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Proofs to two inequality conjectures for a point on the plane of a triangle

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Abstract

We prove two conjectures for a point on the plane of a triangle presented in (Liu in J. Math. Inequal. 8(3):597-611, 2014, doi:10.1007/s11590-013-0708-4) by using the successive difference substitution algorithm NEWTSDS. Compared with the original proof, the new one is simpler and more easily understood. Similar problems can be treated with the same procedure.

MSC: 51M16; 12-04

Keywords: geometry inequality; successive difference substitution; automated reasoning

1 Introduction

Liu [1] put forward four conjectures for a point on the plane of a triangle. For Conjectures 3 and 4 in its last part, we conclude that they are correct. We prove them with the help of a successive difference substitution algorithm called NEWTSDS [2].

As showed in Figure 1, P is a point on the plane of a triangle ABC, which means that P is not limited only inside the triangle. Let (x, y, z) be the barycentric coordinates of P with respect to the triangle ABC. Let D, E, F be the foots of the perpendicular to the three sides BC, CA, AB of $\triangle ABC$ from P, respectively. As in [1], let a, b, c, R_1, R_2, R_3, r_1, r_2, r_3 indicate the lengths of BC, CA, AB, PA, PB, PC, PD, PE, PF, respectively.

Conjectures 3 and 4 in [1] state that

$$\frac{2R_1^2 - r_2^2 - r_3^2}{(b+c)^2} + \frac{2R_2^2 - r_3^2 - r_1^2}{(c+a)^2} + \frac{2R_3^2 - r_1^2 - r_2^2}{(a+b)^2} \geq \frac{3}{8} \tag{1}$$

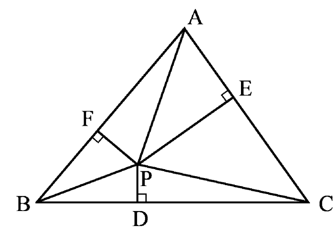
and

$$\frac{R_2^2 + R_3^2 - r_2^2 - r_3^2}{(b+c)^2} + \frac{R_3^2 + R_1^2 - r_3^2 - r_1^2}{(c+a)^2} + \frac{R_1^2 + R_2^2 - r_1^2 - r_2^2}{(a+b)^2} \geq \frac{3}{8}. \tag{2}$$

These two inequalities can be considered as some kind of promotion of the famous Erdös-Mordell inequality [3], which states some relationship between a point and a triangle on the same plane. Mathematicians have presented different proofs, such as [4–6], to the original Erdös-Mordell inequality and some generalizations [7, 8].

The rest of this paper is arranged as follows. In Section 2, a well-known equivalent condition of the nonnegativity of quadratics is recalled, and a computer aided reasoning tool,

Figure 1 Triangle *ABC* and point *P*. Point *P* and triangle *ABC* are on the same plane. The point may be inside or outside the triangle.



the algorithm called NEWTSDS based on the successive difference substitution method, is introduced. The proofs to the two conjectures are discoursed in Section 3. The proofs also show that the algorithm NEWTSDS saves us a lot of effort to avoid tedious calculation, so that we can concentrate on the mainline of the process.

2 Proving tools

In this section, the symbols are not specified, which means that they do not denote anything as in the first section. For any quadratic $f(x) = ax^2 + bx + c$ with $a, b, c, x \in \mathcal{R}$ (real field), we have

$$\forall x \in \mathcal{R}, \quad f(x) \geq 0 \quad \Leftrightarrow \quad a \geq 0 \wedge c \geq 0 \wedge 4ac - b^2 \geq 0. \tag{3}$$

Based on the original successive difference substitution method [9] and its sequels [10], in [2] the algorithm called NEWTSDS is presented, which uses the barycentric matrices obtained by performing elementary row interchanges on the matrix

$$T_n = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ 0 & \frac{1}{2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{n} \\ 0 & \cdots & 0 & \frac{1}{n} \end{bmatrix}_{n \times n}$$

as the substitution matrices. It is a complete tool for the decision of the nonnegativity of polynomials. We use it during the proving process. This is a practical and most efficient method we can perform till now.

3 Main result: proofs to the two conjectures

In this section, all the symbols have the same meanings as in Section 1. Since the barycentric coordinates are a form of homogeneous coordinates, without loss of generality, we can restrict (x, y, z) with the condition

$$x + y + z = 1. \tag{4}$$

There are some other needed substitution equations, which are well known and just listed here without proofs:

$$R_1 = \sqrt{\left(b^2z + c^2y\right) - a^2yz - b^2xz - c^2xy}, \tag{5}$$

$$R_2 = \sqrt{\left(a^2z + c^2x\right) - a^2yz - b^2xz - c^2xy}, \tag{6}$$

$$R_3 = \sqrt{\left(ya^2 + xb^2\right) - a^2yz - b^2xz - c^2xy}, \tag{7}$$

$$r_1 = \left|\frac{2xS}{a}\right|, \qquad r_2 = \left|\frac{2yS}{b}\right|, \qquad r_3 = \left|\frac{2zS}{c}\right|, \tag{8}$$

$$S = \sqrt{s(s-a)(s-b)(s-c)}, \tag{9}$$

$$a = u + v, \qquad b = v + w, \qquad c = w + u. \tag{10}$$

In (9), S and s denote the area and the half of the perimeter of $\triangle ABC$, respectively. In (10), u, v, w are general nonnegative real numbers.

Let f_1 and f_2 denote the differences between the left and right sides of (1) and (2), respectively:

$$f_1 = \frac{2R_1^2 - r_2^2 - r_3^2}{(b+c)^2} + \frac{2R_2^2 - r_3^2 - r_1^2}{(c+a)^2} + \frac{2R_3^2 - r_1^2 - r_2^2}{(a+b)^2} - \frac{3}{8}, \tag{11}$$

$$f_2 = \frac{R_2^2 + R_3^2 - r_2^2 - r_3^2}{(b+c)^2} + \frac{R_3^2 + R_1^2 - r_3^2 - r_1^2}{(c+a)^2} + \frac{R_1^2 + R_2^2 - r_1^2 - r_2^2}{(a+b)^2} - \frac{3}{8}. \tag{12}$$

Obviously, the correctness of (1) and (2) is equivalent to the positive semi-definiteness of (11) and (12), respectively.

