View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Crossref

Li and Li *Journal of Inequalities and Applications* (2015) 2015:356
DOI 10.1186/s13660-015-0881-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Density expansions of extremes from general error distribution with applications

Chunqiao Li and Tingting Li*

*Correspondence:
tinalee@swu.edu.cn
School of Mathematics and Statistics, Southwest University, Chongqing, 400715, China

Abstract

In this paper the higher-order expansions of density of normalized maximum with parent following general error distribution are established. The main results are applied to derive the higher-order expansions of the moments of extremes.

MSC: Primary 60G70; secondary 60F05

Keywords: density expansion; general error distribution; maximum; moment expansion

1 Introduction

In extreme value theory, the quality of convergence of normalized partial maximum of a sample has been studied in recent literature. For the convergence rate of distribution of normalized maximum, we refer to Smith [1], Leadbetter *et al.* [2], de Haan and Resnick [3] for general cases, and specific cases were studied by Hall [4], Nair [5], Peng *et al.* [6] and Jia and Li [7]. Nair [5] derived the higher-order expansions of moments of normalized maximum with parent following normal distribution. Liao *et al.* [8] and Jia *et al.* [9] extended Nair's results to skew-normal distribution and general error distribution, respectively.

The main objective of this paper is to derive the higher-order expansions of density of normalized maximum with parent following the general error distribution. To the best of our knowledge, there are few studies on the rate of convergence of density of normalized maximum except the work of de Haan and Resnick [10] for local limit theorems and Omey [11] for rates of convergence of densities with regular variation with remainders excluding the case we will study in this paper, *i.e.*, the general error distribution.

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with marginal cumulative distribution function (cdf) F_ν following the general error distribution ($F_\nu \sim \mathrm{GED}(\nu)$ for short), and let $M_n = \max_{1\leq k\leq n} X_k$ denote its partial maximum. The probability density function (pdf) of the GED(ν) is given by

$$f_\nu(x) = \frac{\nu \exp(-(1/2)|x/\lambda|^\nu)}{\lambda 2^{1+1/\nu}\Gamma(1/\nu)}, \quad x \in \mathbb{R},$$

where $\nu > 0$ is the shape parameter, $\lambda = [2^{-2/\nu}\Gamma(1/\nu)/\Gamma(3/\nu)]^{1/2}$ and $\Gamma(\cdot)$ denotes the gamma function (Nelson [12]). Note that the GED(2) reduces to the standard normal distribution.

Springer

© 2015 Li and Li. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

For the GED(v), the limiting distribution of maximum M_n and its associated higher-order expansions are given by Peng *et al.* [13] and Jia and Li [7]. Peng *et al.* [6] showed that

$$\lim_{n\to\infty} \mathrm{P}(M_n \le a_n x + b_n) = \Lambda(x) = \exp\bigl(-\exp(-x)\bigr), \quad x \in \mathbb{R} \tag{1.1}$$

provided the norming constants a_n and b_n satisfy the following equations:

$$1 - F_v(b_n) = n^{-1}, \qquad a_n = f(b_n), \tag{1.2}$$

where $f(x) = 2v^{-1}\lambda^v x^{1-v}$. In the sequel, let

$$g_n(x) = n a_n F_v^{n-1}(a_n x + b_n) f_v(a_n x + b_n) \tag{1.3}$$

denote the density of normalized maximum, and

$$\Delta_n\bigl(g_n, \Lambda'; x\bigr) = g_n(x) - \Lambda'(x) \tag{1.4}$$

with $\Lambda'(x) = e^{-x}\Lambda(x)$. By Proposition 2.5 in Resnick [14], $\Delta_n(g_n, \Lambda') \to 0$ as $n \to \infty$. For both applications and theoretical analysis, it is of interest to know the convergence rate of (1.4). This paper focuses on this topic and applies the main results to derive the high-order expansions of moments of extremes.

The paper is organized as follows. Section 2 provides the main results and all proofs are deferred to Section 4. Auxiliary lemmas with proofs are given in Section 3.

2 Main results

In this section, we present the asymptotic expansions of density for the normalized maximum formed by the GED(v) random variables and its applications to the higher-order expansions of moments of extremes.

Theorem 2.1 *Let $F_v(x)$ denote the cdf of* GED(v) *with $v > 0$, then for $v \neq 1$, with norming constants a_n and b_n given by* (1.2), *we have*

$$b_n^v\bigl[b_n^v \Delta_n\bigl(g_n, \Lambda'; x\bigr) - k_v(x)\Lambda'(x)\bigr] \to \omega_v(x)\Lambda'(x) \tag{2.1}$$

as $n \to \infty$, where $k_v(x)$ and $\omega_v(x)$ are respectively given by

$$k_v(x) = k_{v1}(x) + k_{v2}(x), \tag{2.2}$$

$$\omega_v(x) = \bigl(1 - v^{-1}\bigr)\lambda^{2v}\bigl(\omega_{v1}(x) + \omega_{v2}(x)e^{-x} + \omega_{v3}(x)e^{-2x}\bigr) \tag{2.3}$$

with

$$\begin{aligned}
k_{v1}(x) &= -\bigl(1 - v^{-1}\bigr)\lambda^v\bigl(x^2 + 2x\bigr),\\
k_{v2}(x) &= \bigl(1 - v^{-1}\bigr)\lambda^v\bigl(2 + 2x + \bigl(x^2 + 2x\bigr)e^{-x}\bigr),\\
\omega_{v1}(x) &= -4 - 2\bigl(1 - v^{-1}\bigr)x^2 - \frac{2}{3}\bigl(1 - 2v^{-1}\bigr)x^3 + \frac{1}{2}\bigl(1 - v^{-1}\bigr)x^4,
\end{aligned}$$

$$\omega_{\nu 2}(x) = -\frac{4}{\nu}x - \frac{2}{\nu}x^2 - \frac{2}{3}\left(5 - 4\nu^{-1}\right)x^3 - \frac{3}{2}\left(1 - \nu^{-1}\right)x^4,$$
$$\omega_{\nu 3}(x) = \frac{1}{2}\left(1 - \nu^{-1}\right)\left(x^2 + 2x\right)^2.$$

