overline{\mathbf{e}}^{*}_{yz,\kappa(k)}(k) \right\rangle \\
&= \left\langle \overline{\mathbf{x}}(k), \tilde{\overline{\mathbf{x}}}_{\kappa(k)}(k \mid k-1) \right\rangle \left[\left(\mathbf{H}^{*}_{\kappa(k)}(k)\right)^{T} \quad \mathbf{L}^{T}(k) \right] \\
&\quad - \left[\left\langle \overline{\mathbf{x}}(k), \tilde{\overline{\mathbf{w}}}_{\kappa(k)}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right) \right\rangle \left(\mathbf{H}^{*}_{\kappa(k)}(k)\right)^{T} \quad \mathbf{0} \right] \\
&:= \mathbf{P}_{\kappa(k)}(k) \left[\left(\mathbf{H}^{*}_{\kappa(k)}(k)\right)^{T} \quad \mathbf{L}^{T}(k) \right] \\
&\quad - \left[\mathbf{R}_{xw,\kappa(k)}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right) \left(\mathbf{H}^{*}_{\kappa(k)}(k)\right)^{T} \quad \mathbf{0} \right] \\
&:= \left[\mathbf{R}_{xy,\kappa(k)}(k) \quad \mathbf{R}_{xz,\kappa(k)}(k) \right],
\end{aligned}$$

$$\begin{aligned}
&\mathbf{R}^{*}_{eyz,\kappa(k)}(k) \\
&:= \begin{bmatrix} \mathbf{H}^{*}_{\kappa(k)}(k) \\ \mathbf{L}(k) \end{bmatrix} \mathbf{P}_{\kappa(k)}(k) \left[\left(\mathbf{H}^{*}_{\kappa(k)}(k)\right)^{T} \quad \mathbf{L}^{T}(k) \right] + \mathbf{R}^{*}_{vyz,\kappa(k)}(k) \\
&\quad - \begin{bmatrix} \mathbf{H}^{*}_{\kappa(k)}(k)\mathbf{R}^{T}_{xw,\kappa(k)}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right)\left(\mathbf{H}^{*}_{\kappa(k)}(k)\right)^{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\
&\quad - \begin{bmatrix} \mathbf{H}^{*}_{\kappa(k)}(k)\mathbf{R}^{T}_{xw,\kappa(k)}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right)\left(\mathbf{H}^{*}_{\kappa(k)}(k)\right)^{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}^{T},
\end{aligned}$$

$$\mathbf{R}^{*}_{vyz,\kappa(k)}(k) = \begin{bmatrix} \mathbf{R}^{*}_{vyy,\kappa(k)}(k) & \mathbf{0} \\ \mathbf{0} & -\gamma^{2}\mathbf{I} \end{bmatrix}$$

$$\begin{aligned}
&\mathbf{R}_{xw,\kappa(k)}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right) \\
&= \left\langle \overline{\mathbf{x}}(k), \tilde{\overline{\mathbf{w}}}_{\kappa(k)}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right) \right\rangle \\
&= \left\langle \overline{\mathbf{x}}(k), \overline{\mathbf{w}}\left(k, \beta^{k}_{\kappa(k)}\right) \right\rangle \\
&\quad - \left\langle \overline{\mathbf{x}}(k), \hat{\overline{\mathbf{w}}}_{\kappa(k)}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right) \right\rangle \\
&:= \mathbf{Q}\left(k, \beta^{k}_{\kappa(k)}\right) \\
&\quad - \mathbf{P}_{xw,\kappa(k)}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right),
\end{aligned}$$

$$\begin{aligned}
&\mathbf{R}^{*}_{vyy,\kappa(k)}(k) \\
&= \left\langle \tilde{\overline{\mathbf{v}}}^{*}_{y,\kappa(k)-1}(k \mid k-1), \tilde{\overline{\mathbf{v}}}^{*}_{y,\kappa(k)-1}(k \mid k-1) \right\rangle \\
&= \left\langle \overline{\mathbf{v}}^{*}_{y,\kappa(k)-1}(k \mid k-1), \overline{\mathbf{v}}^{*}_{y,\kappa(k)-1}(k \mid k-1) \right\rangle \\
&\quad - \left\langle \tilde{\overline{\mathbf{v}}}^{*}_{y,\kappa(k)-1}(k \mid k-1), \hat{\overline{\mathbf{v}}}^{*}_{y,\kappa(k)-1}(k \mid k-1) \right\rangle \\
&= \mathbf{P}^{*}_{vyy,\kappa(k)}(k) - \mathbf{H}^{*}_{\kappa(k)}(k)\mathbf{R}^{T}_{ww,\kappa(k)}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right) \\
&\quad \times \left(\mathbf{H}^{*}_{\kappa(k)}(k)\right)^{T},
\end{aligned}$$

