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Contributions to the fixed point theory of diagonal operators

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Abstract

In this paper, we introduce the notion of diagonal operator, we present the historical roots of diagonal operators and we give some fixed point theorems for this class of operators. Our approaches are based on the weakly Picard operator technique, difference equation techniques, and some fixed point theorems for multi-valued operators. Some applications to differential and integral equations are given. We also present some research directions.

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1 Introduction and preliminary notions and results

In this section we will present some useful notions and results concerning diagonal operators, coupled fixed point operators, and iterations of some operators generated by the above concepts.

1.1 Diagonal operators

Let X be a nonempty set and $V : X \times X \to X$ be an operator. By definition, the operator $U_V : X \to X$, defined by

$$U_V(x) := V(x,x), \quad \text{for all } x \in X,$$

is called the diagonal operator corresponding to the operator V.

We also consider the following operators generated by an operator $V : X \times X \to X$:

(1) The operator $C_V : X \times X \to X \times X$ defined by

$$C_V(x,y) := \big(V(x,y), V(y,x)\big).$$

By definition an element

$$(x,y) \in F_{C_V} := \big\{(u,v) \in X \times X \mid C_V(u,v) = (u,v)\big\}$$

is called a coupled fixed point of V (see [1]; see also [2, 3]). We remark that

$$(x,x) \in F_{C_V} \quad \Leftrightarrow \quad x \in F_{U_V}.$$

(2) The operator $D_V : X \times X \to X \times X$ defined by $D_V(x,y) := (y, V(x,y))$.
We have $(x,y) \in F_{D_V} \Leftrightarrow x = y$ and $x \in F_{U_V}$.
(3) The operator $T_V : X \to \mathcal{P}(X)$ is defined by

$$T_V(x) := F_{V(\cdot,x)}.$$

It is clear that $F_{U_V} = F_{T_V}$.

The aim of this paper is to present some historical roots of the diagonal operators, to study the fixed points of this class of operators, and to give some applications. Some new research directions are also presented.

More precisely, the plan of the paper is the following:
1. Introduction and preliminary notions and results.
2. Historical roots of the diagonal operators.
3. Iterations of the operators C_V and U_V.
4. Iterations of the operator D_V and the difference equation

$$x_{n+2} = V(x_n, x_{n+1}), \quad n \in \mathbb{N}, x_0, x_1 \in X.$$

5. Fixed point results for the operator T_V.
6. Applications.
7. Research directions.
References.

1.2 *L*-Spaces [4–6]

Following Fréchet [4], we present now the concept of L-space.

Definition 1.1 Let X be a nonempty set. Let

$$s(X) := \{(x_n)_{n\in\mathbb{N}} \mid x_n \in X, n \in \mathbb{N}\}.$$

Let $c(X)$ be a subset of $s(X)$ and $\mathrm{Lim} : c(X) \to X$ be an operator. By definition, the triple $(X, c(X), \mathrm{Lim})$ is called an L-space (denoted by $(X, \to)$) if the following conditions are satisfied:

(i) if $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $\mathrm{Lim}(x_n)_{n\in\mathbb{N}} = x$;
(ii) if $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $\mathrm{Lim}(x_n)_{n\in\mathbb{N}} = x$, then, for all subsequences $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, we have $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and

$$\mathrm{Lim}(x_{n_i})_{i\in\mathbb{N}} = x.$$

By definition, an element of $c(X)$ is said to be a convergent sequence and $\mathrm{Lim}(x_n)_{n\in\mathbb{N}}$ is the limit of this sequence. If $\mathrm{Lim}(x_n)_{n\in\mathbb{N}} = x$, then we will write

$$x_n \to x \quad \text{as } n \to \infty.$$

Remark 1.1 An L-space is any set endowed with a structure implying a notion of convergence for sequences. As examples of L-spaces we mention the following:

(1) If (X,τ) is a Hausdorff topological space, then $(X,\xrightarrow{\tau})$ is an L-space.

(2) If (X,d) is a metric space then $(X,\xrightarrow{d})$ is an L-space.

(3) If $(X,\|\cdot\|)$ is a normed space then $(X,\xrightarrow{\|\cdot\|})$ and $(X,\rightharpoonup)$ are L-spaces.

1.3 Weakly Picard operators [5, 7–11], etc.

Let $(X,\to)$ be an L-space. By definition, $f:X\to X$ is said to be a weakly Picard operator (WPO) if the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations converges for all $x\in X$ and the limit (which may depend on x) is a fixed point of f. If f is weakly Picard operator and $F_f=\{x^*\}$, then, by definition, f is called a Picard operator (PO).

If $f:X\to X$ is WPO, then we define the operator $f^\infty:X\to X$ by

$$f^\infty(x):=\lim_{n\to\infty}f^n(x).$$

Now let (X,d) be a metric space. By definition, a WPO $f:X\to X$ is called ψ-WPO if $\psi:\mathbb{R}_+\to\mathbb{R}_+$ is increasing, continuous in 0 with $\psi(0)=0$, and

$$d\bigl(x,f^\infty(x)\bigr)\le\psi\bigl(d\bigl(x,f(x)\bigr)\bigr),\quad\text{for all } x\in X.$$

1.4 Measures of noncompactness [12–17], etc.

Let X be a Banach space. We will denote by $P_b(X)$ the family of all nonempty bounded subsets of S.

We will use the symbol $\alpha_K:P_b(X)\to\mathbb{R}_+$, for the Kuratowski measure of noncompactness on X, while $\alpha_H:P_b(x)\to\mathbb{R}_+$ will denote the Hausdorff measure of noncompactness on X. The following results are well known.

Darbo's theorem *Let X be a Banach space, $Y\in P_{b,cl,cv}(X)$ and $f:Y\to Y$ be an operator. We suppose that*:

(i) *f is continuous*;

(ii) *there exists $l\in[0,1[$ such that*:

$$\alpha_K\bigl(f(A)\bigr)\le l\cdot\alpha_K(A),\quad\textit{for all } A\in P(Y).$$

Then:

(a) $F_f\neq\emptyset$;

(b) *F_f is a compact subset of Y.*

Sadovskii's theorem *Let X be a Banach space, $Y\in P_{b,cl,cv}(X)$ and $f:Y\to Y$ be an operator. We suppose that*:

(i) *f is continuous*;

(ii) $\alpha_H(f(A))<\alpha_H(A)$, *for all $A\in P(Y)$ with $\alpha_H(A)\neq 0$.*

Then:

(a) $F_f\neq\emptyset$;

(b) *F_f is a compact subset of Y.*

1.5 Fixed point structures [14]

Let $\mathcal{C}$ be a class of structured sets (ordered sets, topological spaces, metric spaces, Banach spaces, Hilbert spaces, ...). Let we denote by Set^* the class of all nonempty sets. For $X \in \mathrm{Set}^*$ we shall use the notations:

$$P(X) := \{Y \subset X \mid Y \neq \emptyset\} \quad \text{and} \quad P(\mathcal{C}) := \{A \in P(X) \mid X \in \mathcal{C}\},$$
$$\mathbb{M}(A,B) := \{f : A \to B \mid f \text{ is an operator}\} \quad \text{and} \quad \mathbb{M}(A) := \mathbb{M}(A,A).$$

Now we consider the following multi-valued operators:

$S : \mathcal{C} \multimap \mathrm{Set}^*, X \multimap S(X) \subset P(X)$,

$M : D_M \subset P(\mathcal{C}) \times P(\mathcal{C}) \multimap \mathbb{M}(P(\mathcal{C}), P(\mathcal{C})), (A,B) \multimap M(A,B) \subset \mathbb{M}(A,B)$.