Remark 2.1 If we choose the norming constants a_n and b_n such that

$$\frac{1 - F_\nu(b_n)}{f_\nu(b_n)} \sim \frac{2\lambda^\nu}{\nu} b_n^{1-\nu}, \qquad a_n = f(b_n) \tag{2.4}$$

with $\nu \neq 1$, then

$$b_n^\nu\left[b_n^\nu \Delta_n\left(g_n, \Lambda'; x\right) - \bar{k}_\nu(x)\Lambda'(x)\right] \to \bar{\omega}_\nu(x)\Lambda'(x) \tag{2.5}$$

as $n \to \infty$, where $\bar{k}_\nu(x)$ and $\bar{\omega}_\nu(x)$ are respectively given by

$$\bar{k}_\nu(x) = \left(1 - \nu^{-1}\right)\lambda^\nu\left(-x^2 + \left(2 + 2x + x^2\right)e^{-x}\right) \tag{2.6}$$

and

$$\bar{\omega}_\nu(x) = \lambda^{2\nu}\left(1 - \nu^{-1}\right)\left(\bar{\omega}_{\nu 1}(x) + \bar{\omega}_{\nu 2}(x)e^{-x} + \bar{\omega}_{\nu 3}(x)e^{-2x}\right) \tag{2.7}$$

with

$$\bar{\omega}_{\nu 1}(x) = \frac{2(2\nu^{-1} - 1)}{3}x^3 + \frac{1 - \nu^{-1}}{2}x^4,$$
$$\bar{\omega}_{\nu 2}(x) = 4\left(\nu^{-1} - 2\right) + 4\left(\nu^{-1} - 2\right)x + 2\left(2\nu^{-1} - 3\right)x^2 + \frac{2}{3}\left(4\nu^{-1} - 5\right)x^3 + \frac{3}{2}\left(\nu^{-1} - 1\right)x^4,$$
$$\bar{\omega}_{\nu 3}(x) = 2\left(1 - \nu^{-1}\right)\left(1 + x + \frac{x^2}{2}\right)^2.$$

For the case of $\nu = 1$, we have the following results.

Theorem 2.2 *For $\nu = 1$, with norming constants $a_n = 2^{-1/2}$ and $b_n = 2^{-1/2}\log(n/2)$, we have*

$$n\left[n\Delta_n\left(g_n, \Lambda'; x\right) - k_1(x)\Lambda'(x)\right] \to \left(\omega_1(x) + \frac{k_1^2(x)}{2}\right)\Lambda'(x) \tag{2.8}$$

as $n \to \infty$, where $k_1(x)$ and $\omega_1(x)$ are respectively given by

$$k_1(x) = -\frac{1}{2}e^{-2x}, \qquad \omega_1(x) = -\frac{1}{3}e^{-3x}. \tag{2.9}$$

To end this section, we apply the higher-order expansions of densities to derive the asymptotic expansions of the moments of extremes. Methods used here are different from those in Nair [5] and Jia *et al.* [9].

In the sequel, for nonnegative integers r, let

$$m_r(n) = \int_{x\in\mathbb{R}} x^r g_n(x)\,dx, \qquad m_r = \int_{x\in\mathbb{R}} x^r \Lambda'(x)\,dx$$

denote respectively the rth moments of $(M_n - b_n)/a_n$ and its limits.

Theorem 2.3 *Let $\{X_n, n \geq 1\}$ be an iid sequence with marginal distribution $F_\nu \sim \mathrm{GED}(\nu)$, then*

(i) *for $\nu \neq 1$, with norming constants a_n and b_n given by* (1.2), *we have*

$$\begin{aligned} & b_n^\nu\big[b_n^\nu\big(m_r(n)-m_r\big)+\big(1-\nu^{-1}\big)\lambda^\nu r(m_{r+1}+2m_r)\big] \\ & \quad \to 2r\lambda^{2\nu}\big(1-\nu^{-1}\big)\bigg[\big(\big(1-\nu^{-1}\big)(r+1)+2\big)m_r+\big(\big(1-\nu^{-1}\big)(r+1)+1\big)m_{r+1} \\ & \qquad + \bigg(\frac{1}{4}\big(1-\nu^{-1}\big)(r-1)+\frac{1}{3}\big(2-\nu^{-1}\big)\bigg)m_{r+2}\bigg] \end{aligned} \tag{2.10}$$

as $n \to \infty$;

(ii) *for $\nu = 1$, with norming constants $a_n = 2^{-1/2}$ and $b_n = 2^{-1/2}\log(n/2)$, we have*

$$n\bigg[n\big(m_r(n)-m_r\big)+(-1)^r\frac{r}{2}\Gamma^{(r-1)}(2)\bigg] \to (-1)^{r-1}\frac{r}{24}\big[8\Gamma^{(r-1)}(3)-3\Gamma^{(r-1)}(4)\big] \tag{2.11}$$

as $n \to \infty$, where $\Gamma^{(r-1)}(t)$ denote the $(r-1)$th derivative of the gamma function at $x = t$.

3 Auxiliary lemmas

In this section we provide auxiliary lemmas which are needed to prove the main results.

Lemma 3.1 *Let $F_\nu(x)$ and $f_\nu(x)$ respectively denote the cdf and pdf of* GED(ν) *with $\nu \neq 1$, for large x, we have*

$$\begin{aligned} & 1-F_\nu(x) \\ & \quad = f_\nu(x)\frac{2\lambda^\nu}{\nu}x^{1-\nu}\big[1+2\big(\nu^{-1}-1\big)\lambda^\nu x^{-\nu}+4\big(\nu^{-1}-1\big)\big(\nu^{-1}-2\big)\lambda^{2\nu}x^{-2\nu}+O\big(x^{-3\nu}\big)\big]. \end{aligned} \tag{3.1}$$

Furthermore, with the norming constants a_n and b_n given by (1.2), *we have*

(i) *for $\nu \neq 1$,*

$$b_n^\nu\big[b_n^\nu\big(F_\nu^n(a_nx+b_n)-\Lambda(x)\big)-\tilde{k}_\nu(x)\Lambda(x)\big] \to \bigg(\tilde{\omega}_\nu(x)+\frac{\tilde{k}_\nu^2(x)}{2}\bigg)\Lambda(x) \tag{3.2}$$

as $n \to \infty$, where $\tilde{k}_\nu(x)$ and $\tilde{\omega}_\nu(x)$ are respectively given by

$$\tilde{k}_\nu(x) = \big(1-\nu^{-1}\big)\lambda^\nu\big(x^2+2x\big)e^{-x}, \tag{3.3}$$

$$\tilde{\omega}_\nu(x) = \big(\nu^{-1}-1\big)\lambda^{2\nu}\bigg(4x+2x^2+\frac{2}{3}\big(2-\nu^{-1}\big)x^3+\frac{1}{2}\big(1-\nu^{-1}\big)x^4\bigg)e^{-x}; \tag{3.4}$$

(ii) *for $\nu = 1$, with norming constants $a_n = 2^{-1/2}$ and $b_n = 2^{-1/2}\log(n/2)$, we have*

$$n\big[n\big(F_1^n(a_nx+b_n)-\Lambda(x)\big)-k_1(x)\Lambda(x)\big] \to \bigg(\omega_1(x)+\frac{k_1^2(x)}{2}\bigg)\Lambda(x) \tag{3.5}$$

as $n \to \infty$, where $k_1(x)$ and $\omega_1(x)$ are those given by (2.9).