$$\begin{aligned}
&\mathbf{P}_{xw,\kappa(k)}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right) \\
&= \left\langle \overline{\mathbf{x}}(k), \hat{\overline{\mathbf{w}}}_{\kappa(k)}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right) \right\rangle \\
&= \left\langle \overline{\mathbf{x}}(k), \hat{\overline{\mathbf{w}}}_{\kappa(k)-1}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right) \right\rangle \\
&\quad + \left\langle \overline{\mathbf{x}}(k), \overline{\mathbf{e}}^{*}_{yz,\kappa(k)-1}(k) \right\rangle \left(\mathbf{R}^{*}_{eyz,\kappa(k)-1}(k)\right)^{-1} \\
&\quad \times \mathbf{R}^{T}_{wyz,\kappa(k)-1}\left(k, \beta^{k}_{\kappa(k)}, k\right) \\
&= \mathbf{P}_{xw,\kappa(k)-1}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right) + \mathbf{R}_{xyz,\kappa(k)-1}(k) \\
&\quad \times \left(\mathbf{R}^{*}_{eyz,\kappa(k)-1}(k)\right)^{-1}\mathbf{R}^{T}_{wyz,\kappa(k)-1}\left(k, \beta^{k}_{\kappa(k)}, k\right), \quad \kappa(k) > 1,
\end{aligned}$$

$$\begin{aligned}
&\mathbf{P}_{xw,1}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right) \\
&= \left\langle \overline{\mathbf{x}}(k), \hat{\overline{\mathbf{w}}}_{1}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right) \right\rangle \\
&= \mathbf{F}(k, k-1) \left\langle \overline{\mathbf{x}}(k-1), \hat{\overline{\mathbf{w}}}_{\kappa(k-1)}\left(k-1, \beta^{k}_{\kappa(k)} \mid k-2\right) \right\rangle \\
&\quad \times \mathbf{F}^{T}(k, k-1) + \left\langle \overline{\mathbf{x}}(k), \overline{\mathbf{e}}^{*}_{yz,\kappa(k-1)}(k-1) \right\rangle \\
&\quad \times \left(\mathbf{R}^{*}_{eyz,\kappa(k-1)}(k-1)\right)^{-1}\mathbf{R}^{T}_{wyz,\kappa(k-1)}\left(k, \beta^{k}_{\kappa(k)}, k-1\right) \\
&= \mathbf{F}(k, k-1)\mathbf{P}_{xw,\kappa(k-1)}\left(k-1, \beta^{k}_{\kappa(k)} \mid k-2\right)\mathbf{F}^{T}(k, k-1) \\
&\quad + \mathbf{R}_{xyz,\kappa(k-1)}(k-1)\left(\mathbf{R}^{*}_{eyz,\kappa(k-1)}(k-1)\right)^{-1} \\
&\quad \times \mathbf{R}^{T}_{wyz,\kappa(k-1)}\left(k, \beta^{k}_{\kappa(k)}, k-1\right),
\end{aligned}$$

$$\begin{aligned}
&\mathbf{R}_{wyz,\kappa(k)-1}\left(k, \beta^{k}_{\kappa(k)}, k\right) \\
&= \left\langle \overline{\mathbf{w}}\left(k, \beta^{k}_{\kappa(k)}\right), \overline{\mathbf{e}}^{*}_{yz,\kappa(k)-1}(k) \right\rangle \\
&= \left\langle \overline{\mathbf{w}}\left(k, \beta^{k}_{\kappa(k)}\right), \begin{bmatrix} \mathbf{H}^{*}_{\kappa(k)-1}(k) \\ \mathbf{L}(k) \end{bmatrix} \tilde{\overline{\mathbf{x}}}_{\kappa(k)-1}(k \mid k-1) \right. \\
&\qquad \left. + \begin{bmatrix} \tilde{\overline{\mathbf{v}}}^{*}_{y,\kappa(k)-1}(k \mid k-1) \\ \overline{\mathbf{v}}_{z,\kappa(k)-1}(k) \end{bmatrix} \right\rangle \\
&= \mathbf{R}^{T}_{xw,\kappa(k)-1}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right) \begin{bmatrix} \mathbf{H}^{*}_{\kappa(k)-1}(k) \\ \mathbf{L}(k) \end{bmatrix}^{T} \\
&\quad - \begin{bmatrix} \mathbf{R}^{T}_{ww,\kappa(k)-1}\left(k, \beta^{k}_{\kappa(k)} k-1\right)\left(\mathbf{H}^{*}_{\kappa(k)-1}(k)\right)^{T} \\ \mathbf{0} \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
&\mathbf{R}_{ww,\kappa(k)-1}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right) \\
&:= \left\langle \overline{\mathbf{w}}\left(k, \beta^{k}_{\kappa(k)}\right), \tilde{\overline{\mathbf{w}}}_{\kappa(k)-1}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right) \right\rangle \\
&= \left\langle \overline{\mathbf{w}}\left(k, \beta^{k}_{\kappa(k)}\right), \overline{\mathbf{w}}\left(k, \beta^{k}_{\kappa(k)}\right) \right\rangle \\
&\quad - \left\langle \overline{\mathbf{w}}\left(k, \beta^{k}_{\kappa(k)}\right), \hat{\overline{\mathbf{w}}}_{\kappa(k)-1}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right) \right\rangle \\
&= \mathbf{Q}\left(k, \beta^{k}_{\kappa(k)}\right) - \mathbf{P}_{ww,\kappa(k)-1}\left(k, \beta^{k}_{\kappa(k)} \mid k-1\right),
\end{aligned}$$