By a fixed point structure on $X \in \mathcal{C}$, we understand a triple $(X, S(X), M)$ with the following properties:

(i) $A \in S(X) \Rightarrow (A,A) \in D_M$;

(ii) $A \in S(X), f \in M(U) \Rightarrow F_f \neq \emptyset$.

The following examples illustrate this notion.

(1) *The fixed point structure (f.p.s.) of Tarski*

Let $\mathcal{C}$ be the class of complete lattices. If $(X, \preceq)$ is a complete lattice,

$$S(X) := \{A \in P(X) \mid (A, \preceq) \text{ is a complete lattice}\}$$

and

$$M(A) := \{f : A \to A \mid f \text{ is increasing}\},$$

then, by Tarski's fixed point theorem, we see that $(X, S(X), M)$ is a f.p.s.

(2) *The f.p.s. of contractions*

Let $\mathcal{C}$ be the class of complete metric spaces. For a complete metric space (X,d), we consider

$$S(X) := P_{cl}(X)$$

and

$$M(A) := \{f : A \to A \mid f \text{ is a contraction}\}.$$

Then the Banach contraction principle implies that $(X, S(X), M)$ is a f.p.s.

(3) *The f.p.s. of Schauder*

Let $\mathcal{C}$ be the class of Banach spaces. For a Banach space X, if we consider

$$S(X) := P_{cp,cv}(X)$$

and

$$M(A) = M(A,A) := C(A,A),$$

then, by Schauder's theorem, we see that $(X, S(X), M)$ is a f.p.s.

A similar notion of fixed point structure can be defined for multi-valued operators (see [14], pp.139-142).

A triple $(X, S(X), M^0)$ is a multi-valued fixed point structure (m.f.p.s.) if the following properties hold:

(i) $S(X) \subset P(X)$ and $S(X) \neq \emptyset$;

(ii) $M^0 : P(X) \multimap \bigcup_{Y \in P(X)} M^0(Y), Y \multimap M^0(Y) \subset \mathbb{M}(Y)$, where $\mathbb{M}(Y)$ is the set of all self multi-valued operators on Y;

(iii) $Y \in S(X), T \in M^0(Y) \Rightarrow F_T \neq \emptyset$.

1.6 Acyclic topological spaces [15, 18]

Let X be a compact metric space and $H_q(x)$ be the q-dimensional Čech homology on $\mathbb{Q}$ of X. By definition, X is called acyclic if $H_q(X) = 0$ for $q \geq 1$ and $H_q(X) \approx \mathbb{Q}$.

The following result is a particular case of the Eilenberg-Montgomery theorem (see [15, 17, 18]).

Theorem 1.1 *Let Y be a compact convex subset of a Banach space E and $T : Y \to P(Y)$ be an upper semi-continuous multi-valued operator with acyclic values. Then $F_T \neq \emptyset$.*

2 Historical roots of the diagonal operators

There are some roots of the diagonal operators as the following examples reveal.

Example 2.1 (Difference equations [19–22]) Let $(X, \to)$ be an L-space and $V : X \times X \to X$ be an operator. We consider the following difference equation:

$$x_{n+2} = V(x_n, x_{n+1}), \quad n \in \mathbb{N}, x_0, x_1 \in X.$$

Let us suppose that $(x_n)_{n\in\mathbb{N}}$ is a solution of this equation with the property that

$$x_n \to x^* \quad \text{as } n \to \infty.$$

If the function V is continuous, then we have $x^* = V(x^*, x^*)$. Thus, x^* is a fixed point of the diagonal operator corresponding to V.

Example 2.2 (Krasnoselskii (1955) [23]) Let X be a Banach space, $Y \in P_{b,cl,cv}(X)$ and $f, g : Y \to Y$ be two operators. We suppose that:

(i) f is a contraction;

(ii) g is complete continuous;

(iii) $f(x) + g(y) \in Y$, for all $x, y \in Y$.

Under these conditions, Krasnoselskii proved that the operator $f + g : Y \to Y$ has at least a fixed point.

If we consider the operator $V : Y \times Y \to Y$, defined by

$$V(x, y) := f(x) + g(y),$$

then $f + g : Y \to Y, x \mapsto f(x) + g(x)$ is the diagonal operator corresponding to V.

Example 2.3 (Browder (1966; [24]; see also [12, 13, 16, 25–28], etc.) Let X be a Banach space, $Y \in P_{b,op}(X)$ and $V : X \times X \to X$ be a continuous operator. Then Browder considered the operator $U : \overline{Y} \to X$ defined by $U(x) := V(x,x)$.

Moreover, Browder introduced the following notions:

(1) U is strictly semicontractive if, for each fixed x in X, $V(\cdot,x)$ is Lipschitzian with constant $l < 1$ and $V(x,\cdot)$ is compact.

(2) U is weakly semicontractive if, for each x in X, the operator $V(\cdot,x)$ is nonexpansive and $V(x,\cdot)$ is compact.

Example 2.4 (Ziebur (1962 and 1965); [29, 30]) Let $b \in \mathbb{R}^m$ and $h, k \in \mathbb{R}^*_+$. One consider a set $\Omega := \prod_{i=1}^m [b_i - k, b_i + k]$ and a function $f \in C([0,h] \times \Omega, \mathbb{R}^m)$. Let us consider the Cauchy problem

$$(\mathrm{C}_1) \qquad \begin{cases} x' = f(t,x), \\ x(0) = x_0. \end{cases}$$

Then Ziebur introduced a function $F \in C([0,h] \times \Omega \times \Omega, \mathbb{R}^m)$ with the following property:

(a) $F(t,x,x) = f(t,x)$, for all $t \in [0,h], x \in \mathbb{R}^m$;

(b) $F(t,\cdot,x)$ is increasing;

(c) $F(t,x,\cdot)$ is decreasing.