Proof See Lemma 1 and Theorem 1 in Jia and Li [7]. □

Lemma 3.2 *Let $F_\nu(x)$ denote the cdf of* GED(ν) *with $\nu > 0$, then*
(i) *for $\nu \neq 1$, with norming constants given by* (1.2), *we have*

$$F_\nu^{n-1}(a_n x + b_n) = \left(1 + \tilde{k}_\nu(x) b_n^{-\nu} + \left(\tilde{\omega}_\nu(x) + \frac{\tilde{k}_\nu^2(x)}{2}\right) b_n^{-2\nu}\big(1 + o(1)\big)\right)\Lambda(x) \tag{3.6}$$

as $n \to \infty$, where $\tilde{k}_\nu(x)$ and $\tilde{\omega}_\nu(x)$ are respectively given by (3.3) *and* (3.4);
(ii) *for $\nu = 1$, with norming constants $a_n = 2^{-1/2}$ and $b_n = 2^{-1/2}\log(n/2)$, we have*

$$F_1^{n-1}(a_n x + b_n) = \left(1 + \frac{1}{n} k_1(x) + \frac{1}{n^2}\left(\omega_1(x) + \frac{k_1^2(x)}{2}\right)\big(1 + o(1)\big)\right)\Lambda(x) \tag{3.7}$$

as $n \to \infty$, where $k_1(x)$ and $\omega_1(x)$ are given by (2.9).

Proof (i) It follows from (3.2) and (3.3) that

$$\begin{aligned} F_\nu^n(a_n x + b_n) &= b_n^{-\nu}\left(b_n^{-\nu}\left(\tilde{\omega}_\nu(x) + \frac{\tilde{k}_\nu^2(x)}{2}\right)\big(1 + o(1)\big) + \tilde{k}_\nu(x)\right)\Lambda(x) + \Lambda(x) \\ &= \left(1 + \tilde{k}_\nu(x) b_n^{-\nu} + \left(\tilde{\omega}_\nu(x) + \frac{\tilde{k}_\nu^2(x)}{2}\right) b_n^{-2\nu}\big(1 + o(1)\big)\right)\Lambda(x). \end{aligned} \tag{3.8}$$

Noting that

$$F_\nu^n(a_n x + b_n) \to \Lambda(x) = \exp\big(-\exp(-x)\big),$$

by taking logarithms, we have

$$n\big(1 - F_\nu(a_n x + b_n)\big) \to e^{-x}.$$

Thus

$$1 - F_\nu(a_n x + b_n) \sim n^{-1} e^{-x} = o\big(b_n^{-3\nu}\big)$$

since $b_n \sim 2^{1/\nu}\lambda(\log n)^{1/\nu}$, which implies

$$\frac{1}{F_\nu(a_n x + b_n)} = 1 + \big(1 - F_\nu(a_n x + b_n)\big)\big(1 + o(1)\big) = 1 + o\big(b_n^{-3\nu}\big). \tag{3.9}$$

The desired result (3.6) follows by (3.8) and (3.9). The proof of (ii) is similar and details are omitted here. □

Lemma 3.3 *Let $f_\nu(x)$ denote the pdf of* GED(ν) *with $\nu \neq 1$, then*

$$f_\nu(x) = \big(1 - F_\nu(x)\big)\frac{\nu}{2\lambda^\nu} x^{\nu-1}\big(1 + 2\big(1 - \nu^{-1}\big)\lambda^\nu x^{-\nu} - 4\big(1 - \nu^{-1}\big)\lambda^{2\nu} x^{-2\nu} + O\big(x^{-3\nu}\big)\big) \tag{3.10}$$

for large x, and

$$\begin{aligned} C_n(x) &:= \frac{a_n f_\nu(a_n x + b_n)}{1 - F_\nu(a_n x + b_n)} \\ &= 1 + \left(1 - \nu^{-1}\right) 2\lambda^\nu (x+1) b_n^{-\nu} + 4\lambda^{2\nu}\left(1 - \nu^{-1}\right)\left(-1 - \frac{x}{\nu} + \frac{1 - 2\nu^{-1}}{2} x^2\right) b_n^{-2\nu} \\ &\quad + O\left(b_n^{-3\nu}\right) \end{aligned} \tag{3.11}$$

as $n \to \infty$.

Proof The desired results follow directly by (3.1). □

Lemma 3.4 *Let*

$$H_\nu(b_n; x) = \frac{1 - F_\nu(a_n x + b_n)}{1 - F_\nu(b_n)} e^x - 1$$

and the norming constants a_n and b_n be given by (1.2), *then*

$$b_n^\nu \left(b_n^\nu H_\nu(b_n; x) - k_{\nu 1}(x)\right) \to \omega_\nu^\circ(x) \tag{3.12}$$

as $n \to \infty$, where $k_{\nu 1}(x)$ is given by (2.2) *and $\omega_\nu^\circ(x)$ is given by*

$$\omega_\nu^\circ(x) = \left(1 - \nu^{-1}\right)\lambda^{2\nu}\left(4x + 2x^2 + \frac{2}{3}\left(2 - \nu^{-1}\right)x^3 + \frac{1}{2}\left(1 - \nu^{-1}\right)x^4\right).$$

Proof Let

$$B_n(x) = \frac{1 + 2(\nu^{-1} - 1)\lambda^\nu (a_n x + b_n)^{-\nu} + 4(\nu^{-1} - 1)(\nu^{-1} - 2)\lambda^{2\nu}(a_n x + b_n)^{-2\nu} + O((a_n x + b_n)^{-3\nu})}{1 + 2(\nu^{-1} - 1)\lambda^\nu (b_n)^{-\nu} + 4(\nu^{-1} - 1)(\nu^{-1} - 2)\lambda^{2\nu}(b_n)^{-2\nu} + O((b_n)^{-3\nu})},$$

it is easy to check that $\lim_{n\to\infty} B_n(x) = 1$ and

$$B_n(x) - 1 = \left(1 + o(1)\right)\left[-4\left(\nu^{-1} - 1\right)\lambda^{2\nu} b_n^{-2\nu} x + O\left(b_n^{-3\nu}\right)\right].$$

Hence,

$$\lim_{n\to\infty} b_n^\nu \left(B_n(x) - 1\right) = 0 \tag{3.13}$$

and

$$\lim_{n\to\infty} b_n^{2\nu}\left(B_\nu(x) - 1\right) = -4\left(\nu^{-1} - 1\right)\lambda^{2\nu} x. \tag{3.14}$$

By (3.1), we have

$$\begin{aligned} &\frac{1 - F_\nu(a_n x + b_n)}{1 - F_\nu(b_n)} e^x \\ &\quad = B_n(x) \exp\left[-\int_0^x \left(\frac{(\nu - 1)a_n}{b_n + a_n t} + \frac{\nu a_n (b_n + a_n t)^{\nu - 1}}{2\lambda^\nu} - 1\right) dt\right] \end{aligned}$$

$$
\begin{aligned}
&= B_n(x)\Biggl\{1-\int_0^x\left(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^\nu}-1\right)dt\\
&\quad+\frac{1}{2}\left[\int_0^x\left(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^\nu}-1\right)dt\right]^2\bigl(1+o(1)\bigr)\Biggr\}.
\end{aligned}
\tag{3.15}
$$