$$\begin{aligned}
&\mathbf{P}_{ww,\kappa(k)}\left(k,\beta_{\kappa(k)}^{k}\mid k-1\right)\\
&:=\left\langle\overline{\mathbf{w}}\left(k,\beta_{\kappa(k)}^{k}\right),\widehat{\overline{\mathbf{w}}}_{\kappa(k)}\left(k,\beta_{\kappa(k)}^{k}\mid k-1\right)\right\rangle\\
&=\left\langle\overline{\mathbf{w}}\left(k,\beta_{\kappa(k)}^{k}\right),\widehat{\overline{\mathbf{w}}}_{\kappa(k)-1}\left(k,\beta_{\kappa(k)}^{k}\mid k-1\right)\right\rangle\\
&\quad+\left\langle\overline{\mathbf{w}}\left(k,\beta_{\kappa(k)}^{k}\right),\overline{\mathbf{e}}_{yz,\kappa(k)-1}^{*}(k)\right\rangle\\
&\quad\times\left(\mathbf{R}_{eyz,\kappa(k)-1}^{*}(k)\right)^{-1}\mathbf{R}_{wyz,\kappa(k)-1}^{T}\left(k,\beta_{\kappa(k)}^{k},k\right)\\
&=\mathbf{P}_{ww,\kappa(k)-1}\left(k,\beta_{\kappa(k)}^{k}\mid k-1\right)+\mathbf{R}_{wyz,\kappa(k)-1}(k)\\
&\quad\times\left(\mathbf{R}_{eyz,\kappa(k)-1}^{*}(k)\right)^{-1}\mathbf{R}_{wyz,\kappa(k)-1}^{T}\left(k,\beta_{\kappa(k)}^{k},k\right),\quad\kappa(k)>1,\\
&\mathbf{P}_{ww,1}\left(k,\beta_{\kappa(k)}^{k}\mid k-1\right)\\
&=\left\langle\overline{\mathbf{w}}\left(k,\beta_{\kappa(k)}^{k}\right),\widehat{\overline{\mathbf{w}}}_{1}\left(k,\beta_{\kappa(k)}^{k}\mid k-1\right)\right\rangle\\
&=\left\langle\overline{\mathbf{w}}\left(k,\beta_{\kappa(k)}^{k}\right),\widehat{\overline{\mathbf{w}}}_{\kappa(k-1)}\left(k,\beta_{\kappa(k)}^{k}\mid k-2\right)\right\rangle\\
&\quad+\left\langle\overline{\mathbf{w}}\left(k,\beta_{\kappa(k)}^{k}\right),\overline{\mathbf{e}}_{yz,\kappa(k-1)}^{*}(k-1)\right\rangle\\
&\quad\times\left(\mathbf{R}_{eyz,\kappa(k-1)}^{*}(k-1)\right)^{-1}\mathbf{R}_{wyz,\kappa(k-1)}^{T}\left(k,\beta_{\kappa(k)}^{k},k-1\right)\\
&=\mathbf{F}(k,k-1)\,\mathbf{P}_{ww,\kappa(k-1)}\left(k-1,\beta_{\kappa(k)}^{k}\mid k-2\right)\mathbf{F}^{T}(k,k-1)\\
&\quad+\mathbf{R}_{wyz,\kappa(k-1)}\left(k,\beta_{\kappa(k)}^{k},k-1\right)\left(\mathbf{R}_{eyz,\kappa(k-1)}^{*}(k-1)\right)^{-1}\\
&\quad\times\mathbf{R}_{wyz,\kappa(k-1)}^{T}\left(k,\beta_{\kappa(k)}^{k},k-1\right),
\end{aligned}\tag{18}$$

and

$$\mathbf{Q}\left(k,\beta_{\kappa(k)}^{k}\right)=\begin{cases}\mathbf{0}, & \beta_{\kappa(k)}^{k}=k;\\ \displaystyle\sum_{i=\beta_{\kappa(k)}^{k}+1}^{k}\mathbf{F}(k,i)\,\mathbf{F}^{T}(k,i), & \beta_{\kappa(k)}^{k}<k.\end{cases}\tag{19}$$

The projection of $\overline{\xi}(k)$ in (14) corresponds to a stationary point of the indefinite quadratic form $J_{\kappa(k)}$ in (4), and the value of $J_{\kappa(k)}$ at this stationary point is

$$\begin{aligned}
\mathbf{J}_{\kappa(k)}^{*}(k)&=\sum_{\substack{i=1,\\ \kappa(i)\neq 0}}^{k}\sum_{j=1}^{\kappa(i)}\left(\mathbf{e}_{yz,j}^{*}(i)\right)^{T}\left(\mathbf{R}_{eyz,j}^{*}(i)\right)^{-1}\mathbf{e}_{yz,j}^{*}(i)\\
&=\mathbf{J}_{\kappa(k)-1}^{*}(k)+\left(\mathbf{e}_{yz,\kappa(k)}^{*}(k)\right)^{T}\left(\mathbf{R}_{eyz,\kappa(k)}^{*}(k)\right)^{-1}\\
&\quad\times\mathbf{e}_{yz,\kappa(k)}^{*}(k),
\end{aligned}\tag{20}$$