The following Cauchy problem was also considered:

$$(\mathrm{C}_2) \qquad \begin{cases} x' = F(t,x,y), \\ y' = F(t,y,x), \end{cases} \qquad (x,y)(0) = (x_0,x_0)$$

and it is proved that if the Cauchy problem (C_2) has a unique solution then the problem (C_1) has a unique solution too and the Picard sequence converges to that solution.

Example 2.5 (Amann (1973, 1977), Opoitsev (1975; [2, 31–33]))

In [2] the author presents the following result 'concerning so-called intervined':

Let X be a chain complete ordered set possessing a least and a greatest element. Let $g : X \times X \to X$ be a mapping such that:

(i) $g(\cdot,y) : X \to X$ is increasing for every $y \in X$;

(ii) $g(x,\cdot) : X \to X$ is decreasing for every $x \in X$.

Then there exist two points $\overline{x}, \widehat{x} \in X$ such that $\overline{x} \preceq \widehat{x}$, $g(\overline{x},\widehat{x}) = \overline{x}$, and $g(\widehat{x},\overline{x}) = \widehat{x}$. Moreover, if $f(x) := g(x,x)$ for all $x \in X$, then

$$F_f \subset [\overline{x},\widehat{x}].$$

Remark 2.1 For other examples on this topic see [1, 33–37], etc.

Example 2.6 (Quasilinear differential equations; see [17, 34, 35, 38], etc.) Diagonal operators also appears by the linearization of a quasilinear differential equations. For example, let us consider the Cauchy problem:

$$\begin{cases} x'(t) = A(t)x(t) + f(t,x(t)), \quad t \in [a,b], \\ x(a) = x_0, \end{cases} \tag{2.1}$$

where $A \in C([a,b], \mathbb{R}^{m \times m})$ and $f \in C([a,b] \times \mathbb{R}^m, \mathbb{R}^m)$.

Then, for each $u \in C([a,b],\mathbb{R}^m)$ with $u(a) = x_0$ we consider the linearized Cauchy problem

$$\begin{cases} x'(t) = A(t)x(t) + f(t,u(t)), & t \in [a,b], \\ x(a) = x_0, \end{cases} \tag{2.2}$$

Let $S \subset C^1([a,b],\mathbb{R}^m)$ be the solution set of the problem (2.1) and let $T(u)$ the solution set of the problem (2.2). Then $S = F_T$.

3 Iterations of C_V and U_V

The following result is the starting point for this section.

Lemma 3.1 *Let $(X,\rightarrow)$ be an L-space and $V : X \times X \to X$ be an operator. We suppose that the operator C_V is WPO. Then we have*:

(a) *U_V is WPO*;
(b) *$C_V^\infty(x,x) = (U_V^\infty(x), U_V^\infty(x))$, for all $x \in X$*;
(c) *if C_V is a PO, then*:
  (1) *$F_{C_V} = \{(x^*,x^*)\}$*;
  (2) *U_V is PO and $F_{U_V} = \{x^*\}$.*

Proof (a) + (b). We remark that $C_V(x,x) = (U_V(x), U_V(x))$. From this we have

$$C_V^n(x,x) = \bigl(U_V^n(x), U_V^n(x)\bigr)$$

and

$$C_V^n(x,x) \to C^\infty(x,x) = \bigl(U_V^\infty(x), U_V^\infty(x)\bigr) \quad \text{as } n \to \infty, \text{ for all } x \in X.$$

(c). Follows from the definition of PO and from (a) + (b).

Now we consider instead of the L-space $(X,\rightarrow)$ a metric space (X,d).

We will consider, on $X \times X$, the following metrics:

$$d_1\bigl((x_1,y_1),(x_2,y_2)\bigr) := d(x_1,x_2) + d(y_1,y_2), \tag{3.1}$$

$$d_2\bigl((x_1,y_1),(x_2,y_2)\bigr) := \bigl[\bigl(d(x_1,x_2)\bigr)^2 + \bigl(d(y_1,y_2)\bigr)^2\bigr]^{\frac{1}{2}}, \tag{3.2}$$

and

$$d_\infty\bigl((x_1,y_1),(x_2,y_2)\bigr) := \max\bigl(d(x_1,x_2), d(y_1,y_2)\bigr). \tag{3.3}$$

□

In the case of metric spaces we have the following result.

Theorem 3.1 *Let (X,d) be a metric space and $V : X \times X \to X$ be an operator. Then*:

(a) *If C_V is ψ – WPO with respect to the metric d_1, then U_V is a θ – WPO where*

$$\theta(r) := \frac{1}{2}\psi(2r), \quad r \in \mathbb{R}_+.$$

(b) *If C_V is $\psi - WPO$ with respect to the metric d_2, then U_V is a $\theta - WPO$ where*

$$\theta(r) := \frac{1}{\sqrt{2}}\psi(\sqrt{2}r), \quad r \in \mathbb{R}_+.$$

(c) *If C_V is $\psi - WPO$ with respect to the metric d_∞, then U_V is a $\psi - WPO$.*

Proof (a). From

$$d_1\big((x,y), C_V^\infty(x,y)\big) \le \psi\big(d_1\big((x,y), C_V(x,y)\big)\big), \quad \text{for all } (x,y) \in X \times X,$$

it follows that

$$d_1\big((x,x), \big(U_V^\infty(x), U_V^\infty(x)\big)\big) \le \psi\big(d_1\big((x,x), \big(V(x,x), V(x,x)\big)\big)\big), \quad \text{for all } x \in X,$$

so

$$d\big(x, U_V^\infty(x)\big) \le \frac{1}{2}\psi\big(2d\big(x, U_V(x)\big)\big), \quad \text{for all } x \in X.$$

(b). If C_V is $\psi - WPO$ with respect to the metric d_2 then

$$d_2\big((x,x), C_V^\infty(x,x)\big) \le \psi\big(d_2\big((x,x), \big(V(x,x), V(x,x)\big)\big)\big), \quad \text{for all } x \in X,$$

which means that

$$\sqrt{2}d\big(x, U_V^\infty(x)\big) \le \psi\big(\sqrt{2}d\big(x, U_V(x)\big)\big), \quad \text{for all } x \in X,$$

so we get the conclusion.

(c). If C_V is $\psi - WPO$ with respect to the metric d_∞ then

$$d_\infty\big((x,x), C_V^\infty(x,x)\big) \le \psi\big(d_\infty\big((x,x), \big(V(x,x), V(x,x)\big)\big)\big), \quad \text{for all } x \in X,$$

so

$$d\big(x, U_V^\infty(x)\big) \le \psi\big(d\big(x, U_V(x)\big)\big), \quad \text{for all } x \in X,$$

which proves that U_V is a $\psi - WPO$. □

The following result is a coupled fixed point theorem in a complete b-metric space, which has as an additional conclusion the fact that the operator C_V is a Picard operator.