Combining (3.13)-(3.15) together, we have

$$
\begin{aligned}
&\lim_{n\to\infty} b_n^\nu H_\nu(b_n;x)\\
&= \lim_{n\to\infty} b_n^\nu\left(B_n(x)\left\{1-\int_0^x\left(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^\nu}-1\right)dt\bigl(1+o(1)\bigr)\right\}-1\right)\\
&= \lim_{n\to\infty}\left(b_n^\nu\bigl(B_n(x)-1\bigr)-b_n^\nu B_n(x)\int_0^x\left(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^\nu}-1\right)dt\right)\\
&= -\lim_{n\to\infty}\int_0^x b_n^\nu\left(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^\nu}-1\right)dt\\
&= -\bigl(1-\nu^{-1}\bigr)\lambda^\nu\bigl(x^2+2x\bigr)\\
&= k_{\nu 1}(x).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\lim_{n\to\infty} b_n^\nu\bigl(b_n^\nu H_\nu(b_n;x)-k_{\nu 1}(x)\bigr)\\
&= \lim_{n\to\infty}\Biggl[b_n^{2\nu}\bigl(B_n(x)-1\bigr)-B_n(x)b_n^\nu\\
&\quad\times\left(\int_0^x b_n^\nu\left(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^\nu}-1\right)dt+k_{\nu 1}(x)\right)\\
&\quad+\frac{1}{2}\bigl(1+o(1)\bigr)B_n(x)b_n^{2\nu}\left(\int_0^x\left(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^\nu}-1\right)dt\right)^2\Biggr]\\
&= -4\bigl(\nu^{-1}-1\bigr)\lambda^{2\nu}x+\frac{2\lambda^{2\nu}(1-\nu^{-1})}{\nu}x^2-\frac{2}{3}\bigl(1-\nu^{-1}\bigr)\bigl(1-2\nu^{-1}\bigr)\lambda^{2\nu}x^3\\
&\quad+\frac{(1-\nu^{-1})^2}{2}\lambda^{2\nu}\bigl(x^2+2x\bigr)^2\\
&= \omega_\nu^\circ(x).
\end{aligned}
$$

The proof is complete. □

Lemma 3.5 *Let $C_n(x)$ be given by* (3.11) *and $D_n(x)$ be denoted by*

$$
D_n(x)=1+\tilde{k}_\nu(x)b_n^{-\nu}+\left(\tilde{\omega}_\nu(x)+\frac{\tilde{k}_\nu^2(x)}{2}\right)b_n^{-2\nu}\bigl(1+o(1)\bigr).
$$

For $\nu\neq 1$ and $-d\log b_n<x<cb_n^{\frac{3}{\nu}}$ with $0<c,d<1$, we have

$$
\begin{aligned}
&\bigl|C_n(x)D_n(x)\bigr|<2,\\
&\bigl|b_n^\nu\bigl(C_n(x)D_n(x)-1\bigr)\bigr|\le 1+\bigl(1+\nu^{-1}\bigr)\lambda^\nu\bigl(2+2|x|+\bigl(x^2+2|x|\bigr)e^{-x}\bigr),
\end{aligned}
$$

$$
\begin{aligned}
&\left|b_n^{\nu}\left[b_n^{\nu}\left(C_n(x)D_n(x)-1\right)-k_{\nu 2}(x)\right]\right| \\
&\quad\le 1+\left(1+\nu^{-1}\right)\lambda^{2\nu}\Bigg[4+\frac{4}{\nu}|x|+\left(2+4\nu^{-1}\right)x^2+\Bigg(\frac{4}{\nu}|x|+\left(4+6\nu^{-1}\right)x^2 \\
&\qquad+\frac{2}{3}\left(1+2\nu^{-1}\right)|x|^3+\frac{1+\nu^{-1}}{2}x^4\Bigg)e^{-x}+\frac{1}{2}\left(1+\nu^{-1}\right)\left(x^2+2|x|\right)^2e^{-2x}\Bigg]
\end{aligned}
$$

for large n, where $k_{\nu 1}(x)$ and $k_{\nu 2}(x)$ are given by (2.2).

Proof The desired results follow from Lemmas 3.2 and 3.3. □

The following Mills' inequalities are from the GED(ν) in Jia *et al.* [9], which will be used later.

Lemma 3.6 *Let $F_\nu(x)$ and $f_\nu(x)$ denote the cdf and pdf of* GED(ν), *respectively. Then*

(i) *for $\nu>1$ and all $x>0$, we have*

$$\frac{2\lambda^\nu}{\nu}x^{1-\nu}\left(1+\frac{2(\nu-1)\lambda^\nu}{\nu}x^{-\nu}\right)^{-1}<\frac{1-F_\nu(x)}{f_\nu(x)}<\frac{2\lambda^\nu}{\nu}x^{1-\nu}; \tag{3.16}$$

(ii) *for $0<\nu<1$ and all $x>\lambda[2(1/\nu-1)]^{1/\nu}$, we have*

$$\frac{2\lambda^\nu}{\nu}x^{1-\nu}<\frac{1-F_\nu(x)}{f_\nu(x)}<\frac{2\lambda^\nu}{\nu}x^{1-\nu}\left(1+\frac{2(\nu-1)\lambda^\nu}{\nu}x^{-\nu}\right)^{-1}. \tag{3.17}$$

Lemma 3.7 *Let the norming constant b_n be given by* (1.2), *for any constant $0<c<1$ and arbitrary nonnegative integers i, j and k, we have*

$$\lim_{n\to\infty} b_n^i\int_{cb_n^{\frac{\nu}{3}}}^{\infty}|x|^je^{-kx}\Lambda(x)\,dx=0, \tag{3.18}$$

$$\lim_{n\to\infty} b_n^i\int_{cb_n^{\frac{\nu}{3}}}^{\infty}|x|^jg_n(x)\,dx=0 \tag{3.19}$$

if $\nu\neq 1$.

Proof By arguments similar to Lemma 3.3 in Jia *et al.* [9], we can get (3.18). The rest is to prove (3.19). By (3.16) and (3.17), and Lemma 3.5, we have

$$
\begin{aligned}
&b_n^i\int_{cb_n^{\frac{\nu}{3}}}^{\infty}|x|^jg_n(x)\,dx \\
&\quad=b_n^i\int_{cb_n^{\frac{\nu}{3}}}^{\infty}|x|^jna_nF_\nu^{n-1}(a_nx+b_n)f_\nu(a_nx+b_n)\,dx \\
&\quad=b_n^i\int_{cb_n^{\frac{\nu}{3}}}^{\infty}|x|^j\frac{1-F_\nu(a_nx+b_n)}{1-F_\nu(b_n)}e^xC_n(x)D_n(x)\Lambda'(x)\,dx \\
&\quad\le 4b_n^i\int_{cb_n^{\frac{\nu}{3}}}^{\infty}|x|^je^{-x}\Lambda(x)\,dx \\
&\quad\to 0
\end{aligned}
$$

as $n\to\infty$. The proof is complete. □

Lemma 3.8 *Assume that the shape parameter* $\nu \neq 1$, *then for any constant* $0 < d < 1$ *and arbitrary nonnegative integers* i, j *and* k, *we have*

$$\lim_{n\to\infty} b_n^i \int_{-\infty}^{-d\log b_n} |x|^j e^{-kx} \Lambda(x)\,dx = 0, \tag{3.20}$$

$$\lim_{n\to\infty} b_n^i \int_{-\infty}^{-d\log b_n} |x|^j g_n(x)\,dx = 0. \tag{3.21}$$

Proof By arguments similar to that of Lemma 3.7, we have

$$\begin{aligned} &b_n^i \int_{-\infty}^{-d\log b_n} |x|^j e^{-kx} \Lambda(x)\,dx \\ &\quad \le b_n^i \exp\left(-\frac{b_n^d}{2}\right) \int_1^{\infty} x^j e^{kx} \exp\left(-\frac{e^x}{2}\right) dx \\ &\quad \to 0 \end{aligned}$$

as $n \to \infty$ since $\int_1^\infty x^j e^{kx} \exp(-\frac{e^x}{2})\,dx < \infty$.