where $\mathbf{J}_{0}^{*}(k)=\mathbf{J}_{\kappa(k-1)}^{*}(k-1)$,

$$\begin{aligned}
\mathbf{e}_{yz,\kappa(k)}^{*}(k)&=\begin{bmatrix}\mathbf{e}_{y,\kappa(k)}^{*}(k\mid k-1)\\ \mathbf{e}_{z,\kappa(k)}(k\mid k-1)\end{bmatrix}\\
&=\begin{bmatrix}\mathbf{y}_{\alpha_{\kappa(k)}^{k}}\left(\beta_{\kappa(k)}^{k}\right)-\widehat{\mathbf{y}}_{\kappa(k)}^{*}(k\mid k-1)\\ \widehat{\mathbf{z}}_{\kappa(k)}(k\mid k)-\widehat{\mathbf{z}}_{\kappa(k)}(k\mid k-1)\end{bmatrix},\\
\widehat{\mathbf{y}}_{\kappa(k)}^{*}(k\mid k-1)&=\mathbf{H}_{\kappa(k)}^{*}(k)\,\widehat{\mathbf{x}}_{\kappa(k)}(k\mid k-1)\\
&\quad-\mathbf{H}_{\kappa(k)}^{*}(k)\,\widehat{\mathbf{w}}_{\kappa(k)}\left(k,\beta_{\kappa(k)}^{k}\mid k-1\right),\\
\widehat{\mathbf{z}}_{\kappa(k)}(k\mid k-1)&=\mathbf{L}(k)\,\widehat{\mathbf{x}}_{\kappa(k)}(k\mid k-1),\\
\widehat{\mathbf{w}}_{j}\left(i,\beta_{j}^{i}\mid i-1\right)&=\widehat{\mathbf{w}}_{j-1}\left(i,\beta_{j}^{i}\mid i-1\right)+\mathbf{R}_{wyz,j-1}\left(i,\beta_{j}^{k},i\right)\\
&\quad\times\left(\mathbf{R}_{eyz,j-1}^{*}(i)\right)^{-1}\mathbf{e}_{yz,j-1}^{*}(i),\quad j>1,\\
\widehat{\mathbf{w}}_{1}\left(i,\beta_{j}^{i}\mid i-1\right)&=\widehat{\mathbf{w}}_{\kappa(i-1)}\left(i,\beta_{j}^{i}\mid i-2\right)\\
&\quad+\mathbf{R}_{wyz,\kappa(i-1)}\left(i,\beta_{j}^{k},i-1\right)\\
&\quad\times\left(\mathbf{R}_{eyz,\kappa(i-1)}^{*}(i-1)\right)^{-1}\\
&\quad\times\mathbf{e}_{yz,\kappa(i-1)}^{*}(i-1),\\
\widehat{\mathbf{w}}_{\kappa(i-1)}\left(i,\beta_{j}^{i}\mid i-2\right)&=\mathbf{F}(i,i-1)\,\widehat{\mathbf{w}}_{\kappa(i-1)}\left(i-1,\beta_{j}^{i}\mid i-2\right),\\
\widehat{\mathbf{x}}_{\kappa(k)}(k\mid k-1)&=\widehat{\mathbf{x}}_{\kappa(k)-1}(k\mid k),\quad\kappa(k)>0,\\
\widehat{\mathbf{x}}_{0}(k\mid k-1)&=\mathbf{F}(k,k-1)\,\widehat{\mathbf{x}}_{\kappa(k-1)}(k-1\mid k-1).
\end{aligned}\tag{21}$$

In (20), $\mathbf{R}_{eyz,\kappa(k)}^{*}(k)$ can also be expressed as follows:

$$\mathbf{R}_{eyz,\kappa(k)}^{*}(k):=\begin{bmatrix}\mathbf{R}_{yy,\kappa(k)}(k) & \mathbf{R}_{yz,\kappa(k)}(k)\\ \mathbf{R}_{zy,\kappa(k)}(k) & \mathbf{R}_{zz,\kappa(k)}(k)\end{bmatrix};\tag{22}$$

then,

$$\begin{aligned}
&\left(\mathbf{R}_{eyz,\kappa(k)}^{*}(k)\right)^{-1}\\
&=\begin{bmatrix}\mathbf{R}_{yy,\kappa(k)}(k) & \mathbf{R}_{yz,\kappa(k)}(k)\\ \mathbf{R}_{zy,\kappa(k)}(k) & \mathbf{R}_{zz,\kappa(k)}(k)\end{bmatrix}^{-1}\\
&=\begin{bmatrix}\mathbf{I} & -\mathbf{R}_{yy,\kappa(k)}^{-1}(k)\,\mathbf{R}_{yz,\kappa(k)}(k)\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\\
&\quad\times\begin{bmatrix}\mathbf{R}_{yy,\kappa(k)}(k) & \mathbf{0}\\ \mathbf{0} & \mathbf{R}_{zz,\kappa(k)}(k)-\mathbf{R}_{zy,\kappa(k)}(k)\mathbf{R}_{yy,\kappa(k)}^{-1}(k)\mathbf{R}_{yz,\kappa(k)}(k)\end{bmatrix}^{-1}\\
&\quad\times\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{R}_{zy,\kappa(k)}(k)\,\mathbf{R}_{yy,\kappa(k)}^{-1}(k) & \mathbf{I}\end{bmatrix}.
\end{aligned}\tag{23}$$