Substituting (4)-(10) into (11) and (12) and removing the nonnegative denominator, we get the equivalent polynomials f_{1n} and f_{2n} of u, v, w, x, y:

$$f_{1n} = g_1x^2 + g_2x + g_3, \qquad f_{2n} = h_1x^2 + h_2x + h_3,$$

where g_1, h_1 are polynomials of u, v, w, and g_i, h_i $(i = 2, 3)$ are polynomials of u, v, w, y. This means the positive semi-definiteness of f_1 and f_2 is equivalent to the positive semi-definiteness of f_{1n} and f_{2n}, respectively, when $u, v, w \geq 0$. For better performance, we write all the long equations in Section 5.

Remark 1 The barycentric coordinate system is a widely used tool, which transforms geometric problems to algebraic ones for which lots of computer-aided solvers are available. As shown in the following steps, we can focus ourselves on the main line of proofs and employ some automated reasoning package to deal with the less important branches.

According to condition (3), when $x, y \in \mathcal{R}$ and $u, v, w \geq 0$, f_{1n}, f_{2n} are both nonnegative if and only if g_1, h_1, g_3, h_3, $\Delta_1 = 4g_1g_3 - g_2^2$ and $\Delta_2 = 4h_1h_3 - h_2^2$ are all nonnegative.

From the expressions it is easy to find that $g_1 = h_1$ and they are both nonnegative. In the following two subsections, we discourse the nonnegativity of g_3, h_3, Δ_1, and Δ_2 when $y \in \mathcal{R}$, $u, v, w \geq 0$.

3.1 Nonnegativity of g_3 and h_3

Since g_3 and h_3 are both quadratics of y, we can rewrite them as follows:

$$g_3 = (u+v)^2\left(g_{31}y^2 + g_{32}y + g_{33}\right), \tag{13}$$

$$h_3 = (u+v)^2\left(h_{31}y^2 + h_{32}y + h_{33}\right). \tag{14}$$

From the expressions we find that $h_{31} = g_{31}$, g_{33}, and h_{33} are all nonnegative. Denote

$$\delta_g = 4g_{31}g_{33} - g_{32}^2 = 64(w+u)^2(v+w)^2(v+2u+w)^2(u+w+2v)^2\delta_{g3},$$
$$\delta_h = 4h_{31}h_{33} - h_{32}^2 = 64(w+u)^2(v+w)^2\delta_{h3}.$$

Following the NEWTSDS algorithm, we can get the positive semi-definiteness of δ_{g3} and δ_{h3} by iterating four loops and five loops, respectively. Hence, we get the nonnegativity of g_3 and h_3 according to (3).

Remark 2 We can perform the substitution procedure based on the successive difference substitution method by computer. For ternary forms, in every loop, the number of substitutions is at most $3! = 6$ to the power of the ordinal of the current loop, which means at most 6^n substitution steps are needed for the nth loop. Actually, many examples show that fewer substitutions are needed. Bounds on the number of steps of substitutions for algorithms based on the successive difference substitution method are presented in [11, 12]. The TSDS5 package for the Maple platform developed by L Yang and Y Yao is available at http://staff.uestc.edu.cn/huangfangjian/?page_id=27. We can use it to check the nonnegativity of polynomials or get a counterpoint when it does not hold.

3.2 Nonnegativity of Δ_1 and Δ_2

Since Δ_1 and Δ_2 are quadratics of y too, we can perform similar operations as before. First, factor and rewrite Δ_1 and Δ_2 as follows:

$$\Delta_1 = 64(v+u)^2(w+u)^2(v+w)^2\left(\varphi_1 y^2 + \varphi_2 y + \varphi_3\right),$$
$$\Delta_2 = 64(v+u)^2(w+u)^2(v+w)^2\left(\psi_1 y^2 + \psi_2 y + \psi_3\right).$$

Then by simple calculation we have

$$\varphi_1 = 64u^2v^2w^2(w+v+u)^2\varphi_{11},$$
$$\varphi_2 = 128wvu(v+w)^2(v+w+u)(v+u+2w)^2(u+w+2v)^2\varphi_{21},$$
$$\varphi_3 = (v+w)^2(v+u+2w)^2(u+w+2v)^2\varphi_{31},$$
$$\psi_1 = \varphi_1,$$
$$\psi_2 = 32wvu(v+w)^2(v+w+u)\psi_{21},$$
$$\psi_3 = (v+w)^2\psi_{31}.$$

From the expression of φ_{11} we find that all the coefficients are positive. Then we have the nonnegativity of φ_1 and ψ_1. Using NEWTSDS, we can obtain the positive semi-definiteness of φ_{31} and ψ_{31} by iterating four and five loops, respectively. Following this, we acquire the nonnegativity of φ_3 and ψ_3. Denote

$$\delta_\varphi = 4\varphi_1\varphi_3 - \varphi_2^2,$$
$$\delta_\psi = 4\psi_1\psi_3 - \psi_2^2.$$

Computation shows that

$$\begin{aligned}\delta_\varphi &= \delta_\psi \\ &= 256v^2u^2w^2(v+w)^2(u+w+2v)^2(u+v+w)^2 \\ &\quad \cdot (u+v+2w)^2(2u+w+v)^2\delta_{\varphi 1}\delta_{\varphi 2}.\end{aligned}$$

Only one loop of NEWTSDS is needed for $\delta_{\varphi 1}$ to check its nonnegativity. From the expression it is easy to find that $\delta_{\varphi 2}$ is positive semi-definite because all the coefficients of $\delta_{\varphi 2}$ with respect to u are positive semi-definite. So we have the nonnegativity of δ_φ and δ_ψ. According to condition (3), Δ_1 and Δ_2 are both nonnegative because φ_1, φ_3, δ_φ, ψ_1, ψ_3, δ_ψ are all nonnegative.