For assertion (3.21), we only consider the case of $\nu > 1$ since the proof of the case of $0 < \nu < 1$ is similar. Rewrite

$$\begin{aligned} b_n^i \int_{-\infty}^{-d\log b_n} |x|^j g_n(x)\,dx &= b_n^i \int_{-\infty}^{-\frac{3\nu\lambda^{-\nu} b_n^\nu}{2}} |x|^j g_n(x)\,dx + b_n^i \int_{-\frac{3\nu\lambda^{-\nu} b_n^\nu}{2}}^{-\frac{\nu\lambda^{-\nu} b_n^{\frac{\nu-1}{2}}}{2}} |x|^j g_n(x)\,dx \\ &\quad + b_n^i \int_{-\frac{\nu\lambda^{-\nu} b_n^{\frac{\nu-1}{2}}}{2}}^{-d\log b_n} |x|^j g_n(x)\,dx \\ &= I_n + II_n + III_n. \end{aligned}$$

First note that $\int_{\mathbb{R}} |x|^j f_\nu(x)\,dx < \infty$ and the symmetry of f_ν implies $F_\nu(-x) + F_\nu(x) = 1$. By using (1.2) and (3.16) we have

$$\begin{aligned} I_n &< 2^j \nu\lambda^{-\nu} b_n^{i+j+\nu} \bigl(1 - F_\nu(2b_n)\bigr)^{n-2} \\ &< 2^j \nu\lambda^{-\nu} b_n^{i+j+\nu} \exp\left((n-2)\left(c - (\nu-1)\log b_n - \frac{(2b_n)^\nu}{2\lambda^\nu}\right)\right) \\ &\to 0 \end{aligned} \tag{3.22}$$

as $n \to \infty$, where $c = (1 - \nu - 1/\nu)\log 2 + (\nu - 1)\log\lambda - \log\Gamma(1/\nu)$.

To show $II_n \to 0$ and $III_n \to 0$, we consider the case of $\nu > 2$ first. By using the following inequalities

$$1 - \nu x < (1-x)^\nu < 1 - \nu x + \frac{\nu(\nu-1)}{2}x^2, \quad 0 < x < \frac{1}{2}, \nu > 2, \tag{3.23}$$

we can get

$$n a_n f_\nu\bigl(b_n - b_n^{\frac{1-\nu}{2}}\bigr) < 2\exp\left(\frac{\nu}{2\lambda^\nu} b_n^{\frac{\nu-1}{2}}\right)$$

and

$$\begin{aligned}
&\frac{1-F_\nu(b_n-b_n^{\frac{1-\nu}{2}})}{1-F_\nu(b_n)} \\
&\quad > \frac{\frac{2\lambda^\nu}{\nu}(b_n-b_n^{\frac{1-\nu}{2}})^{1-\nu}(1+\frac{2(\nu-1)\lambda^\nu}{\nu}(b_n-b_n^{\frac{1-\nu}{2}})^{-\nu})^{-1}f_\nu(b_n-b_n^{\frac{1-\nu}{2}})}{\frac{2\lambda^\nu}{\nu}b_n^{1-\nu}f_\nu(b_n)} \\
&\quad > \frac{(1-b_n^{-\frac{1+\nu}{2}})^{1-\nu}}{1+\frac{2(\nu-1)\lambda^\nu}{\nu}(b_n-b_n^{\frac{1-\nu}{2}})^{-\nu}}\exp\biggl(\frac{\nu}{2\lambda^\nu}b_n^{\frac{\nu-1}{2}}-\frac{\nu(\nu-1)}{4\lambda^\nu}b_n^{-1}\biggr) \\
&\quad > \frac{1}{2}\exp\biggl(\frac{\nu}{2\lambda^\nu}b_n^{\frac{\nu-1}{2}}\biggr)
\end{aligned}$$

for large n, which implies that

$$\begin{aligned}
II_n &< b_n^i\biggl(\frac{3\nu\lambda^{-\nu}b_n^\nu}{2}\biggr)^j na_nf_\nu\biggl(-a_n\frac{\nu\lambda^{-\nu}b_n^{\frac{\nu-1}{2}}}{2}+b_n\biggr)F_\nu^{n-1}\biggl(-a_n\frac{\nu\lambda^{-\nu}b_n^{\frac{\nu-1}{2}}}{2}+b_n\biggr)\frac{3\nu\lambda^{-\nu}b_n^\nu}{2} \\
&< b_n^i\biggl(\frac{3\nu\lambda^{-\nu}b_n^\nu}{2}\biggr)^{j+1} na_nf_\nu\biggl(-a_n\frac{\nu\lambda^{-\nu}b_n^{\frac{\nu-1}{2}}}{2}+b_n\biggr)\exp\bigl(-(n-1)\bigl(1-F_\nu\bigl(b_n-b_n^{\frac{1-\nu}{2}}\bigr)\bigr)\bigr) \\
&= 2b_n^i\biggl(\frac{3\nu\lambda^{-\nu}b_n^\nu}{2}\biggr)^{j+1}\frac{\lambda 2^{1+\frac{1}{\nu}}\Gamma(\frac{1}{\nu})}{\nu}\exp\biggl(\frac{b_n^\nu+\nu b_n^{\frac{\nu-1}{2}}}{2\lambda^\nu}-\frac{1}{2}\exp\biggl(\frac{\nu}{2\lambda^\nu}b_n^{\frac{\nu-1}{2}}\biggr)\biggr) \\
&\to 0
\end{aligned} \tag{3.24}$$

as $n\to\infty$.