Letting $\widehat{\mathbf{z}}^*_{\kappa(k)}(k \mid k) = \widehat{\mathbf{z}}_{\kappa(k)}(k \mid k-1) - \mathbf{R}_{zy,\kappa(k)}(k)\mathbf{R}^{-1}_{yy,\kappa(k)}(k)\mathbf{e}^*_{y,\kappa(k)}(k \mid k-1)$, the last term in (20) is

$$\begin{aligned}
&\mathbf{e}^*_{yz,\kappa(k)}(k)\left(\mathbf{R}^*_{eyz,\kappa(k)}(k)\right)^{-1}\mathbf{e}^*_{yz,\kappa(k)}(k) \\
&= \left(\mathbf{e}^*_{y,\kappa(k)}(k \mid k-1)\right)^T \mathbf{R}^{-1}_{yy,\kappa(k)}(k)\,\mathbf{e}^*_{y,\kappa(k)}(k \mid k-1) \\
&\quad + \left(\widehat{\mathbf{z}}_{\kappa(k)}(k \mid k) - \widehat{\mathbf{z}}^*_{\kappa(k)}(k \mid k)\right)^T \\
&\quad \times \left(\mathbf{R}_{zz,\kappa(k)}(k) - \mathbf{R}_{zy,\kappa(k)}(k)\,\mathbf{R}^{-1}_{yy,\kappa(k)}(k)\,\mathbf{R}_{yz,\kappa(k)}(k)\right)^{-1} \\
&\quad \times \left(\widehat{\mathbf{z}}_{\kappa(k)}(k \mid k) - \widehat{\mathbf{z}}^*_{\kappa(k)}(k \mid k)\right).
\end{aligned} \tag{24}$$

According to Lemma 12 in [21], $\mathbf{J}^*_{\kappa(k)}(k)$ is the minimum of $\mathbf{J}_{\kappa(k)}(k)$ if and only if $\mathbf{R}^*_{eyz,\kappa(k)}(k)$ and $\mathbf{R}^*_{vyz,\kappa(k)}(k)$ have the same inertia. Considering the block triangular factorization of $\mathbf{R}^*_{eyz,\kappa(k)}(k)$ as shown in (23), the minimum condition can also be given by

$$\mathbf{R}_{zz,\kappa(k)}(k) - \mathbf{R}_{zy,\kappa(k)}(k)\,\mathbf{R}^{-1}_{yy,\kappa(k)}(k)\,\mathbf{R}_{yz,\kappa(k)}(k) < \mathbf{0}. \tag{25}$$

Therefore, a choice of $\widehat{\mathbf{z}}_{\kappa(k)}(k \mid k)$ to guarantee $\mathbf{J}^*_{\kappa(k)}(k) > 0$ is $\widehat{\mathbf{z}}_{\kappa(k)}(k \mid k) = \widehat{\mathbf{z}}^*_{\kappa(k)}(k \mid k)$, and the minimum of $\mathbf{J}_{\kappa(k)}(k)$ is

$$\begin{aligned}
\mathbf{J}^*_{\kappa(k)}(k) &= \mathbf{J}^*_{\kappa(k)-1}(k) + \mathbf{e}^*_{y,\kappa(k)}(k \mid k-1) \\
&\quad \times \left(\mathbf{R}_{yy,\kappa(k)}(k)\right)^{-1}\mathbf{e}^*_{y,\kappa(k)}(k \mid k-1).
\end{aligned} \tag{26}$$

Then, the estimation of the signal to be estimated is

$$\widehat{\mathbf{z}}(k \mid k) = \mathbf{L}(k)\,\widehat{\mathbf{x}}_{\kappa(k)}(k \mid k) \tag{27}$$

in which

$$\widehat{\mathbf{x}}_{\kappa(k)}(k \mid k) = \widehat{\mathbf{x}}_{\kappa(k)}(k \mid k-1) + \mathbf{K}_{y,\kappa(k)}(k)\,\mathbf{e}^*_{y,\kappa(k)}(k \mid k-1), \tag{28}$$

$$\mathbf{K}_{y,\kappa(k)}(k) = \mathbf{R}_{xy,\kappa(k)}(k)\,\mathbf{R}^{-1}_{yy,\kappa(k)}(k). \tag{29}$$

The parameters $\mathbf{R}_{xy,\kappa(k)}(k)$ and $\mathbf{R}^{-1}_{yy,\kappa(k)}(k)$ in (29) can be obtained by iterating the equations in (18).

In summary, the unified finite horizon H_∞ fusion filtering algorithm is given by

$$\begin{aligned}
\widehat{\mathbf{z}}(k \mid k) &= \mathbf{L}(k)\,\widehat{\mathbf{x}}_{\kappa(k)}(k \mid k) \\
\widehat{\mathbf{x}}_{\kappa(k)}(k \mid k) &= \widehat{\mathbf{x}}_{\kappa(k)-1}(k \mid k) + \mathbf{K}_{y,\kappa(k)}(k)\,\mathbf{e}^*_{y,\kappa(k)}(k \mid k-1) \\
\widehat{\mathbf{x}}_0(k \mid k) &= \mathbf{F}(k,k-1)\,\widehat{\mathbf{x}}_{\kappa(k-1)}(k-1 \mid k-1),
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
&\mathbf{K}_{y,\kappa(k)}(k) = \mathbf{R}_{xy,\kappa(k)}(k)\,\mathbf{R}^{-1}_{yy,\kappa(k)}(k) \\
&\mathbf{e}^*_{y,\kappa(k)}(k \mid k-1) \\
&\quad = \mathbf{y}_{\alpha^k_{\kappa(k)}}\left(\beta^k_{\kappa(k)}\right) - \widehat{\mathbf{y}}^*_{\kappa(k)}(k \mid k-1) \\
&\quad = \mathbf{y}_{\alpha^k_{\kappa(k)}}\left(\beta^k_{\kappa(k)}\right) - \mathbf{H}^*_{\kappa(k)}(k)\,\widehat{\mathbf{x}}_{\kappa(k)}(k \mid k-1) \\
&\qquad - \mathbf{H}^*_{\kappa(k)}(k)\,\widehat{\mathbf{w}}_{\kappa(k)}\left(k, \beta^k_{\kappa(k)} \mid k-1\right).
\end{aligned} \tag{31}$$

The Riccati equations are given as follows.