So, the nonnegativity of all g_1, g_3, h_1, h_3, Δ_1, and Δ_2 is achieved. This, we obtain the positive semi-definiteness of f_{1n} and f_{2n} based on the equivalence (3). Equivalently, we acquire the nonnegativity of f_1 and f_2. Furthermore, we finish the proofs of conjectures (1) and (2).

4 Conclusions

In this paper, two interesting conjectures are proved. These two inequalities are about a point and a triangle on the same plane. They describe some relationships between the lengths of three sides of the triangle, the distances from the point to the three vertices and to the three sides of the triangle. The whole procedure is achieved by using the traditional analytical method and some computer-aided tool. First, we transform these two geometric problems into algebraic ones by using the barycentric coordinate system. After analyzing f_{1n} and f_{2n} as general polynomials of x, we find their equivalent problems, the nonnegativity of g_1, g_3, h_1, h_3, Δ_1, and Δ_2. Actually, this is a process of reducing indeterminate. Repeating this process to eliminate y, we get some polynomials of three variables u, v, w. After that, NEWTSDS is employed to check their positive semi-definiteness, and then we finish the proofs of the conjectures. We can solve some other problems in a similar way just using an analytical method to control the main line and leave branches to the computer. It is easy to read and easy to reproduce.

5 Summarization of long expressions

We summarize all the explicit expressions that are too long to be listed in the main body:

$$\begin{aligned}g_1 &= 16(v+w)^2\big((8v^2+8w^2+(v-w)^2)u^8+(56v^3+40v^2w+40vw^2 \\ &\quad +56w^3)u^7+(144v^4+222v^3w+248v^2w^2+222vw^3+144w^4)u^6 \\ &\quad +(456v^4w+194v^5+456vw^4+770v^3w^2+194w^5+770v^2w^3)u^5 \\ &\quad +(144v^6+468v^5w+1174v^4w^2+1{,}688v^3w^3+1{,}174v^2w^4+468vw^5 \\ &\quad +144w^6)u^4+(56v^7+246v^6w+954v^5w^2+2{,}016v^4w^3+2{,}016v^3w^4 \\ &\quad +954v^2w^5+246vw^6+56w^7)u^3+(9v^8+56v^7w+412v^6w^2+1{,}298v^5w^3 \\ &\quad +1{,}870v^4w^4+1{,}298v^3w^5+412v^2w^6+56vw^7+9w^8)u^2+(2v^8w+88v^7w^2 \\ &\quad +422v^6w^3+844v^5w^4+844v^4w^5+422v^3w^6+88v^2w^7+2vw^8)u+9v^8w^2 \\ &\quad +56v^7w^3+194v^5w^5+144v^4w^6+56v^3w^7+144v^6w^4+9v^2w^8\big), \qquad (15)\end{aligned}$$

$$
\begin{aligned}
g_2 = {} & 32(v+w)^2(v+u)^2\Big(\big((9v-9w)u^7+\big(47v^2-56w^2+17vw\big)u^6+\big(97v^3 \\
& +161v^2w-144w^3-34vw^2\big)u^5+\big(295wv^3+248w^2v^2-194w^4-124w^3v \\
& +97v^4\big)u^4+\big(432v^3w^2-144w^5-124vw^4+268v^2w^3+217v^4w+47v^5\big)u^3 \\
& +\big(248w^4v^2-56w^6+9v^6+432w^3v^3+61v^5w-34vw^5+256w^2v^4\big)u^2 \\
& +\big(295v^3w^4-9w^7+161v^2w^5+17vw^6+2v^6w+61v^5w^2+217v^4w^3\big)u \\
& +47v^5w^3+97v^3w^5+9v^6w^2+97v^4w^4+47v^2w^6+9vw^7\big)y-\big(23u^2v^2w^2 \\
& -u^5w+2uw^5-2u^4w^2-u^3w^3+u^2w^4+8vw^5+w^6+25u^2w^3v+22uw^4v \\
& +13u^3wv^2+2w^2v^4+14w^4v^2+7u^3v^3+7u^4v^2+2u^2v^4+9w^3v^3+7u^2wv^3 \\
& +vu^5+3u^4wv+9uw^2v^3+13u^3w^2v+27uw^3v^2\big)(u+w+2v)^2\Big), \qquad (16)
\end{aligned}
$$

$$
g_3 = (v+u)^2\big(g_{31}y^2+g_{32}y+g_{33}\big), \qquad (17)
$$

$$
\begin{aligned}
g_{31} = {} & 16\big(8\big(u^2+v^2\big)+(u-v)^2\big)w^8+128(u+v)\big(6\big(u^2+v^2\big)+(u-v)^2\big)w^7 \\
& +\big(2{,}304u^4+3{,}968v^2u^2+3{,}552v^3u+3{,}552vu^3+2{,}304v^4\big)w^6 \\
& +32(u+v)\big(97u^4+131vu^3+254v^2u^2+131v^3u+97v^4\big)w^5+\big(2{,}304u^6 \\
& +18{,}784v^2u^4+27{,}008v^3u^3+18{,}784v^4u^2+7{,}488v^5u+7{,}488vu^5+2{,}304v^6\big)w^4 \\
& +32(u+v)\big(28u^6+95vu^5+382v^2u^4+626v^3u^3+382v^4u^2+95v^5u \\
& +28v^6\big)w^3+\big(144u^8+6{,}592v^2u^6+20{,}768v^3u^5+29{,}920v^4u^4+6{,}592v^6u^2 \\
& +20{,}768v^5u^3+896v^7u+896vu^7+144v^8\big)w^2 \\
& +32vu(u+v)\big(u^6+43vu^5+168v^2u^4+254v^3u^3+168v^4u^2+43v^5u+v^6\big)w \\
& +16v^2u^2\big(9v^4+38v^3u+59v^2u^2+38vu^3+9u^4\big)(u+v)^2, \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
g_{32} = {} & -32(v+w)^2(u+w+2v)^2\big(\big(2w^2+2u^2\big)v^4+(w+u)\big(7w^2+9u^2\big)v^3 \\
& +\big(23u^2w^2+7w^4+14u^4+27u^3w+13uw^3\big)v^2+(w+u)\big(w^4+2uw^3 \\
& +11u^2w^2+14u^3w+8u^4\big)v+u(u-w)\big(u^2+uw+w^2\big)(w+u)^2\big), \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
g_{33} = {} & (v+w)^2(u+w+2v)^2\big(\big(26w^2+26u^2+3(u-w)^2\big)v^4+2\big(46\big(w^2 \\
& +u^2\big)+9(u-w)^2\big)(w+u)v^3+\big(121w^4+121u^4+202u^2w^2+190u^3w \\
& +190uw^3\big)v^2+2(w+u)\big(14w^4+31uw^3+66u^2w^2+31u^3w+14u^4\big)v \\
& +\big(4u^4+4u^3w+29u^2w^2+4uw^3+4w^4\big)(w+u)^2\big), \qquad (20)
\end{aligned}
$$