Similarly, for $-\frac{\nu\lambda^{-\nu}b_n^{\frac{\nu-1}{2}}}{2}<x<-d\log b_n$, we have

$$\begin{aligned}
&na_nf_\nu(a_nx+b_n) \\
&\quad < \biggl(1+\frac{2(\nu-1)\lambda^\nu}{\nu}b_n^{-\nu}\biggr)\frac{f_\nu(a_nx+b_n)}{f_\nu(b_n)} \\
&\quad < \biggl(1+\frac{2(\nu-1)\lambda^\nu}{\nu}b_n^{-\nu}\biggr)\exp\biggl(-\frac{b_n^\nu}{2\lambda^\nu}2\lambda^\nu b_n^{-\nu}x\biggr) \\
&\quad < 2e^{-x}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1-F_\nu(a_nx+b_n)}{1-F_\nu(b_n)} \\
&\quad > \frac{\frac{2\lambda^\nu}{\nu}(a_nx+b_n)^{1-\nu}(1+\frac{2(\nu-1)\lambda^\nu}{\nu}(a_nx+b_n)^{-\nu})^{-1}f_\nu(a_nx+b_n)}{\frac{2\lambda^\nu}{\nu}b_n^{1-\nu}f_\nu(b_n)} \\
&\quad > \frac{(1+\frac{2\lambda^\nu}{\nu}b_n^{-\nu}x)^{1-\nu}}{1+\frac{2(\nu-1)\lambda^\nu}{\nu}(a_nx+b_n)^{-\nu}}\exp\biggl(-x-\frac{\nu-1}{\nu}\lambda^\nu b_n^{-\nu}x^2\biggr) \\
&\quad > \frac{1}{2}e^{-x}
\end{aligned}$$

for large n by using (3.23). Then

$$\begin{aligned} III_n &< 2b_n^i \int_{-\frac{\nu\lambda^{-\nu}b_n^{\frac{\nu-1}{2}}}{2}}^{-d\log b_n} |x|^j e^{-x} \exp\left(-\frac{1}{2}e^{-x}\right) dx \\ &< 2b_n^i \exp\left(-\frac{e^{d\log b_n}}{4}\right) \int_{-\frac{\nu\lambda^{-\nu}b_n^{\frac{\nu-1}{2}}}{2}}^{-d\log b_n} |x|^j e^{-x} \exp\left(-\frac{1}{4}e^{-x}\right) dx \\ &\to 0 \end{aligned} \tag{3.25}$$

as $n \to \infty$.

Combining with (3.22)-(3.25), the assertion (3.21) is derived for $\nu > 2$. Similar proofs for the case of $1 < \nu \leq 2$ and details are omitted here. The proof is complete. □

Lemma 3.9 *Let* $\alpha = \min(1, \nu)$ *as* $\nu \neq 1$. *For large n and* $-d\log b_n < x < cb_n^{\frac{\nu}{3}}$, *both* $x^r b_n^\nu \Delta_n(g_n, \Lambda'; x)$ *and* $x^r b_n^\nu [b_n^\nu \Delta_n(g_n, \Lambda'; x) - k_\nu(x)\Lambda'(x)]$ *are bounded by integrable functions independent of n, with* $r > 0$, $0 < c < 1$ *and* $0 < d < \alpha$, *where* a_n *and* b_n *are given by* (1.2), *and* $k_\nu(x)$ *is given by* (2.2).

Proof We only consider the case of $\nu > 1$. For the case of $0 < \nu < 1$, the proofs are similar and details are omitted here. Rewrite

$$b_n^\nu \Delta_n(g_n, \Lambda'; x) = b_n^\nu H_\nu(b_n; x) e^{-x}\Lambda(x) + b_n^\nu \big(C_n(x)D_n(x) - 1\big) e^{-x}\Lambda(x),$$

where $C_n(x)$ is given by (3.11), $H_\nu(b_n; x)$ and $D_n(x)$ are respectively defined in Lemma 3.4 and Lemma 3.5. Note that $\int_{-\infty}^{\infty} x^k e^{-tx} \exp(-e^{-x})\, dx = (-1)^k \Gamma^{(k)}(t)$ is finite for $t > 0$ and nonnegative integers k. Lemma 3.5 shows that $b_n^\nu (C_n(x)D_n(x) - 1)e^{-x}\Lambda(x)$ is bounded by integrable function independent of n. The rest is to prove that $b_n^\nu H_\nu(b_n; x)$ is bounded by $m(x)$, where $m(x)$ is a polynomial on x. Rewrite

$$b_n^\nu H_\nu(b_n; x) = I_n(x) + J_n(x), \tag{3.26}$$

where

$$\begin{aligned} I_n(x) &= b_n^\nu \big(B_n(x) - 1\big), \\ J_n(x) &= b_n^\nu B_n(x) \int_0^x \left(\frac{(\nu-1)a_n}{b_n + a_n t} + \frac{\nu a_n (b_n + a_n t)^{\nu-1}}{2\lambda^\nu} - 1 \right) dt. \end{aligned}$$

For $-d\log b_n < x < cb_n^{\frac{\nu}{3}}$, from Lemma 3.4 it follows that

$$\big|I_n(x)\big| < 1 \tag{3.27}$$

and

$$\big|J_n(x)\big| < 1 + \left[\frac{\nu^{-1}}{2^{-1}\nu\lambda^{-\nu} - d}|x| + \big(1 - \nu^{-1}\big)\lambda^\nu x^2 + \frac{4}{3}\big(1 - \nu^{-1}\big)\big|1 - 2\nu^{-1}\big|\lambda^{2\nu}|x|^3 \right]. \tag{3.28}$$

Hence, the desired result (3.26) follows by combining (3.16), (3.27) and (3.28) together.

Rewrite

$$\begin{aligned}
&b_n^{\nu}\big[b_n^{\nu}\Delta_n\big(g_n,\Lambda';x\big)-k_{\nu}(x)\Lambda'(x)\big]\\
&\quad=b_n^{\nu}\left[b_n^{\nu}\left(\frac{1-F_{\nu}(a_nx+b_n)}{1-F_{\nu}(b_n)}e^{x}C_n(x)D_n(x)-1\right)-k_{\nu}(x)\right]\Lambda'(x)\\
&\quad=b_n^{\nu}\left[b_n^{\nu}C_n(x)D_n(x)\left(\frac{1-F_{\nu}(a_nx+b_n)}{1-F_{\nu}(b_n)}e^{x}-1\right)\right.\\
&\qquad\left.+b_n^{\nu}\big(C_n(x)D_n(x)-1\big)-\big(k_{\nu1}(x)+k_{\nu2}(x)\big)\right]\Lambda'(x)\\
&\quad=b_n^{\nu}\big[b_n^{\nu}C_n(x)D_n(x)\big(H_{\nu}(b_n;x)-b_n^{-\nu}k_{\nu1}(x)\big)+b_n^{\nu}\big(C_n(x)D_n(x)-1-b_n^{-\nu}k_{\nu2}(x)\big)\\
&\qquad+\big(C_n(x)D_n(x)-1\big)k_{\nu1}(x)\big]\Lambda'(x).
\end{aligned}$$