(1) If the next measurement arrives at fusion filter in the fusion period $[k, k+1)$, the Ricatti equation is

$$\mathbf{P}_{\kappa(k)+1}(k) = \mathbf{P}_{\kappa(k)}(k) - \mathbf{K}_{\kappa(k)}(k)\,\mathbf{R}^T_{xyz,\kappa(k)}(k). \tag{32}$$

(2) Otherwise, the next measurement arrives at fusion filter in the fusion period $[k+1, k+2)$. The Ricatti equation is

$$\begin{aligned}
\mathbf{P}_1(k+1) &= \mathbf{F}(k+1,k)\left(\mathbf{P}_{\kappa(k)}(k) - \mathbf{K}_{\kappa(k)}(k)\,\mathbf{R}^T_{xy,\kappa(k)}(k)\right) \\
&\quad \times \mathbf{F}^T(k+1,k) + \mathbf{I}.
\end{aligned} \tag{33}$$

Remark 4. In ideal communication networks, all the measurements arrive at the fusion filter in time; namely, the measurements are all ITMs. In this case, the above unified finite horizon H_∞ fusion filtering algorithm will be simplified into the following sequential finite horizon H_∞ fusion filtering algorithm, which can deal with the ITMs in real time according to their sampled sequence

$$\widehat{\mathbf{z}}(k \mid k) = \mathbf{L}(k)\,\widehat{\mathbf{x}}_N(k \mid k), \tag{34}$$

in which

$$\begin{aligned}
\widehat{\mathbf{x}}_l(k \mid k) &= \widehat{\mathbf{x}}_{l-1}(k \mid k) \\
&\quad + \mathbf{K}_{y,l}(k)\,\mathbf{e}_{y,l}(k \mid k-1), \quad l = 1,2,\ldots,N, \\
\widehat{\mathbf{x}}_0(k \mid k) &= \mathbf{F}(k,k-1)\,\widehat{\mathbf{x}}_N(k-1 \mid k-1), \\
\mathbf{K}_{y,l}(k) &= \mathbf{H}_l(k)\,\mathbf{P}_l(k)\left[\mathbf{H}_l(k)\,\mathbf{P}_l(k)\,\mathbf{H}^T_l(k) + \mathbf{I}\right]^{-1}, \\
\mathbf{e}_{y,l}(k \mid k-1) &= \mathbf{y}_l(k) - \mathbf{H}_l(k)\,\widehat{\mathbf{x}}_{l-1}(k \mid k-1).
\end{aligned} \tag{35}$$

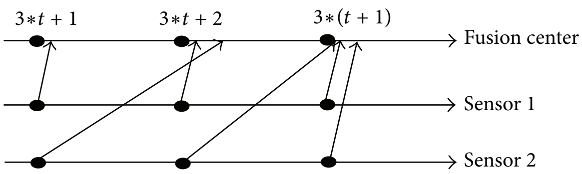

FIGURE 1: The measurement arrival scenes.

The corresponding Riccati equations are

$$\mathbf{P}_1(k) = \mathbf{F}(k,k-1) \times \left(\mathbf{P}_N(k-1) - \mathbf{R}_{xyz,l}(k)\mathbf{R}^{-1}_{eyz,l}(k) \times \mathbf{R}^T_{xyz,N}(k-1)\right) \times \mathbf{F}^T(k,k-1) + \mathbf{I}$$

$$\mathbf{P}_{l+1}(k) = \mathbf{P}_l(k) - \mathbf{R}_{xyz,l}(k)\mathbf{R}^{-1}_{eyz,l}(k)\mathbf{R}^T_{xyz,l}(k), \quad (36)$$

$$\mathbf{R}_{xyz,l}(k) = \mathbf{P}_l(k)\left[(\mathbf{H}_l(k))^T \quad \mathbf{L}^T(k)\right],$$

$$\mathbf{R}_{eyz,l}(k) := \begin{bmatrix}\mathbf{H}_l(k)\\ \mathbf{L}(k)\end{bmatrix}\mathbf{P}_l(k) \times \left[(\mathbf{H}_l(k))^T \quad \mathbf{L}^T(k)\right] + \begin{bmatrix} I & \mathbf{0}\\ \mathbf{0} & -\gamma^2\mathbf{I}\end{bmatrix}.$$

4. Simulation

In order to illustrate the viability and the effectiveness of the proposed method, the discrete dynamical system as shown in (1) is considered in this section, in which $\mathbf{F}(k,k-1) = \left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]$, $\mathbf{H}_i = [1,1], (i = 1,2)$, $L = [0.1, 0]$, $\gamma = 1.5$. Moreover, the initial value is selected as $\mathbf{x}_0 = [1,5]$, and $\mathbf{P}_0 = \left[\begin{smallmatrix}1 & 1/2\\ 1/2 & 2\end{smallmatrix}\right]$.