$$
h_1 = g_1,
$$

$$
\begin{aligned}
h_2 = {} & 16(v+w)^2(u+v)^2\Big(\big((18v-18w)u^7+\big(-112w^2+94v^2+34wv\big)u^6 \\
& +\big(-68vw^2-288w^3+322v^2w+194v^3\big)u^5+\big(590v^3w+496v^2w^2 \\
& -248vw^3+194v^4-388w^4\big)u^4+\big(536v^2w^3+434v^4w-248vw^4+94v^5 \\
& -288w^5+864v^3w^2\big)u^3+\big(122v^5w+512v^4w^2-112w^6-68vw^5+864v^3w^3 \\
& +18v^6+496v^2w^4\big)u^2+\big(434v^4w^3+590v^3w^4+122v^5w^2+34vw^6+4v^6w \\
& -18w^7+322v^2w^5\big)u+94v^2w^6+18vw^7+94v^5w^3+194v^4w^4+194v^3w^5
\end{aligned}
$$

$$+ 18v^6w^2\big)y + 8(w-v)u^7 + \big(43w^2 - 45v^2 - 14wv\big)u^6 + \big(82w^3 - 16vw^2$$
$$- 96v^3 - 130v^2w\big)u^5 + \big(-134vw^3 + 47w^4 - 200v^3w - 259v^2w^2 - 98v^4\big)u^4$$
$$- \big(542v^2w^3 + 377vw^4 + 49v^5 + 99v^4w + 278v^3w^2 + 47w^5\big)u^3$$
$$+ \big(3v^5w - 10v^6 - 83v^4w^2 - 447vw^5 - 470v^3w^3 - 82w^6 - 743v^2w^4\big)u^2$$
$$+ w(w+v)\big(-43w^5 - 204vw^4 - 312v^2w^3 - 178v^3w^2 - 19v^4w + 12v^5\big)u$$
$$- w^2(w+2v)\big(8w^3 + 21vw^2 + 17v^2w + 5v^3\big)(w+v)^2\big), \quad (21)$$

$$h_3 = (u+v)^2\big(h_{31}y^2 + h_{32}y + h_{33}\big), \quad (22)$$

$$h_{31} = g_{31},$$

$$h_{32} = -16(v+w)^2\big(\big(10u^2 + 10w^2 - 12uw\big)v^6 + (w+u)\big(49w^2 - 52uw$$
$$+ 59u^2\big)v^5 + \big(98w^4 + 147u^4 + 83u^2w^2 + 99uw^3 + 197u^3w\big)v^4$$
$$+ 2\big(48w^4 + 52uw^3 + 87u^2w^2 + 148u^3w + 97u^4\big)(w+u)v^3 + \big(141u^4$$
$$+ 234u^3w + 134u^2w^2 + 40uw^3 + 45w^4\big)(w+u)^2v^2 + \big(53u^5 + 141u^4w$$
$$+ 112u^3w^2 + 12u^2w^3 - 2uw^4 + 8w^5\big)(w+u)^2v + u(-w+u)\big(8w^4$$
$$+ 35uw^3 + 55u^2w^2 + 35u^3w + 8u^4\big)(w+u)^2\big), \quad (23)$$

$$h_{33} = (v+w)^2\big(\big(8\big(w^2+u^2\big) + 60(u-w)^2\big)v^6$$
$$+ 4(w+u)\big(14\big(w^2+u^2\big) + 73(u-w)^2\big)v^5$$
$$+ \big(444\big(w^4+u^4\big) + 357\big(u^2-w^2\big)^2 + 316uw\big(u^2+w^2\big)\big)v^4$$
$$+ 2(w+u)\big(420\big(w^4+u^4\big) + 93\big(u^2-w^2\big)^2 + 292uw\big(u^2+w^2\big)\big)v^3$$
$$+ \big(753u^4 + 614u^3w + 78u^2w^2 + 614uw^3 + 753w^4\big)(w+u)^2v^2$$
$$+ 6\big(50u^4 + 53u^3w + 12u^2w^2 + 53uw^3 + 50w^4\big)(w+u)^3v + \big(52u^6 + 172u^5w$$
$$+ 177u^4w^2 + 98u^3w^3 + 177u^2w^4 + 172uw^5 + 52w^6\big)(w+u)^2\big), \quad (24)$$