Theorem 3.2 ([39]) *Let (X,d) be a complete b-metric space with constant $s \ge 1$. Let $V : X \times X \to X$ be an operator. Assume that there exists $k \in (0,1)$ such that, for all $(x,y), (u,v) \in X \times X$, we have*

$$d\big(V(x,y), V(u,v)\big) + d\big(V(y,x), V(v,u)\big) \le k\big[d(x,u) + d(y,v)\big].$$

Then there exists a unique solution $(x^*, y^*) \in X \times X$ *of the following coupled fixed point problem* :

$$\begin{cases} x = V(x,y), \\ y = V(y,x). \end{cases} \tag{3.4}$$

and, for any initial point $(x_0, y_0) \in X \times X$, *the sequences* $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ *defined, for* $n \in \mathbb{N}$, *by*

$$\begin{cases} x_{n+1} = V(x_n, y_n), \\ y_{n+1} = V(y_n, x_n), \end{cases} \tag{3.5}$$

converge to x^* *and, respectively, to* y^* *as* $n \to \infty$.

In particular, the operator $C_V : X \times X \to X \times X$ *given by* $C_V(x,y) := (V(x,y), V(y,x))$ *is a Picard operator.*

Proof For the sake of completeness we present here the sketch of the proof. We introduce on $Z := X \times X$ the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$ defined by

$$\tilde{d}\big((x,y),(u,v)\big) := d(x,u) + d(y,v).$$

Notice that, as before, $\tilde{d}$ is a b-metric on Z with the same constant $s \geq 1$ and, if the space (X,d) is complete, then $(Z,\tilde{d})$ is complete too.

We consider now the operator $F : Z \to Z$ given by

$$F(x,y) := \big(V(x,y), V(y,x)\big).$$

It is easy to prove now that F is a contraction in $(Z,\tilde{d})$ with constant $k \in (0,1)$, *i.e.*,

$$\tilde{d}\big(F(z),F(w)\big) \leq k\tilde{d}(z,w), \quad \text{for all } z,w \in Z.$$

Thus, we can apply for F the b-metric space version of the contraction principle given by Czerwik (see, for example, Theorem 12.2, p.115 in [40]) and we get the conclusion. □

Another result involves the coupled fixed point problem in a complete metric space under a contraction condition on the graphic of the operator. In this case, we will see that C_V is a weakly Picard operator. Let us also point out that we denote $V^2(x,y) := V(V(x,y),V(y,x))$ and $V^2(y,x) := V(V(y,x),V(x,y))$, while the graphic of an operator $U : X \to X$ is denoted by $\mathrm{Graph}(U) := \{(x,y) \in X \times X : y = U(x)\}$.

Theorem 3.3 *Let* (X,d) *be a complete metric space and* $V : X \times X \to X$ *be an operator. Assume that there exists* $k \in (0,1)$ *such that, for all* $(x,y) \in X \times X$, *we have*

$$d\big(V(x,y),V^2(x,y)\big) + d\big(V(y,x),V^2(y,x)\big) \leq k\big[d\big(x,V(x,y)\big) + d\big(y,V(y,x)\big)\big].$$

Then there exists at least one solution $(x^*, y^*) \in X \times X$ *of the coupled fixed point problem* (3.4) *and, for any initial point* $(x_0, y_0) \in X \times X$, *the sequences* $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ *defined by* (3.5) *converge to* x^* *and, respectively, to* y^* *as* $n \to \infty$.

In particular, $C_V : X \times X \to X \times X$ *given by* $C_V(x,y) := (V(x,y), V(y,x))$ *is a weakly Picard operator.*

Proof We consider again on $Z := X \times X$ the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$ defined by

$$\tilde{d}\big((x,y),(u,v)\big) := d(x,u) + d(y,v).$$

As before, $(Z,\tilde{d})$ is a complete metric space.

We consider now the operator $F : Z \to Z$ given by

$$F(x,y) := \big(V(x,y), V(y,x)\big).$$

It is easy to prove now that F is a graphic contraction in $(Z,\tilde{d})$ with constant $k \in (0,1)$, *i.e.*,

$$\tilde{d}\big(F(z),F^2(z)\big) \le k\tilde{d}\big(z,F(z)\big), \quad \text{for all } z = (x,y) \in Z.$$

Thus, the conclusion follows by the graphic contraction principle; see, for example, [8] or [10]. □

Remark 3.1 It is worth to mention that the above results can easily be considered in the framework of an ordered metric space X, under contraction type conditions imposed for comparable elements (with respect to a partial order relation $\preceq$ on X); see for example [3, 39, 41–43], etc.

We will consider now some qualitative properties concerning the behavior of an operator $A : X \to X$, where (X,d) is a metric space. More precisely, we consider the following notions:

(i) *the fixed point equation*

$$x = A(x), \quad x \in X$$

is called *well-posed* if $F_A = \{x_A^*\}$ and for any $x_n \in X$, $n \in \mathbb{N}$ a sequence in X such that

$$d\big(x_n, A(x_n)\big) \to 0 \quad \text{as } n \to \infty,$$

we have

$$x_n \to x_A^* \quad \text{as } n \to \infty.$$

(ii) The operator A has the *Ostrowski property* (or the operator A has the limit shadowing property) if $F_A = \{x^*\}$ and for any $x_n \in X$, $n \in \mathbb{N}$ a sequence in X such that

$$d\big(x_{n+1}, A(x_n)\big) \to 0 \quad \text{as } n \to \infty,$$

we have

$$x_n \to x^* \quad \text{as } n \to \infty.$$

(iii) *The fixed point equation*

$$x = A(x), \quad x \in X$$

is *generalized Ulam-Hyers stable* if there exists an increasing function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$, continuous in 0 with $\theta(0) = 0$, and for each $\varepsilon > 0$ and for each solution y^* of the inequality

$$d\big(y, A(y)\big) \leq \varepsilon$$

there exists a solution x^* of the fixed point equation with

$$d\big(x^*, y^*\big) \leq \theta(\varepsilon).$$

By the above notions, we have the following result.