The measurement arrival scenes are designed as follows. All the measurements are sampled in time. For Sensor 1, all the measurements arrive at the fusion filter in time. For Sensor 2, the measurements sampled at the moments with indexes modulo 3 equal to 1 or 2 arrive at the fusion filter with one-step delay, and other measurements arrive in time, as shown in Figure 1.

In this simulation, two simulation results are compared. The first one is the result of the proposed method in the above measurement arrival scene, which is for short marked as "Algorithm 1" in this section. The other one is the simulation result of the sequential fusion filtering method in the scene that all the measurements arrive in time, as shown in Remark 4, which is marked as "Algorithm 2."

According to the simulation results shown in Table 1, Figures 1 and 2, the following performances of the proposed algorithm are illustrated.

(1) The proposed algorithm could deal with different kinds of arrived measurements: ITMs (the measurements sampled by Sensor 1 and the ones sampled by Sensor 2 at the sampled moments with indexes divided by 3 exactly), ISDMs (the measurements sampled by Sensor 2 at the sampled moments with indexes modulo 3 equal to 1), and OOSMs (the measurements sampled by Sensor 2 at the sampled moments with indexes modulo 3 equal to 2).

(2) For Algorithm 1, at the fusion time with the indexes divided by 3 exactly, the delay measurements can all arrive at the fusion center. At this fusion time, the fusion filtering results of Algorithm 1are better than the ones of Algorithm 2. The mean absolute estimation errors at these fusion times of Algorithm 1 are 0.3787, while the one of Algorithm 2 is 0.3832. It is because more amount of information is applied to update the estimate at this fusion time in Algorithm 1 (Figure 3).

(3) It is implied that the proposed algorithm could deal with the delay measurements effectively.

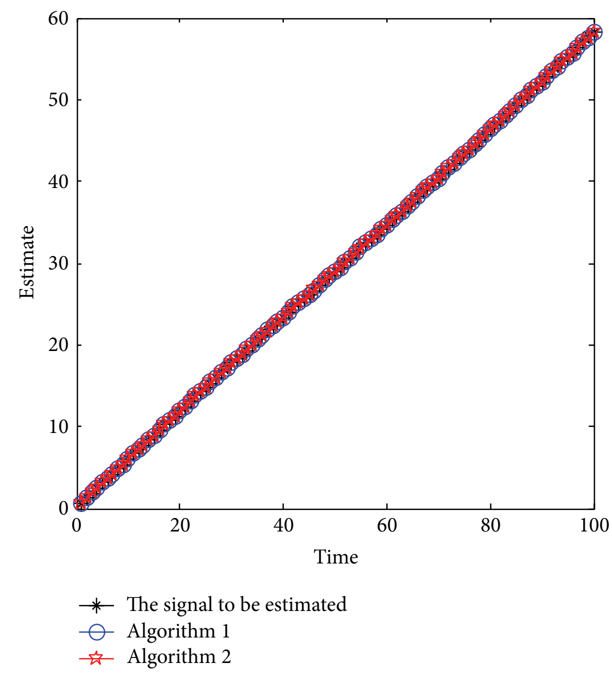

FIGURE 2: The estimate curves.

TABLE 1: The mean absolute estimation errors at filtering time with the indexes are divided by 5 exactly.

	Algorithm 1	Algorithm 2
The mean absolute estimation errors	0.3787	0.3832

5. Conclusion

In this paper, a unified finite horizon H_∞ filtering method is proposed for general networked dynamical systems, the fusion filter of which could receive none, one, or multiple measurements in a fusion period. According to the complex arrival scenes of networked measurements, a novel H_∞ performance criterion function is built to restrain the H_∞ filtering process. Based on the projection method in

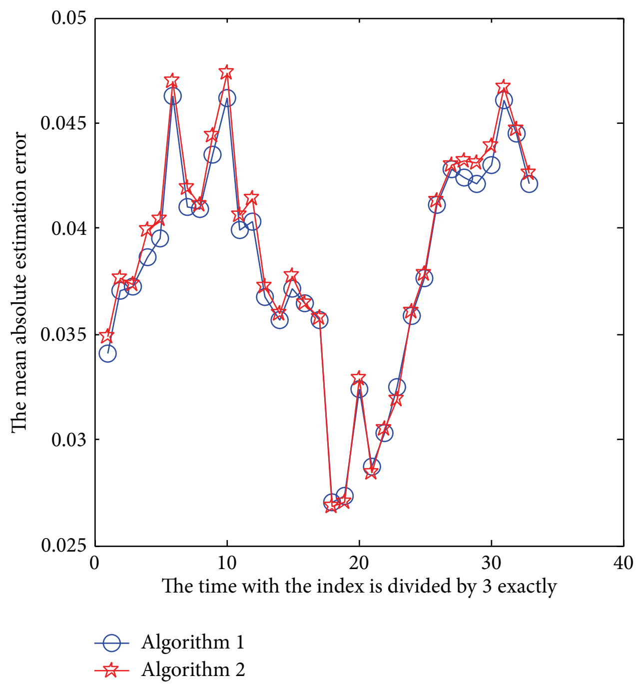

Figure 3: The absolute estimation error curves at the fusion time with the indexes are divided by 3 exactly.