$$\delta_{g3} = \big(5w^2 + 5v^2 - 6vw\big)u^{10} + \big(34v^3 + 44w^3 + 10w^2v - 64v^2w\big)u^9$$
$$+ \big(133v^4 + 196w^4 - 372v^3w - 67v^2w^2 + 330vw^3\big)u^8 + \big(306v^5$$
$$+ 538w^5 - 686v^3w^2 + 1{,}398v^2w^3 + 1{,}596vw^4 - 976v^4w\big)u^7$$
$$+ \big(404v^6 + 888w^6 + 2{,}834v^3w^3 - 1{,}462v^5w + 5{,}830v^2w^4 + 3{,}780vw^5$$
$$- 1{,}666v^4w^2\big)u^6 + \big(306v^7 + 832w^7 + 4{,}926vw^6 + 10{,}740v^2w^5$$
$$+ 11{,}016v^3w^4 + 3{,}970v^4w^3 - 1{,}462v^6w - 2{,}104v^5w^2\big)u^5 + \big(133v^8$$
$$+ 421w^8 + 16{,}686v^3w^5 - 1{,}666v^6w^2 + 3{,}970v^5w^3$$
$$+ 10{,}996v^2w^6 - 976v^7w + 3{,}568vw^7 + 13{,}444v^4w^4\big)u^4 + \big(116w^9$$
$$+ 13{,}932v^3w^6 + 2{,}834v^6w^3 - 372v^8w + 34v^9 + 6{,}608v^2w^7 + 16{,}686v^4w^5$$
$$+ 11{,}016v^5w^4 - 686v^7w^2 + 1{,}448vw^8\big)u^3 + \big(20w^{10} + 5v^{10} + 332w^9v$$
$$- 64v^9w + 5{,}830v^6w^4 + 10{,}740v^5w^5 + 6{,}608v^3w^7 - 67v^8w^2 + 2{,}294v^2w^8$$

$$
\begin{aligned}
&+ 10{,}996v^4w^6 + 1{,}398v^7w^3\big)u^2 - 2vw(2w+v)(w+v)\big(-6w^7 - 74vw^6 \\
&- 248v^2w^5 - 483v^3w^4 - 383v^4w^3 - 129v^5w^2 - 14v^6w + 3v^7\big)u \\
&+ v^2w^2\big(5w^4 + 14vw^3 + 47v^2w^2 + 14v^3w + 5v^4\big)(2w+v)^2(w+v)^2, \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
\delta_{h3} = {}& \big(212w^2 + 212u^2 - 408wu\big)v^{14} + \big(2{,}220w^3 + 2{,}220u^3 - 1{,}916w^2u - 2{,}300wu^2\big)v^{13} \\
&+ \big(10{,}805w^4 + 10{,}805u^4 - 19{,}714w^2u^2 - 2{,}732wu^3 + 2{,}068w^3u\big)v^{12} + \big(32{,}490w^5 \\
&+ 32{,}490u^5 - 69{,}996w^2u^3 + 13{,}570wu^4 + 41{,}314w^4u - 45{,}964w^3u^2\big)v^{11} \\
&+ \big(67{,}597w^6 + 67{,}597u^6 + 162{,}758w^5u - 144{,}081w^2u^4 + 66{,}038wu^5 \\
&+ 18{,}191w^4u^2 - 221{,}412w^3u^3\big)v^{10} + \big(102{,}990u^7 + 102{,}990w^7 - 206{,}730w^2u^5 \\
&- 211{,}210w^4u^3 + 147{,}758wu^6 - 546{,}970w^3u^4 + 374{,}070w^6u + 351{,}630w^5u^2\big)v^9 \\
&+ \big(118{,}172u^8 + 118{,}172w^8 + 417{,}882w^5u^3 - 236{,}233w^2u^6 - 691{,}326w^4u^4 \\
&+ 588{,}982w^7u + 212{,}666wu^7 - 923{,}410w^3u^5 + 988{,}871w^6u^2\big)v^8 + \big(102{,}990u^9 \\
&+ 102{,}990w^9 + 268{,}456w^5u^4 - 242{,}348w^2u^7 + 212{,}666wu^8 - 1{,}213{,}872w^4u^5 \\
&+ 671{,}426w^8u + 1{,}623{,}572w^7u^2 - 1{,}177{,}156w^3u^6 + 1{,}712{,}716w^6u^3\big)v^7 \\
&+ \big(67{,}597u^{10} + 67{,}597w^{10} + 562{,}874w^9u + 1{,}809{,}843w^8u^2 - 236{,}233w^2u^8 \\
&+ 147{,}758wu^9 - 1{,}442{,}604w^4u^6 + 2{,}908{,}276w^7u^3 - 1{,}177{,}156w^3u^7 \\
&+ 2{,}249{,}152w^6u^4 + 72{,}256w^5u^5\big)v^6 + 2(w+u)\big(16{,}245w^{10} + 155{,}512w^9u \\
&+ 554{,}255w^8u^2 + 984{,}618w^7u^3 + 917{,}616w^6u^4 + 301{,}498w^5u^5 - 265{,}370w^4u^6 \\
&- 341{,}566w^3u^7 - 120{,}139w^2u^8 + 16{,}774wu^9 + 16{,}245u^{10}\big)v^5 \\
&+ (w+u)^2v^4\big(10{,}805u^{10} - 8{,}040wu^9 - 138{,}806w^2u^8 - 261{,}318w^3u^7 \\
&- 29{,}884w^4u^6 + 589{,}542w^5u^5 + 1{,}099{,}952w^6u^4 + 1{,}015{,}022w^7u^3 \\
&+ 512{,}862w^8u^2 + 125{,}664w^9u + 10{,}805w^{10}\big) + 2(w+u)^2v^3\big(1{,}110u^{11} - 3{,}586wu^{10} \\
&- 28{,}936w^2u^9 - 49{,}248w^3u^8 + 21{,}827w^4u^7 + 214{,}535w^5u^6 + 405{,}461w^6u^5 \\
&+ 428{,}681w^7u^4 + 276{,}050w^8u^3 + 102{,}424w^9u^2 + 18{,}420w^{10}u + 1{,}110w^{11}\big) \\
&+ \big(212w^{12} + 212u^{12} + 177{,}228w^9u^3 + 371{,}841w^8u^4 + 498{,}624w^7u^5 \\
&- 14{,}284w^3u^9 + 61{,}237w^4u^8 + 48{,}698w^{10}u^2 + 6{,}124w^{11}u + 440{,}754w^6u^6 \\
&- 14{,}478w^2u^{10} - 2{,}724wu^{11} + 243{,}440w^5u^7\big)(w+u)^2v^2 \\
&- 2wuv(w+u)^3\big(204u^{10} + 346wu^9 - 2{,}684w^2u^8 - 13{,}847w^3u^7 - 32{,}132w^4u^6 \\
&- 46{,}414w^5u^5 - 45{,}006w^6u^4 - 29{,}321w^7u^3 - 12{,}042w^8u^2 \\
&- 2{,}662w^9u - 204w^{10}\big) + w^2u^2\big(212u^8 + 1{,}372wu^7 + 4{,}045w^2u^6 + 7{,}230w^3u^5 \\
&+ 8{,}707w^4u^4 + 7{,}230w^5u^3 + 4{,}045w^6u^2 + 1{,}372w^7u + 212w^8\big)(w+u)^4, \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
\varphi_{11} = {}& 80v^{10} + (752w + 752u)v^9 + \big(6{,}042uw + 3{,}245w^2 + 3{,}245u^2\big)v^8 \\
&+ 4(u+w)\big(2{,}109u^2 + 3{,}442uw + 2{,}109w^2\big)v^7 + \big(69{,}240u^2w^2 + 48{,}988u^3w \\
&+ 48{,}988uw^3 + 14{,}576w^4 + 14{,}576u^4\big)v^6 + 2(u+w)\big(8{,}713u^4 + 27{,}030u^3w
\end{aligned}
$$