By Lemma 3.5, we only need to estimate the bound of $b_n^{\nu}[b_n^{\nu}C_n(x)D_n(x)(H_{\nu}(b_n;x)-b_n^{-\nu}k_{\nu1}(x))]$. Rewrite

$$b_n^{\nu}\big[b_n^{\nu}C_n(x)D_n(x)\big(H_{\nu}(b_n;x)-b_n^{-\nu}k_{\nu1}(x)\big)\big]=H_n(x)-K_n(x)+L_n(x),$$

where

$$\begin{aligned}
&H_n(x)=b_n^{2\nu}\big(B_n(x)-1\big),\\
&K_n(x)=b_n^{2\nu}\left(B_n(x)\int_0^x\left(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^{\nu}}-1\right)dt+b_n^{-\nu}k_{\nu1}(x)\right),\\
&L_n(x)=\frac{1}{2}\big(1+o(1)\big)B_n(x)b_n^{2\nu}\left(\int_0^x\left(\frac{(\nu-1)a_n}{b_n+a_nt}+\frac{\nu a_n(b_n+a_nt)^{\nu-1}}{2\lambda^{\nu}}-1\right)dt\right)^2.
\end{aligned}$$

For $0<x<cb_n^{\frac{\nu}{3}}$, by using $1-\nu y<(1+y)^{-\nu}<1$ for $\nu>0$ and $y>0$, we have

$$\big|H_n(x)\big|<1+8\lambda^{2\nu}\big(1+8\lambda^{\nu}\big)x$$

due to Lemma 3.4. If $-d\log b_n<x<0$, by using $1+\nu y<(1+y)^{\nu}<1$ for $\nu>1$ and $-1<y<0$, Lemma 3.4 shows that

$$\big|H_n(x)\big|<1+16\lambda^{2\nu}\big(1+8\lambda^{\nu}\big)|x|$$

for large n. Similarly,

$$\big|K_n(x)\big|<2\big(1-\nu^{-1}\big)\left|\frac{\nu}{2\lambda^{\nu}}-d\right|^{-1}\lambda^{\nu}x^2+\frac{4}{3}\big(1-\nu^{-1}\big)\big|1-2\nu^{-1}\big|\lambda^{2\nu}|x|^3, \tag{3.29}$$

$$\big|L_n(x)\big|<1+\frac{1}{2}\left[\frac{\nu^{-1}}{2^{-1}\nu\lambda^{-\nu}-d}|x|+\big(1-\nu^{-1}\big)\lambda^{\nu}x^2+\frac{4}{3}\big(1-\nu^{-1}\big)\big|1-2\nu^{-1}\big|\lambda^{2\nu}|x|^3\right]^2 \tag{3.30}$$

as $-d\log b_n<x<cb_n^{\frac{\nu}{3}}$. Hence, we derive the desired result by combining (3.16), (3.29) and (3.30) together.

The proof is complete. □

For $\nu = 1$, note that the GED(1) is the Laplace distribution with pdf given by

$$f_1(x) = 2^{-\frac{1}{2}} \exp\bigl(-2^{\frac{1}{2}}|x|\bigr), \quad x \in \mathbb{R},$$

and its distributional tail can be written as

$$1 - F_1(x) = 2^{-\frac{1}{2}} f_1(x) = 2^{-1} \exp\bigl(-2^{\frac{1}{2}}\bigr) \exp\biggl(-\int_1^x \frac{1}{f(t)}\, dt\biggr), \quad x > 0$$

with $f(t) = 2^{-\frac{1}{2}}$. For the Laplace distribution, similar to the case of $\nu > 1$, we have the following two results.

Lemma 3.10 *For* $0 < d < 1$ *and an arbitrary nonnegative real number* j, *we have*

$$\lim_{n\to\infty} n^2 \int_{-\infty}^{-db_n^{\frac{1}{2}}} |x|^j e^{-kx} \Lambda(x)\, dx = 0, \quad k = 1, 2, \ldots, \tag{3.31}$$

$$\lim_{n\to\infty} n^2 \int_{-\infty}^{-db_n^{\frac{1}{2}}} |x|^j F_1^n(a_n x + b_n)\, dx = 0. \tag{3.32}$$

Lemma 3.11 *For* $x > -db_n^{\frac{1}{2}}$, *both* $x^r n((F_1^n(a_n x + b_n))' - \Lambda'(x))$ *and* $x^r n[n((F_1^n(a_n x + b_n))' - \Lambda'(x)) + \frac{1}{2}e^{-2x}\Lambda'(x)]$ *are bounded by integrable functions independent of* n, *where* $r > 0$ *and* $0 < d < 1$.

4 Proofs of the main results

Proof of Theorem 2.1 From Lemma 3.3 and Lemma 3.5 it follows that

$$\begin{aligned}
&C_n(x)D_n(x) \\
&\quad = 1 + k_{\nu 2}(x)b_n^{-\nu} + \bigl(1 - \nu^{-1}\bigr)\lambda^{2\nu}\biggl[-4 - \frac{4}{\nu}x + 2\bigl(1 - 2\nu^{-1}\bigr)x^2 + \biggl(-\frac{4}{\nu}x + 2\bigl(2 - 3\nu^{-1}\bigr)x^2 \\
&\qquad + \frac{2(1 - 2\nu^{-1})}{3}x^3 - \frac{1 - \nu^{-1}}{2}x^4\biggr)e^{-x} + \frac{(1 - \nu^{-1})}{2}\bigl(x^2 + 2x\bigr)^2 e^{-2x}\biggr]b_n^{-2\nu} + O\bigl(b_n^{-3\nu}\bigr),
\end{aligned} \tag{4.1}$$

where $k_{\nu 2}(x)$ is given by (2.2). Note that (3.12) shows

$$\frac{1 - F_\nu(a_n x + b_n)}{1 - F_\nu(b_n)} e^x = 1 + k_{\nu 1}(x)b_n^{-\nu} + \omega_\nu^\circ(x)b_n^{-2\nu} + O\bigl(b_n^{-3\nu}\bigr). \tag{4.2}$$

Hence,

$$\begin{aligned}
&\Delta_n\bigl(g_n, \Lambda'; x\bigr) \\
&\quad = \biggl(\frac{1 - F_\nu(a_n x + b_n)}{1 - F_\nu(b_n)} e^x C_n(x)D_n(x) - 1\biggr)\Lambda'(x) \\
&\quad = \biggl[\bigl(1 - \nu^{-1}\bigr)\lambda^\nu\bigl(2 - x^2 + \bigl(x^2 + 2x\bigr)e^{-x}\bigr)b_n^{-\nu} + \bigl(1 - \nu^{-1}\bigr)\lambda^{2\nu}\biggl(-4 - 2\bigl(1 - \nu^{-1}\bigr)x^2 \\
&\qquad - \frac{2(1 - 2\nu^{-1})}{3}x^3 + \frac{1 - \nu^{-1}}{2}x^4 + \biggl(-\frac{4}{\nu}x - \frac{2}{\nu}x^2 - \frac{2(5 - 4\nu^{-1})}{3}x^3 - \frac{3(1 - \nu^{-1})}{2}x^4\biggr)e^{-x}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{1-\nu^{-1}}{2}\left(x^2+2x\right)^2 e^{-2x}\Big)b_n^{-2\nu} + O\left(b_n^{-3\nu}\right)\Big]\Lambda'(x) \\
&= \left[k_\nu(x)b_n^{-\nu} + \left(1-\nu^{-1}\right)\lambda^{2\nu}\left(\omega_{\nu1}(x)+\omega_{\nu2}(x)e^{-x}+\omega_{\nu3}(x)e^{-2x}\right)b_n^{-2\nu} + O\left(b_n^{-3\nu}\right)\right]\Lambda'(x) \\
&= \left(k_\nu(x)b_n^{-\nu} + \omega_\nu(x)b_n^{-2\nu} + O\left(b_n^{-3\nu}\right)\right)\Lambda'(x)
\end{aligned}
$$

implying that

$$
\lim_{n\to\infty} b_n^\nu\left[b_n^\nu\Delta_n\left(g_n,\Lambda';x\right) - k_\nu(x)\Lambda'(x)\right] = \omega_\nu(x)\Lambda'(x).
$$