Krein space, a novel H_∞ filtering method is proposed to uniformly deal with various delay measurements and ITMs in the centralized fusion frame. Otherwise, there are several interesting future directions along the line of this work:

(1) how to deal with the networked measurements in the distributed fusion frame in a uniform manner,

(2) how to deal with various quantified delay measurements for networked multisensor systems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The paper was supported by the National Natural Science Foundation of China under Grants 61333005, 61172133, 61174112, 61304258, and 61273075.

References

[1] Q. Wu, X. Litrico, and M. B. Alexandre, "Data reconciliation of an open channel flow network using modal decomposition," in *Proceedings of the 47th IEEE Conferenece on Decision and Control (CDC '08)*, pp. 3903–3910, Cancún, Mexico, December 2008.

[2] Z. He, *Research on comprehensive energy saving and coordinative optimization control of distributed pumping stations in urban drainage systems [Ph.D. thesis]*, Zhejiang University, Hangzhou, China, 2009.

[3] Y. Zhang, C. Liu, and X. Mu, "On stochastic finite-time control of discrete-time fuzzy systems with packet dropout," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 752950, 18 pages, 2012.

[4] C. Wen, R. Liu, and T. Chen, "Linear unbiased state estimation with random one-step sensor delay," *Circuits, Systems, & Signal Processing*, vol. 26, no. 4, pp. 573–590, 2007.

[5] C. Su and C. Lu, "Interconnected network state estimation using randomly delayed measurements," *IEEE Transactions on Power Systems*, vol. 16, no. 4, pp. 870–878, 2001.

[6] H. Song, L. Yu, and W. Zhang, "H_∞ filtering of network-based systems with random delay," *Signal Processing*, vol. 89, no. 4, pp. 615–622, 2009.

[7] B. Shen, Z. Wang, H. Shu, and G. Wei, "H_∞ filtering for nonlinear discrete-time stochastic systems with randomly varying sensor delays," *Automatica*, vol. 45, no. 4, pp. 1032–1037, 2009.

[8] K. Zhang, X. R. Li, and Y. Zhu, "Optimal update with out-of-sequence measurements," *IEEE Transactions on Signal Processing*, vol. 53, no. 6, pp. 1992–2004, 2005.

[9] Q. Ge, T. Xu, X. Feng, and C. Wen, "Universal delayed Kalman filter with measurement weighted summation for the linear time invariant system," *Chinese Journal of Electronics*, vol. 20, no. 1, pp. 67–72, 2011.

[10] C. Wen, Q. Ge, and X. Feng, "Hybrid filter with predict-estimator and compensator for the linear time invariant delayed system," *Journal of Electronics*, vol. 26, no. 5, pp. 666–672, 2009.

[11] H. Zhang, G. Feng, and C. Han, "Linear estimation for random delay systems," in *Proceedings of the 49th IEEE Conference on Decision and Control (CDC '10)*, pp. 449–454, Atlanta, Ga, USA, 2010.

[12] H. Zhang, L. Xie, H. S. Zhang, and Y. C. Soh, "A reorganized innovation approach to linear estimation," *IEEE Transaction on Automatic Control*, vol. 49, no. 10, pp. 1810–1814, 2004.

[13] B. Chen, L. Yu, and W. Zhang, "Robust Kalman filtering for uncertain state delay systems with random observation delays and missing measurements," *IET Control Theory & Applications*, vol. 5, no. 17, pp. 1945–1954, 2011.

[14] S. Sun, "Linear minimum variance estimators for systems with bounded random measurement delays and packet dropouts," *Signal Processing*, vol. 89, no. 7, pp. 1457–1466, 2009.

[15] S. Sun, L. Xie, W. Xiao, and Y. C. Soh, "Optimal linear estimation for systems with multiple packet dropouts," *Automatica*, vol. 44, no. 5, pp. 1333–1342, 2008.

[16] M. Sahebsara, T. Chen, and S. L. Shah, "Optimal H_∞ filtering in networked control systems with multiple packet dropouts," *Systems & Control Letters*, vol. 57, no. 9, pp. 696–702, 2008.

[17] M. Moayedi, Y. K. Foo, and Y. C. Soh, "Optimal and suboptimal minimum-variance filtering in networked systems with mixed uncertainties of random sensor delays, packet dropouts and missing measurements," *International Journal of Control, Automation and Systems*, vol. 8, no. 6, pp. 1179–1188, 2010.

[18] H. Gao and T. Chen, "H_∞ estimation for uncertain systems with limited communication capacity," *IEEE Transactions on Automatic Control*, vol. 52, no. 11, pp. 2070–2084, 2007.

[19] X. Shen, E. Song, Y. Zhu, and Y. Luo, "Globally optimal distributed Kalman fusion with local out-of-sequence-measurement updates," *IEEE Transactions on Automatic Control*, vol. 54, no. 8, pp. 1928–1934, 2009.

[20] X. Shen, Y. Zhu, E. Song, and Y. Luo, "Optimal centralized update with multiple local out-of-sequence measurements," *IEEE Transactions on Signal Processing*, vol. 57, no. 4, pp. 1551–1562, 2009.

[21] B. Hassibi, A. H. Sayed, and T. Kailath, "Linear estimation in Krein spaces. II. Applications," *IEEE Transactions on Automatic Control*, vol. 41, no. 1, pp. 34–49, 1996.