$$
\begin{aligned}
&+ 37{,}466u^2w^2 + 27{,}030uw^3 + 8{,}713w^4\bigr)v^5 + \bigl(14{,}576w^6 + 14{,}576u^6 + 71{,}486uw^5 \\
&+ 71{,}486u^5w + 200{,}992u^3w^3 + 157{,}438u^4w^2 + 157{,}438u^2w^4\bigr)v^4 \\
&+ 4(u+w)\bigl(2{,}109u^6 + 10{,}138u^5w + 22{,}110u^4w^2 + 28{,}138u^3w^3 + 22{,}110u^2w^4 \\
&+ 10{,}138uw^5 + 2{,}109w^6\bigr)v^3 + \bigl(69{,}240u^2w^6 + 22{,}204u^7w + 69{,}240u^6w^2 \\
&+ 22{,}204uw^7 + 128{,}992u^3w^5 + 128{,}992u^5w^3 + 157{,}438u^4w^4 + 3{,}245u^8 \\
&+ 3{,}245w^8\bigr)v^2 + 2(u+w)\bigl(376u^8 + 2{,}645u^7w + 8{,}457u^6w^2 + 16{,}037u^5w^3 \\
&+ 19{,}706u^4w^4 + 16{,}037u^3w^5 + 8{,}457u^2w^6 + 2{,}645uw^7 + 376w^8\bigr)v \\
&+ \bigl(80w^8 + 592uw^7 + 1{,}981u^2w^6 + 3{,}882u^3w^5 + 4{,}831u^4w^4 \\
&+ 3{,}882u^5w^3 + 1{,}981u^6w^2 + 592u^7w + 80u^8\bigr)(u+w)^2, \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
\varphi_{21} ={}& (w-u)v^7 + \bigl(6w^2 - 9u^2 - 5uw\bigr)v^6 + \bigl(16w^3 - 35u^3 - 10uw^2 - 51u^2w\bigr)v^5 \\
&+ \bigl(22w^4 - 75u^4 - 9uw^3 - 110u^2w^2 - 150u^3w\bigr)v^4 + \bigl(-93u^5 - 226u^4w \\
&- 9uw^4 - 133u^2w^3 - 251u^3w^2 + 16w^5\bigr)v^3 + \bigl(6w^6 - 65u^6 - 110u^2w^4 \\
&- 292u^4w^2 - 194u^5w - 251u^3w^3 - 10uw^5\bigr)v^2 + \bigl(w^7 - 24u^7 - 95u^6w \\
&- 226u^4w^3 - 150u^3w^4 - 51u^2w^5 - 5uw - 194u^5w^2\bigr)v \\
&- u(w+u)\bigl(w^4 + 4uw^3 + 7u^2w^2 + 4u^3w + u^4\bigr)(w+2u)^2, \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
\varphi_{31} ={}& \bigl(5u^2 - 6uw + 5w^2\bigr)v^{10} + \bigl(10u^2w - 64uw^2 + 34w^3 + 44u^3\bigr)v^9 \\
&+ \bigl(330u^3w - 67u^2w^2 + 196u^4 + 133w^4 - 372uw^3\bigr)v^8 + \bigl(-686u^2w^3 + 1{,}596u^4w \\
&+ 1{,}398u^3w^2 - 976uw^4 + 306w^5 + 538u^5\bigr)v^7 + \bigl(3{,}780u^5w - 1{,}462uw^5 \\
&- 1{,}666u^2w^4 + 5{,}830u^4w^2 + 2{,}834u^3w^3 + 404w^6 + 888u^6\bigr)v^6 \\
&+ \bigl(10{,}740u^5w^2 + 4{,}926u^6w + 11{,}016u^4w^3 + 3{,}970u^3w^4 - 1{,}462uw^6 \\
&- 2{,}104u^2w^5 + 306w^7 + 832u^7\bigr)v^5 + \bigl(16{,}686u^5w^3 + 13{,}444u^4w^4 \\
&+ 10{,}996u^6w^2 + 3{,}568u^7w - 1{,}666u^2w^6 - 976uw^7 \\
&+ 3{,}970u^3w^5 + 421u^8 + 133w^8\bigr)v^4 + \bigl(34w^9 + 16{,}686u^5w^4 + 13{,}932u^6w^3 \\
&+ 6{,}608u^7w^2 + 1{,}448u^8w + 11{,}016u^4w^5 - 372uw^8 + 2{,}834u^3w^6 \\
&- 686u^2w^7 + 116u^9\bigr)v^3 + \bigl(5w^{10} + 10{,}996u^6w^4 + 6{,}608u^7w^3 + 2{,}294u^8w^2 \\
&+ 10{,}740u^5w^5 + 5{,}830u^4w^6 + 332u^9w + 1{,}398u^3w^7 - 67u^2w^8 \\
&+ 20u^{10} - 64w^9u\bigr)v^2 + 2uw(w+2u)(w+u)\bigl(-3w^7 + 14uw^6 + 129u^2w^5 \\
&+ 383u^3w^4 + 483u^4w^3 + 248u^5w^2 + 74u^6w + 6u^7\bigr)v \\
&+ u^2w^2\bigl(5w^4 + 14uw^3 + 47u^2w^2 + 14u^3w + 5u^4\bigr)(w+u)^2(w+2u)^2, \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
\psi_{21} ={}& (64w - 64u)v^{11} + \bigl(-640u^2 + 576w^2 + 32uw\bigr)v^{10} + \bigl(2{,}377w^3 - 2{,}953u^3 \\
&- 2{,}409u^2w + 2{,}281uw^2\bigr)v^9 + \bigl(10{,}263uw^3 - 8{,}309u^4 + 5{,}932w^4 - 13{,}547u^3w \\
&- 1{,}195u^2w^2\bigr)v^8 + \bigl(9{,}952w^5 + 9{,}436u^2w^3 - 38{,}556u^4w - 27{,}700u^3w^2 \\
&+ 23{,}776uw^4 - 15{,}884u^5\bigr)v^7 + \bigl(11{,}770w^6 - 21{,}722u^6 + 25{,}702u^2w^4
\end{aligned}
$$