Theorem 3.4 *Let (X, d) be a metric space and $V : X \times X \to X$ be an operator. Then*:

(a) *If the fixed point equations for C_V is well-posed and $F_{C_V} = \{(x^*, x^*)\}$, then the fixed point equations for U_V is well-posed.*

(b) *If the operator C_V has the Ostrowski property and $F_{C_V} = \{(x^*, x^*)\}$, then the operator U_V has the Ostrowski property.*

(c) *If the fixed point equations for C_V is generalized Ulam-Hyers stable and all the fixed points of C_V are of the form (x^*, x^*), then the fixed point equations for U_V is generalized Ulam-Hyers stable.*

Proof (a). If $F_{C_V} = \{(x^*, x^*)\}$ then $F_{U_V} = \{x^*\}$. Let $(x_n)_{n\in\mathbb{N}} \subset X$ such that $d(x_n, U_V(x_n)) \to 0$ as $n \to +\infty$. Then

$$d_*\big((x_n, x_n), C_V(x_n, x_n)\big) \to 0, \quad n \to +\infty,$$

where d_* is one of the metrics on $X \times X$ defined by (3.1)-(3.3). Since the fixed point equation for C_V is well-posed

$$d_*\big((x_n, x_n), \big(x^*, x^*\big)\big) \to 0, \quad n \to +\infty,$$

therefore

$$d\big(x_n, x^*\big) \to 0, \quad n \to +\infty,$$

so the fixed point equation for U_V is well-posed.

(b). Let $(x_n)_{n\in\mathbb{N}} \subset X$ such that $d(x_n, U_V(x_{n+1})) \to 0$ as $n \to +\infty$. Then

$$d_*\big((x_n, x_n), C_V(x_{n+1}, x_{n+1})\big) \to 0, \quad n \to +\infty,$$

where d_* is one of the metrics on $X \times X$ defined by (3.1)-(3.3). Since the operator C_V has the Ostrowski property

$$d_*\big((x_n, x_n), \big(x^*, x^*\big)\big) \to 0, \quad n \to +\infty,$$

therefore

$$d(x_n, x^*) \to 0, \quad n \to +\infty,$$

so the operator U_V has the Ostrowski property.

(c). If $(x^*, x^*) \in F_{C_V}$ then $x^* \in F_{U_V}$. Let $\varepsilon > 0$ and $y^* \in X$ be a solution of the inequality

$$d(y, U_V(y)) \leq \varepsilon,$$

then

$$d_1((y^*, y^*), C_V(y^*, y^*)) = 2d(y^*, U_V(y^*)) \leq 2\varepsilon.$$

From the Ulam-Hyers stability of the fixed point equation for C_V we see that there exists $(x^*, x^*) \in F_{C_V}$ such that

$$d_1((y^*, y^*), (x^*, x^*)) \leq \theta(2\varepsilon) \quad \Longleftrightarrow \quad 2d(y^*, x^*) \leq \theta(2\varepsilon),$$

so

$$d(y^*, x^*) \leq \theta_1(\varepsilon),$$

where $\theta_1(t) = \frac{1}{2}\theta(2t)$ which proves that the fixed point equations for U_V is generalized Ulam-Hyers stable.

If we replace the metric d_1 on $X \times X$ defined by (3.1) with the metric d_2 or d_∞ on $X \times X$ defined by (3.2), respectively, by (3.3), we get the same conclusion but instead of $\theta_1(t)$ we have a different function, in the case of d_2 we have $\theta_2(t) = \frac{1}{\sqrt{2}}\theta(\sqrt{2}t)$, and in the case of d_∞ we have $\theta_\infty(t) = \theta(t)$. □

Remark 3.2 For other considerations on the operator C_V see [39, 44–56], etc.

4 Iterations of the operator D_V and the difference equation $x_{n+2} = V(x_n, x_{n+1})$, $n \in \mathbb{N}$, $x_0, x_1 \in X$

Let X be a nonempty set and $V : X \times X \to X$. Let $(x_n, y_n) := D_V^n(x_0, y_0)$, $(x_0, y_0) \in X \times X$. We remark that $(x_n)_{n\in\mathbb{N}}$ is a solution of the difference equation

$$x_{n+2} = V(x_n, x_{n+1}), \quad n \in \mathbb{N}, x_0, x_1 = y_0 \in X. \tag{4.1}$$

Moreover, we have the following.

Lemma 4.1 *Let $(X, \to)$ be an L-space and $V : X \times X \to X$ a continuous operator. Then the following statements are equivalent*:

(i) *D_V is Picard with $F_{D_V} = \{(x^*, x^*)\}$*;

(ii) *D_V^2 is Picard operator with $F_{D_V^2} = \{(x^*, x^*)\}$*;

(iii) x^* *is globally asymptotically stable solution of the difference equation*

$$x_{n+2} = V(x_n, x_{n+1}), \quad n \in \mathbb{N}.$$

Proof (i) ⇔ (ii). See Lemma 2.2 in [57]. (i) ⇔ (iii). See [20, 21]. □

Theorem 4.1 *Let* (X,d) *be a complete metric space,* $V : X \times X \to X$ *and* $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$. *We suppose that*:

(i) φ *is increasing*;
(ii) $\sum_{k=0}^{\infty} \phi^k(r) < +\infty$, *where* $\phi(r) = \varphi(r,r)$, $r \in \mathbb{R}_+$;
(iii) $\varphi(r,0) + \varphi(0,r) \le \phi(r)$, $r \in \mathbb{R}_+$;
(iv) $d(V(x_0,x_1), V(x_1,x_2)) \le \varphi(d(x_0,x_1), d(x_1,x_2))$, *for all* $x_0, x_1, x_2 \in X$.

Then

(a) $F_{U_V} = \{x^*\}$.
(b) *If* $(x_n)_{n\in\mathbb{N}}$ *is a solution of the difference equation* (4.1) *then* $x_n \to x^*$ *as* $n \to +\infty$.

Proof Let $x_0, x_1 \in X$ and $(x_n)_{n\in\mathbb{N}}$ be defined by the difference equation (4.1). We have

$$\begin{aligned} d(x_2,x_3) &= d\bigl(V(x_0,x_1), V(x_1,x_2)\bigr) \le \varphi\bigl(d(x_0,x_1), d(x_1,x_2)\bigr) \\ &\le \phi\bigl(d_\infty\bigl((x_0,x_1),(x_1,x_2)\bigr)\bigr), \\ d(x_3,x_4) &= d\bigl(V(x_1,x_2), V(x_2,x_3)\bigr) \le \varphi\bigl(d(x_1,x_2), d(x_2,x_3)\bigr) \\ &\le \varphi\bigl(d_\infty\bigl((x_0,x_1),(x_1,x_2)\bigr), \phi\bigl(d_\infty\bigl((x_0,x_1),(x_1,x_2)\bigr)\bigr)\bigr) \\ &\le \phi\bigl(d_\infty\bigl((x_0,x_1),(x_1,x_2)\bigr)\bigr), \\ d(x_4,x_5) &= d\bigl(V(x_2,x_3), V(x_3,x_4)\bigr) \le \varphi\bigl(d(x_2,x_3), d(x_3,x_4)\bigr) \\ &\le \varphi\bigl(\phi\bigl(d_\infty\bigl((x_0,x_1),(x_1,x_2)\bigr)\bigr), \phi\bigl(d_\infty\bigl((x_0,x_1),(x_1,x_2)\bigr)\bigr)\bigr) \\ &= \phi^2\bigl(d_\infty\bigl((x_0,x_1),(x_1,x_2)\bigr)\bigr). \end{aligned}$$

