Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 494067, 7 pages
http://dx.doi.org/10.1155/2013/494067

Research Article

The Asymptotic Behavior for a Class of Impulsive Delay Differential Equations

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Received 27 January 2013; Accepted 21 March 2013

Academic Editor: Chuangxia Huang

This paper is concerned with asymptotical behavior for a class of impulsive delay differential equations. The new criteria for determining attracting sets and attracting basin of the impulsive system are obtained by developing the properties of quasi-invariant sets. Examples and numerical simulations are given to illustrate the effectiveness of our results. In addition, we show that the impulsive effects may play a key role to these asymptotical properties even though the solutions of corresponding nonimpulsive systems are unbounded.

1. Introduction

Impulsive delay differential equations have attracted increasing interests since time delays and impulsive effects commonly exist in many fields such as population dynamics, automatic control, drug administration, and communication networks [1–4]. In past two decades, its asymptotical behaviors such as stability and attractivity of the equilibrium point or periodical solutions have been deeply studied for impulsive functional differential equations (see, [5–18]).

However, under impulsive perturbation, the solutions may not be attracted to an equilibrium point or periodical trajectory but to some bounded region. In this case, it is interesting to investigate the attracting set and attracting basin, that is, the region attracting the solutions and the range in which initial values vary when remaining the attractivity for impulsive delay differential equations. In [19], Xu and Yang first give the method to estimate global attracting set and invariant set for impulsive delayed systems by developing delayed differential inequalities. The techniques are further developed to study global attractivity for some complex impulsive systems such as impulsive neutral differential equations [20, 21] and impulsive stochastic systems [22]. But the techniques and methods given in the existing publications are invalid for determining locally attracting set and attracting basin for impulsive delay differential equations.

In this paper, our objective is to mainly discuss the asymptotical behavior on (locally) attracting set and its attracting basin for a class of impulsive delay differential equations. Based on the quasi-invariant properties, we estimate the existence range of attracting set and attracting basin of the impulsive delay systems by solving algebraic equations and employing differential inequality technique. Examples are given to illustrate the effectiveness of our method and show that the asymptotic behavior of the impulsive systems may be different from one of the corresponding continuous systems.

2. Preliminaries

Let N be the set of all positive integers, R^n the space of n-dimensional real column vectors, and $R^{m\times n}$ the set of $m \times n$ real matrices. For $A, B \in R^{m\times n}$ or $A, B \in R^n$, $A \geq B(A \leq B, A > B, A < B)$ means that each pair of corresponding elements of A and B satisfies the inequality "$\geq(\leq, >, <)$." $R^n_+ = \{x \in R^n \mid x \geq 0\}$, $E = (1, 1, \ldots, 1)^T \in R^n$, and I denotes an $n \times n$ unit matrix.

Let $\tau > 0$ and $t_0 < t_1 < t_2 < \cdots$ be the fixed points with $\lim_{k\to\infty} t_k = \infty$ (called impulsive moments).

$C[X, Y]$ denotes the space of continuous mappings from the topological space X to the topological space Y. Let $C \triangleq C[[-\tau, 0], R^n]$ especially.

Morever, PC $\overset{\Delta}{=} \{\phi : [-\tau, 0] \to R^n \mid \phi(t^+) = \phi(t)$ for $t \in [-\tau, 0)$, $\phi(t^-)$ exists for $t \in (-\tau, 0]$, $\phi(t^-) = \phi(t)$ for all but at most a finite number of points $t \in (-\tau, 0]\}$. PC is a space of piecewise right-hand continuous functions which is a nature extension of the phrase space C.

We define $\mathrm{PC}[[t_0, \infty), R^n] \overset{\Delta}{=} \{\psi : [t_0, \infty) \to R^n \mid \psi(t)$ is continuous at $t \neq t_k$, $\psi(t_k^+)$ and $\psi(t_k^-)$ exist, $\psi(t_k) = \psi(t_k^+)$, for $k \in N\}$.