$$
\begin{aligned}
&- 87{,}718u^4w^2 - 36{,}348u^3w^3 - 70{,}452u^5w + 34{,}992uw^5\big)v^6 \\
&+ \big(9{,}952w^7 - 21{,}722u^7 + 33{,}684u^2w^5 - 135{,}354u^4w^3 \\
&- 39{,}116u^3w^4 - 155{,}370u^5w^2 - 89{,}370u^6w + 34{,}992uw^6\big)v^5 \\
&+ \big(-15{,}884u^8 + 5{,}932w^8 - 39{,}116u^3w^5 + 25{,}702u^2w^6 \\
&- 224{,}070u^5w^3 - 155{,}910u^4w^4 - 179{,}720u^6w^2 \\
&- 80{,}502u^7w + 23{,}776uw^7\big)v^4 - (w+u)\big(-2{,}377w^8 - 7{,}886uw^7 - 1{,}550u^2w^6 \\
&+ 37{,}898u^3w^5 + 97{,}456u^4w^4 + 126{,}614u^5w^3 + 97{,}282u^6w^2 \\
&+ 42{,}558u^7w + 8{,}309u^8\big)v^3 - \big(2{,}953u^8 + 15{,}609u^7w + 36{,}963u^6w^2 \\
&+ 50{,}305u^5w^3 + 42{,}147u^4w^4 + 20{,}771u^3w^5 + 4{,}029u^2w^6 \\
&- 1{,}129uw^7 - 576w^8\big)(w+u)^2v^2 - \big(640u^9 + 4{,}206u^8w + 12{,}463u^7w^2 \\
&+ 21{,}735u^6w^3 + 24{,}569u^5w^4 + 18{,}497u^4w^5 + 8{,}889u^3w^6 \\
&+ 2{,}281u^2w^7 + 96uw^8 - 64w^9\big)(w+u)^2v - u\big(w^2 + uw + u^2\big)\big(64w^6 + 384uw^5 \\
&+ 969u^2w^4 + 1{,}297u^3w^3 + 969u^4w^2 + 384u^5w + 64u^6\big)(w+u)^3, \qquad (30)
\end{aligned}
$$

$$
\begin{aligned}
\psi_{31} = {}& \big(-408wu + 212w^2 + 212u^2\big)v^{14} + \big(2{,}220w^3 + 2{,}220u^3 - 2{,}300w^2u \\
&- 1{,}916wu^2\big)v^{13} + \big(10{,}805w^4 - 2{,}732w^3u + 10{,}805u^4 - 19{,}714w^2u^2 \\
&+ 2{,}068wu^3\big)v^{12} + \big(32{,}490w^5 + 13{,}570w^4u - 69{,}996w^3u^2 - 45{,}964w^2u^3 \\
&+ 41{,}314wu^4 + 32{,}490u^5\big)v^{11} + \big(67{,}597w^6 + 66{,}038w^5u - 144{,}081w^4u^2 \\
&- 221{,}412w^3u^3 + 18{,}191w^2u^4 + 162{,}758wu^5 + 67{,}597u^6\big)v^{10} \\
&+ \big(102{,}990u^7 + 102{,}990w^7 - 546{,}970w^4u^3 + 374{,}070wu^6 \\
&- 211{,}210w^3u^4 + 147{,}758w^6u - 206{,}730w^5u^2 + 351{,}630w^2u^5\big)v^9 + \big(118{,}172u^8 \\
&- 691{,}326w^4u^4 + 212{,}666w^7u + 588{,}982wu^7 + 417{,}882w^3u^5 - 236{,}233w^6u^2 \\
&- 923{,}410w^5u^3 + 118{,}172w^8 + 988{,}871w^2u^6\big)v^8 + \big(102{,}990u^9 \\
&+ 268{,}456w^4u^5 + 212{,}666w^8u - 242{,}348w^7u^2 + 1{,}712{,}716w^3u^6 \\
&- 1{,}177{,}156w^6u^3 - 1{,}213{,}872w^5u^4 + 102{,}990w^9 + 1{,}623{,}572w^2u^7 \\
&+ 671{,}426wu^8\big)v^7 + \big(2{,}249{,}152w^4u^6 - 1{,}177{,}156w^7u^3 + 2{,}908{,}276w^3u^7 \\
&- 1{,}442{,}604w^6u^4 + 72{,}256w^5u^5 + 147{,}758w^9u + 67{,}597u^{10} + 67{,}597w^{10} \\
&- 236{,}233w^8u^2 + 1{,}809{,}843w^2u^8 + 562{,}874wu^9\big)v^6 \\
&+ 2(w+u)\big(16{,}245w^{10} + 16{,}774w^9u - 120{,}139w^8u^2 - 341{,}566w^7u^3 \\
&- 265{,}370w^6u^4 + 301{,}498w^5u^5 + 917{,}616w^4u^6 + 984{,}618w^3u^7 \\
&+ 554{,}255w^2u^8 + 155{,}512wu^9 + 16{,}245u^{10}\big)v^5 + \big(10{,}805u^{10} + 125{,}664wu^9 \\
&+ 512{,}862w^2u^8 + 1{,}015{,}022w^3u^7 + 1{,}099{,}952w^4u^6 + 589{,}542w^5u^5 \\
&- 29{,}884w^6u^4 - 261{,}318w^7u^3 - 138{,}806w^8u^2 - 8{,}040w^9u \\
&+ 10{,}805w^{10}\big)(w+u)^2v^4 + 2\big(1{,}110u^{11} + 18{,}420wu^{10} + 102{,}424w^2u^9
\end{aligned}
$$