The proof is complete. □

Proof of Theorem 2.3 For $\nu\neq 1$, by Lemmas 3.5-3.9 and the dominated convergence theorem, we have

$$
\begin{aligned}
& b_n^\nu\left(m_r(n)-m_r\right) \\
&= b_n^\nu\int_{-\infty}^{-d\log b_n} x^r\Delta_n\left(g_n,\Lambda';x\right)dx + b_n^\nu\int_{-d\log b_n}^{cb_n^{\frac{\nu}{3}}} x^r\Delta_n\left(g_n,\Lambda';x\right)dx \\
&\quad + b_n^\nu\int_{cb_n^{\frac{\nu}{3}}}^{\infty} x^r\Delta_n\left(g_n,\Lambda';x\right)dx \\
&\to \int_{-\infty}^{\infty} x^r k_\nu(x)\Lambda'(x)\,dx \\
&= -\left(1-\nu^{-1}\right)\lambda^\nu r(2m_r+m_{r+1})
\end{aligned}
$$

and

$$
\begin{aligned}
& b_n^\nu\left[b_n^\nu\left(m_r(n)-m_r\right) + \left(1-\nu^{-1}\right)\lambda^\nu r(2m_r+m_{r+1})\right] \\
&= \int_{-\infty}^{\infty} x^r b_n^\nu\left[b_n^\nu\Delta_n\left(g_n,\Lambda';x\right)-k_\nu(x)\Lambda'(x)\right]dx \\
&= \int_{-\infty}^{-d\log b_n} x^r b_n^{2\nu}\Delta_n\left(g_n,\Lambda';x\right)dx - \int_{-\infty}^{-d\log b_n} x^r b_n^\nu k_\nu(x)\Lambda'(x)\,dx \\
&\quad - \int_{cb_n^{\frac{\nu}{3}}}^{\infty} x^r b_n^\nu k_\nu(x)\Lambda'(x)\,dx + \int_{-d\log b_n}^{cb_n^{\frac{\nu}{3}}} x^r b_n^\nu\left[b_n^\nu\Delta_n\left(g_n,\Lambda';x\right)-k_\nu(x)\Lambda'(x)\right]dx \\
&\quad + \int_{cb_n^{\frac{\nu}{3}}}^{\infty} x^r b_n^{2\nu}\Delta_n\left(g_n,\Lambda';x\right)dx \\
&\to \int_{-\infty}^{\infty} x^r\omega_\nu(x)\Lambda'(x)\,dx \\
&= 2r\lambda^{2\nu}\left(1-\nu^{-1}\right)\Big[\left(\left(1-\nu^{-1}\right)(r+1)+2\right)m_r + \left(\left(1-\nu^{-1}\right)(r+1)+1\right)m_{r+1} \\
&\quad + \left(\frac{1}{4}\left(1-\nu^{-1}\right)(r-1)+\frac{1}{3}\left(2-\nu^{-1}\right)\right)m_{r+2}\Big]
\end{aligned}
$$

as $n\to\infty$.

For the case of $\nu = 1$, note that

$$\int_{-\infty}^{\infty} x^k e^{-mx} \Lambda(x)\, dx = (-1)^k \Gamma^{(k)}(m).$$

Combining with Lemmas 3.10 and 3.11 and the dominated convergence theorem, we can derive the desired results.

The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CL drafted the manuscript and TL revised the whole paper critically. Both authors were involved in the proof of the main results of the paper. All authors read and approved the final manuscript.

Acknowledgements
We would like to appreciate the reviewers for reading the paper and making helpful comments that improved the original paper. This work was supported by the National Natural Science Foundation of China (11171275), the Natural Science Foundation Project of CQ (cstc2012jjA00029), the Fundamental Research Funds for the Central Universities (XDJK2013C021) and the Doctoral Grant of Southwest University (SWU113012).

Received: 12 May 2015 Accepted: 1 November 2015 Published online: 10 November 2015

References
1. Smith, LR: Uniform rates of convergence in extreme-value theory. Adv. Appl. Probab. **14**, 600-622 (1982)
2. Leadbetter, MR, Lindgren, G, Rootzén, H: Extremes and Related Properties of Random Sequences and Processes. Springer, New York (1983)
3. de Haan, L, Resnick, SI: Second-order regular variation and rates of convergence in extreme-value theory. Ann. Probab. **1**, 97-124 (1996)
4. Hall, P: On the rate of convergence of normal extremes. J. Appl. Probab. **16**, 433-439 (1979)
5. Nair, KA: Asymptotic distribution and moments of normal extremes. Ann. Probab. **9**, 150-153 (1981)
6. Peng, Z, Nadarajah, S, Lin, F: Convergence rate of extremes for the general error distribution. J. Appl. Probab. **47**, 668-679 (2010)
7. Jia, P, Li, T: Higher-order expansions for distributions of extremes from general error distribution. J. Inequal. Appl. **2014**, 213 (2014)
8. Liao, X, Peng, Z, Nadarajah, S: Asymptotic expansions of the moments of skew-normal extremes. Stat. Probab. Lett. **83**, 1321-1329 (2013)
9. Jia, P, Liao, X, Peng, Z: Asymptotic expansions of the moments of extremes from general error distribution. J. Math. Anal. Appl. **422**, 1131-1145 (2015)
10. de Haan, L, Resnick, SI: Local limit theorems for sample extremes. Ann. Probab. **10**, 396-413 (1982)
11. Omey, E: Rates of convergence for densities in extreme value theory. Ann. Probab. **2**, 479-486 (1988)
12. Nelson, DB: Conditional heteroskedasticity in asset returns: a new approach. Econometrica **59**, 347-370 (1991)
13. Peng, Z, Tong, B, Nadarajah, S: Tail behavior of the general error distribution. Commun. Stat., Theory Methods **38**, 1884-1892 (2009)
14. Resnick, SI: Extreme Values, Regular Variation, and Point Processes. Springer, New York (1987)

Submit your manuscript to a SpringerOpen journal and benefit from:

► Convenient online submission
► Rigorous peer review
► Immediate publication on acceptance
► Open access: articles freely available online
► High visibility within the field
► Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com