By induction we get

$$d(x_n, x_{n+1}) \le \phi^{[\frac{n}{2}]}\bigl(d_\infty\bigl((x_0,x_1),(x_1,x_2)\bigr)\bigr),$$

thus

$$\begin{aligned} d(x_n, x_{n+p}) &\le \sum_{i=0}^{p-1} d(x_{n+i}, x_{n+i+1}) \le \sum_{i=0}^{p-1} \phi^{[\frac{n+i}{2}]}\bigl(d_\infty\bigl((x_0,x_1),(x_1,x_2)\bigr)\bigr) \\ &\le 2 \sum_{j=[\frac{n}{2}]}^{[\frac{n+p-1}{2}]} \phi^j\bigl(d_\infty\bigl((x_0,x_1),(x_1,x_2)\bigr)\bigr) \to 0 \quad \text{as } n,p \to +\infty, \end{aligned}$$

so $(x_n)_{n\in\mathbb{N}}$ is fundamental, therefore there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to +\infty$.

By the continuity assumption on φ in 0 we have

$$\begin{aligned} &d(x^*, V(x^*,x^*)) \\ &\quad\le d(x^*,x_n) + d(x_n, V(x_{n-1},x^*)) + d(V(x_{n-1},x^*), V(x^*,x^*)) \\ &\quad\le d(x^*,x_n) + \varphi(d(x_{n-2},x_{n-1}), d(x_{n-1},x^*)) + \varphi(d(x_{n-1},x^*), d(x^*,x^*)), \end{aligned}$$

thus $x^* = V(x^*,x^*)$, which means that $x^* \in F_{U_V}$.

If there exist $x^*, y^* \in F_{U_V}$ then

$$\begin{aligned} d(x^*,y^*) &= d(V(x^*,x^*), V(y^*,y^*)) \\ &\le d(V(x^*,x^*), V(x^*,y^*)) + d(V(x^*,y^*), V(y^*,y^*)) \\ &\le \varphi(0, d(x^*,y^*)) + \varphi(d(x^*,y^*), 0) \le \phi(d(x^*,y^*)), \end{aligned}$$

but $\phi(r) < r$ for all $r \in \mathbb{R}_+^*$, so $d(x^*,y^*) = 0$. □

Theorem 4.2 *Let (X,d) be a complete metric space and $V : X \times X \to X$. We suppose that there exist $l_1, l_2 \in \mathbb{R}_+$, $l_1 + l_2 < 1$ such that*

$$d(V(x_1,y_1), V(x_2,y_2)) \le l_1 d(x_1,x_2) + l_2 d(y_1,y_2), \quad \text{for all } x_i, y_i \in X, i = 1,2.$$

Then

(a) *D_V^2 is a $(l_1 + l_2)$-contraction;*
(b) *$F_{D_V} = \{(x^*,x^*)\}$ and $F_{U_V} = \{x^*\}$;*
(c) *if $(x_n)_{n\in\mathbb{N}}$ is a solution of the difference equation*

$$x_{n+2} = V(x_n, x_{n+1}), \quad n \in \mathbb{N},$$

then $x_n \to x^*$ as $x \to \infty$.

Proof (a). Let us consider the complete metric space $(X \times X, d_\infty)$ where d_∞ is defined by (3.3). We have

$$\begin{aligned} &d_\infty(D_V^2(x_1,y_1), D_V^2(x_2,y_2)) \\ &\quad= \max\{d(V(x_1,y_1), V(x_2,y_2)), d(V(y_1, V(x_1,y_1)), V(y_2, V(x_2,y_2)))\} \end{aligned}$$

and

$$\begin{aligned} &d(V(x_1,y_1), V(x_2,y_2)) \le (l_1 + l_2) \cdot d_\infty((x_1,y_1),(x_2,y_2)), \\ &d(V(y_1, V(x_1,y_1)), V(y_2, V(x_2,y_2))) \\ &\quad\le l_1 d(y_1,y_2) + l_2 d(V(x_1,y_1), V(x_2,y_2)) \\ &\quad\le (l_1 + l_2) \cdot d_\infty((x_1,y_1),(x_2,y_2)). \end{aligned}$$

Thus

$$d_\infty\big(D_V^2(x_1,y_1),D_V^2(x_2,y_2)\big) \le (l_1+l_2)\cdot d_\infty\big((x_1,y_1),(x_2,y_2)\big),$$

so D_V^2 is an (l_1+l_2)-contraction.

From Lemma 4.1 we get (b) and (c). □

For related results concerning the difference equation (4.1) see [19–22, 57, 58], etc.

5 Fixed point results for the operator T_V

A possible approach for the study of the fixed points of the operator T_V is given by the following general result.

Lemma 5.1 *Let X be a nonempty set, $(X,S_1(X),M_1)$ be a f.p.s. on X and $(X,S_2(X),M_2^0)$ be a m.f.p.s. on X. Let $Y\in S_1(X)\cap S_2(X)$ and $V:Y\times Y\to Y$. We suppose that*:

(i) $S_1(X)\cap S_2(X)\neq\emptyset$;

(ii) $V(\cdot,x)\in M_1(Y)$, *for each* $x\in Y$;

(iii) $T_V\in M_1^0(Y)$.

Then $F_{T_V}\neq\emptyset$ and $F_{U_V}=F_{T_V}$.

Proof Since $Y\in S_1(X)\cap S_2(X)$ and using (ii) we obtain $T_V(x)\neq\emptyset$, for each $x\in Y$. Since $Y\in S_2(X)$ and using (iii) we get $F_{T_V}\neq\emptyset$. On the other hand, $F_{U_V}=F_{T_V}$. □

In particular, we have the following consequences of the above approach.

Theorem 5.1 *Let X be a Banach space and $Y\in P_{cp,cv}(X)$. Let $V:Y\times Y\to Y$ be an operator such that*:

(i) $V:Y\times Y\to Y$ *is continuous*;

(ii) *the set* $\{u\in Y\mid u=V(u,x)\}$ *is convex, for each* $x\in Y$.

Then $F_{T_V}\neq\emptyset$, i.e., there exists $x^*\in Y$ such that $x^*=V(x^*,x^*)$.