$$
\begin{aligned}
&+ 276{,}050w^3u^8 + 428{,}681w^4u^7 + 405{,}461w^5u^6 + 214{,}535w^6u^5 \\
&+ 21{,}827w^7u^4 - 49{,}248w^8u^3 - 28{,}936w^9u^2 - 3{,}586w^{10}u + 1{,}110w^{11}\big)(w+u)^2v^3 \\
&+ \big(48{,}698w^2u^{10} + 212w^{12} + 212u^{12} - 2{,}724w^{11}u + 440{,}754w^6u^6 + 371{,}841w^4u^8 \\
&- 14{,}478w^{10}u^2 + 6{,}124wu^{11} + 61{,}237w^8u^4 - 14{,}284w^9u^3 \\
&+ 498{,}624w^5u^7 + 243{,}440w^7u^5 + 177{,}228w^3u^9\big)(w+u)^2v^2 \\
&+ 2(w+u)^3vwu\big(204u^{10} + 2{,}662wu^9 + 12{,}042w^2u^8 + 29{,}321w^3u^7 + 45{,}006w^4u^6 \\
&+ 46{,}414w^5u^5 + 32{,}132w^6u^4 + 13{,}847w^7u^3 + 2{,}684w^8u^2 - 346w^9u - 204w^{10}\big) \\
&+ w^2u^2\big(212u^8 + 1{,}372wu^7 + 4{,}045w^2u^6 + 7{,}230w^3u^5 + 8{,}707w^4u^4 + 7{,}230w^5u^3 \\
&+ 4{,}045w^6u^2 + 1{,}372w^7u + 212w^8\big)(w+u)^4, \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
\delta_{\varphi 1} = {}& 16w^{10} + (48u + 48v)w^9 + \big(89u^2 + 89v^2 - 78vu\big)w^8 \\
&+ 4(u+v)\big(25u^2 - 86vu + 25v^2\big)w^7 + \big(48u^4 - 232v^2u^2 - 180v^3u \\
&- 180vu^3 + 48v^4\big)w^6 + 2(u+v)\big(5u^4 - 18vu^3 + 50v^2u^2 - 18v^3u + 5v^4\big)w^5 \\
&+ \big(48u^6 + 198v^2u^4 + 288v^3u^3 + 198v^4u^2 - 26v^5u - 26vu^5 + 48v^6\big)w^4 \\
&+ 4(u+v)\big(25u^6 - 70vu^5 + 86v^2u^4 - 14v^3u^3 + 86v^4u^2 - 70v^5u + 25v^6\big)w^3 \\
&+ \big(-232v^2u^6 + 64v^3u^5 + 198v^4u^4 - 232v^6u^2 + 64v^5u^3 - 244v^7u \\
&- 244vu^7 + 89u^8 + 89v^8\big)w^2 + 2(u+v)\big(24u^8 - 63vu^7 - 59v^2u^6 - 31v^3u^5 \\
&+ 18v^4u^4 - 31v^5u^3 - 59v^6u^2 - 63v^7u + 24v^8\big)w + \big(16v^8 + 16v^7u + 41v^6u^2 + 2v^5u^3 \\
&+ 3v^4u^4 + 2v^3u^5 + 41v^2u^6 + 16vu^7 + 16u^8\big)(u+v)^2, \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
\delta_{\varphi 2} = {}& \big(8\big(v^2 + w^2\big) + (w-v)^2\big)u^8 + 8(w+v)\big(6\big(v^2 + w^2\big) + (w-v)^2\big)u^7 \\
&+ \big(144w^4 + 144v^4 + 248v^2w^2 + 222v^3w + 222vw^3\big)u^6 \\
&+ 2(w+v)\big(97w^4 + 131vw^3 + 254v^2w^2 + 131v^3w + 97v^4\big)u^5 \\
&+ \big(144w^6 + 144v^6 + 1{,}174v^2w^4 + 1{,}688v^3w^3 + 1{,}174v^4w^2 + 468v^5w + 468vw^5\big)u^4 \\
&+ 2(w+v)\big(28w^6 + 95vw^5 + 382v^2w^4 + 626v^3w^3 + 382v^4w^2 + 95v^5w + 28v^6\big)u^3 \\
&+ \big(9v^8 + 9w^8 + 56v^7w + 412v^2w^6 + 1{,}298v^3w^5 + 1{,}870v^4w^4 + 1{,}298v^5w^3 \\
&+ 412v^6w^2 + 56vw^7\big)u^2 + 2vw(w+v)\big(w^6 + 43vw^5 + 168v^2w^4 + 254v^3w^3 \\
&+ 168v^4w^2 + 43v^5w + v^6\big)u + v^2w^2\big(9v^4 + 38v^3w + 59v^2w^2 \\
&+ 38vw^3 + 9w^4\big)(w+v)^2. \qquad (33)
\end{aligned}
$$

Competing interests

The author declares that he does not have any commercial or associative interest that represents a conflict of interests in connection with the work submitted.

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