Proof Since $V(\cdot,x):Y\to Y$ is continuous and $Y\in P_{cp,cv}(X)$, by Schauder's fixed point theorem, we get $F_{V(\cdot,x)}\neq\emptyset$, for each $x\in Y$. Moreover, by (ii), the set $F_{V(\cdot,x)}$ is convex, for each $x\in Y$. On the other hand, by the continuity of V, we see that the set $\{(x,u)\in Y\times Y\mid u=V(u,x)\}$ is closed in $Y\times Y$. Thus, the multi-valued operator $T_V:Y\to P(Y)$ given by $T_V(x):=F_{V(\cdot,x)}$ has a closed graphic. Since the co-domain Y is compact, T_V is upper semi-continuous on Y. Hence, we get $T_V:Y\to P_{cp,cv}(Y)$ and it is upper semi-continuous. By Bohnenblust-Karlin's fixed point theorem we get $F_{T_V}\neq\emptyset$. □

Theorem 5.2 *Let X be a uniformly convex Banach space and $Y\in P_{cp,cv}(X)$. Let $V:Y\times Y\to Y$ be an operator such that*:

(i) $V:Y\times Y\to Y$ *is continuous*;

(ii) $V(\cdot,x):Y\to Y$ *is nonexpansive, for each* $x\in Y$.

Then $F_{T_V}\neq\emptyset$.

Proof Since $V(\cdot,x):Y\to Y$ is nonexpansive and $Y\in P_{cp,cv}(X)$, by Browder-Ghöde-Kirk's fixed point theorem, we see that the set $F_{V(\cdot,x)}$ is nonempty and convex, for each $x\in Y$. On the other hand, by the continuity of V, we see that the set $\{(x,u) \in Y \times Y \mid u = V(u,x)\}$ is closed in $Y \times Y$. Thus, the multi-valued operator $T_V : Y \to P(Y)$ given by $T_V(x) := F_{V(\cdot,x)}$ has a closed graphic. Since the co-domain Y is compact, T_V is upper semi-continuous on Y. Hence, we get $T_V : Y \to P_{cp,cv}(Y)$ and it is upper semi-continuous. Our conclusion follows by Bohnenblust-Karlin's fixed point theorem. □

Theorem 5.3 *Let* $(X, \|\cdot\|)$ *be a Banach space and* $Y \in P_{cl,cv}(X)$. *Let* $V : Y \times Y \to Y$ *be an operator such that*:

(i) *there exists* $\alpha \in (0,1)$ *such that, for each* $x \in X$, *we have*

$$\big\| V(u,x) - V(v,x) \big\| \leq \alpha \|u - v\|, \quad \textit{for all } u, v \in Y;$$

(ii) *for each* $u \in Y$ *the operator* $V(u,\cdot) : Y \to Y$ *is continuous*;

(iii) *for each* $u \in Y$ *the set* $V(u,Y)$ *is relatively compact.*

Then $F_{T_V} \neq \emptyset$.

Proof Since, for every $x \in Y$, the operator $V(\cdot,x) : Y \to Y$ is a contraction, for each $x \in Y$ there exists a unique $u^* = u^*(x) \in Y$ such that $V(u^*,x) = u^*$. Thus, the operator $T_V : Y \to Y$ given by $T_V(x) := u^*(x)$ is a self single-valued operator on Y. Notice that

$$T_V(x) = V\big(u^*(x), x\big) \quad \text{for each } x \in Y.$$

Moreover, T_V is continuous since

$$\begin{aligned}
&\big\| T_V(x_1) - T_V(x_2) \big\| \\
&\quad = \big\| V\big(u^*(x_1), x_1\big) - V\big(u^*(x_2), x_2\big) \big\| \\
&\quad \leq \big\| V\big(u^*(x_1), x_1\big) - V\big(u^*(x_2), x_1\big) \big\| + \big\| V\big(u^*(x_2), x_1\big) - V\big(u^*(x_2), x_2\big) \big\| \\
&\quad \leq \alpha \big\| u^*(x_1) - u^*(x_2) \big\| + \big\| V\big(u^*(x_2), x_1\big) - V\big(u^*(x_2), x_2\big) \big\| \\
&\quad = \alpha \big\| T_V(x_1) - T_V(x_2) \big\| + \big\| V\big(u^*(x_2), x_1\big) - V\big(u^*(x_2), x_2\big) \big\|.
\end{aligned}$$

As a consequence, we get the continuity of the single-valued opertor T_V:

$$\big\| T_V(x_1) - T_V(x_2) \big\| \leq \frac{1}{1-\alpha} \cdot \big\| V\big(u^*(x_2), x_1\big) - V\big(u^*(x_2), x_2\big) \big\| \to 0 \quad \text{as } x_1 \to x_2.$$

By (iii), $T_V(Y)$ is relatively compact. Thus, by Schauder's fixed point theorem, there exists $x^* \in Y$ such that $x^* = T_V(x^*)$. As a consequence, $x^* = u^* = V(u^*, x^*) = V(x^*, x^*)$. □

Theorem 5.4 *Let* $(X, \|\cdot\|)$ *be a Banach space and* $Y \in P_{cp,cv}(X)$. *Let* $V : Y \times Y \to Y$ *be an operator such that*:

(i) *there exists* $\alpha \in (0,1)$ *such that, for each* $x \in X$, *we have*

$$\big\| V(u,x) - V\big(V(u,x), x\big) \big\| \leq \alpha \big\| u - V(u,x) \big\|, \quad \textit{for all } x, u \in Y;$$

(ii) $V : Y \times Y \to Y$ *is continuous*;

(iii) *the set* $\{u \in Y \mid u = V(u,x)\}$ *is convex, for each* $x \in Y$.

Then $F_{T_V} \neq \emptyset$.

Proof Notice first that, for every $x \in Y$, the operator $V(\cdot,x) : Y \to Y$ is a graphic contraction. Thus, for each $x \in Y$, the set $F_{V(\cdot,x)}$ is nonempty. Moreover, by the continuity of V, the set $F_{V(\cdot,x)}$ is closed. Thus, the operator $T_V : Y \to P(Y)$ given by $T_V(x) := F_{V(\cdot,x)}$ is a multi-valued operator with closed graph. Since Y is compact, we see that T_V is upper semi-continuous on Y with compact and (by (iii)) convex values. The conclusion follows by Bohnenblust-Karlin's fixed point theorem. □

Another result of this type can be reached using the above mentioned particular variant of the Eilenberg-Montgomery theorem; see Theorem 1.1.

Theorem 5.5 *Let* $(X, \|\cdot\|)$ *be a Banach space and* $Y \in P_{cp,cv}(X)$. *Let* $V : Y \times Y \to Y$ *be an operator such that*:

(i) *for each* $x \in X$ *the operator* $V(\cdot,x)$ *is nonexpansive and compact*;
(ii) *the operator* $V : Y \times Y \to Y$ *is continuous*.

Then $F_{T_V} \neq \emptyset$.

Proof Notice first that, by Theorem 1.63 in [59], the set $F_{V(\cdot,x)}$ is nonempty and acyclic, for each $x \in Y$. On the other hand, by the continuity of V, we see that the set $\{(x,u) \in Y \times Y \mid u = V(u,x)\}$ is closed in $Y \times Y$. Thus, the multi-valued operator $T_V : Y \to P(Y)$ given by $T_V(x) := F_{V(\cdot,x)}$ has a closed graphic. Since the co-domain Y is compact, T_V is upper semi-continuous on Y. Hence, we see that $T_V : Y \to P(Y)$ has acyclic values and it is upper semi-continuous. The conclusion follows by Theorem 1.1. □

Remark 5.1 For the case of the Eilenberg-Montgomery fixed point theorem see [15, 34, 35], etc.

Remark 5.2 For the fixed point theory of multivalued operators see [14, 15, 17, 18], etc.

6 Applications

6.1 Fredholm type integral equations

Let $\Omega \subset \mathbb{R}^m$ be a bounded domain and $C(\overline{\Omega})$ be the Banach space with

$$\|x\|_\infty := \max_{t \in \overline{\Omega}} \bigl|x(t)\bigr|.$$

We consider the integral equation, in $C(\overline{\Omega})$,

$$x(t) = \int_\Omega K\bigl(t,s,x(s),x(s)\bigr)\,ds, \quad t \in \overline{\Omega}, \tag{6.1}$$

where $K \in C(\overline{\Omega}, \overline{\Omega} \times \mathbb{R}^2)$.

Let us consider now the operator $V : C(\overline{\Omega}) \times C(\overline{\Omega}) \to C(\overline{\Omega})$ defined by

$$V(x,y)(t) := \int_\Omega K\bigl(t,s,x(s),y(s)\bigr)\,ds.$$

By Theorem 4.1 we have the following.

Theorem 6.1 *We suppose that*:

(i) $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ *satisfies conditions* (i)-(iii) *in Theorem* 4.1;

(ii) $|K(t,s,u,v) - K(t,s,v,w)| \leq \frac{1}{\mathrm{mes}(\Omega)}\varphi(|u-v|,|v-w|)$ *for all* $t,s \in \overline{\Omega}$, $u,v,w \in \mathbb{R}$.

Then:

(a) *Equation* (6.1) *has a unique solution*, $x^* \in C(\overline{\Omega})$.

(b) *The sequence* $(x_n)_{n\in\mathbb{N}}$, *defined by*

$$x_{n+2}(t) = \int_\Omega K\bigl(t,s,x_n(s),x_{n+1}(s)\bigr)\,ds, \quad t \in \overline{\Omega},$$

converges to x^* *for all* $x_0, x_1 \in C(\overline{\Omega})$.

6.2 A periodic boundary value problem

We will consider now a periodic boundary value problem of the following type:

$$\begin{cases} -x'' = f(t,x,x), & t \in [a,b], \\ x(a) = 0, & x(b) = 0, \end{cases} \tag{6.2}$$

where $f : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is a given continuous function.

This problem is equivalent to a Fredholm type integral equation of the following form:

$$x(t) = \int_a^b G(t,s)f\bigl(s,x(s),x(s)\bigr), \quad t \in [a,b],$$

where $G : [a,b] \times [a,b] \to \mathbb{R}_+$ is the corresponding Green function.

Let us define now the operator $V : C[a,b] \times C[a,b] \to C[a,b]$ defined by

$$V(x,y)(t) := \int_a^b G(t,s)f\bigl(s,x(s),y(s)\bigr)\,ds.$$

By Theorem 4.1 we have the following.

Theorem 6.2 *We suppose that*:

(i) $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ *satisfies conditions* (i)-(iii) *in Theorem* 4.1;

(ii) $|f(s,u,v) - f(s,v,w)| \leq \frac{8}{(b-a)^2} \cdot \varphi(|u-v|,|v-w|)$ *for all* $t,s \in [a,b]$, $u,v,w \in \mathbb{R}$.

Then:

(a) *The boundary value problem* (6.2) *has a unique solution*, x^*.

(b) *The sequence* $(x_n)_{n\in\mathbb{N}}$, *defined by*

$$x_{n+2}(t) = \int_a^b f\bigl(s,x_n(s),x_{n+1}(s)\bigr)\,ds, \quad t \in [a,b]$$

converges to x^* *for all* $x_0, x_1 \in C[a,b]$.

6.3 Other applications

Other applications of the abstract results given in this paper can be obtained for the case of functional-differential equations and functional-integral equations (or inclusions) which appear in [6, 21, 36, 44, 53, 60], etc.

7 Research directions

7.1 Mixed monotone operators

Let $(X, \leq)$ be an ordered set and $U : X \to X$. Under these conditions there exists $V : X \times X \to X$ such that:

(i) $V(\cdot, x)$ is increasing;

(ii) $V(x, \cdot)$ is decreasing;

(iii) $U = U_V$.

References: [2, 29, 30], etc.

7.2 Difference equations for diagonal operators

Let $(X, \to)$ be an L-space and $U : X \to X$ an operator. Under which conditions is U a diagonal operator with respect to some $V : X \times X \to X$ such that each solution $(x_n)_{n\in\mathbb{N}}$ of the difference equation

$$x_{n+1} = V(x_n, x_{n+1}), \quad n \in \mathbb{N},$$

converges to a fixed point of U?

References: [19, 20, 22, 57, 61, 62], etc.

7.3 Fixed point structures approach to diagonal operators

Let $(X, S(X), M)$ be a fixed point structure on X, $Y \in S(X)$ and $V : Y \times Y \to Y$. Under which conditions on V do we have $U_V \in M(Y)$?

Commentaries Let X be a Banach space, $(X, P_{b,cl,cv}(X), M)$ the fixed point structure of Darbo. Let $Y \in P_{b,cl,cv}(X)$ and $V : Y \times Y \to Y$. We suppose that:

(i) V is continuous;

(ii) $V(\cdot, y)$ is a l-contraction, for all $y \in Y$;

(iii) $V(x, \cdot)$ is compact, for all $x \in Y$.

Then $U_V \in M(Y)$ (see [12, 13, 27]).

References: [12, 24–27], etc.

Competing interests

All the authors have read Springer Open's guidance on competing interests and we confirm that none of the authors have any competing interests in the manuscript.

Authors' contributions

All the authors have made equal contributions to this work. All authors read and approved the final manuscript.

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