For $x \in R^n$, $A \in R^{n\times n}$, $\phi \in C$ or $\phi \in$ PC, we define

$$[x]^+ = (|x_1|, |x_2|, \dots, |x_n|)^T,$$
$$[A]^+ = \left(|a_{ij}|\right)_{n\times n}, \tag{1}$$
$$[\phi]^+_\tau = (\|\phi_1\|_\tau, \|\phi_2\|_\tau, \dots, \|\phi_n\|_\tau),$$

where $\| \phi_i\|_\tau = \sup_{s\in[-\tau,0]} \| \phi_i(s) \|$ and $\| \cdot \|$ is an norm in R^n.

In this paper, we will consider a impulsive delay differential equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(t, x_t), \quad t \neq t_k, t \geq t_0, \\ \Delta x &= Bx(t_k^-) + I_k(t_k^-, x(t_k^-)), \quad k \in N, \end{aligned} \tag{2}$$

where $\dot{x}(t)$ denotes the right-hand derivative of $x(t)$, $\Delta x = x(t_k^+) - x(t_k^-)$, $x(t_k) = x(t_k^+)$, $A = \mathrm{diag}\{a_1, a_2, \dots, a_n\}$, $B = \mathrm{diag}\{b_1, b_2, \dots, b_n\}$, $f \in C[[t_{k-1}, t_k) \times \mathrm{PC}, R^n]$, and the limit $\lim_{(t,\phi)\to(t_k^-,\varphi)} f(t, \phi) = f(t_k^-, \varphi)$ exists, $I_k \in C[[t_0, \infty) \times R^n, R^n]$, and $x_t \in$ PC is defined by $x_t(s) = x(t+s)$, $s \in [-\tau, 0]$.

A function $x(t) : [t_0 - \tau, \infty) \to R^n$ is called to be a solution of (2) through (t_0, ϕ), if $x(t) \in \mathrm{PC}[[t_0, \infty), R^n]$ as $t \geq t_0$, and satisfies (2) with the initial condition

$$x(t_0 + s) = \phi(s), \quad s \in [-\tau, 0], \phi \in \mathrm{PC}. \tag{3}$$

Throughout the paper, we always assume that for any $\phi \in$ PC, system (2) has at least one solution through (t_0, ϕ), denoted by $x(t, t_0, \phi)$ or $x_t(t_0, \phi)$ (simply $x(t)$ and x_t if no confusion should occur), where $x_t(t_0, \phi) = x(t + s, t_0, \phi) \in$ PC, $s \in [-\tau, 0]$.

In this paper, we need the following definitions involving attracting set, attracting basin, the quasi-invariant set of impulsive systems, and monotonous vector functions.

Definition 1. The set $S \subset$ PC is called to be an attracting set of (2), and $D \subset$ PC is called an attraction basin of S, if for any initial value $\phi \in D$, the solution $x_t(t_0, \phi)$ converges to S as $t \to +\infty$. That is,

$$\mathrm{dist}(x_t(t_0, \phi), S) \longrightarrow 0, \quad \text{as}\, t \longrightarrow +\infty, \tag{4}$$

where $\mathrm{dist}(\varphi, S) = \inf_{\psi\in S} \mathrm{dist}(\varphi, \psi)$, $\mathrm{dist}(\varphi, \psi) = \sup_{s\in[-\tau,0]} \|\varphi(s) - \psi(s)\|$, for $\varphi \in$ PC.

Definition 2. The set $D \subset$ PC is called to be a positive quasi-invariant set of (2), if there is a positive diagonal matrix $L = \mathrm{diag}\{l_i\}$ such that for any initial value $\phi \in D$, the solutions $x_t(t_0, \phi)$ satisfy $Lx_t(t_0, \phi) \in D$, for $t \geq t_0$. When $L = I$(identity matrix) especially, the set D is called positively invariant.

Definition 3. Let $\Omega \subset R^n$. The vector function $F(x) : \Omega \to R^n$ is called to be monotonically nondecreasing in Ω, if for any $x', x'' \in \Omega$, $x' \leq x''$ implies $F(x') \leq F(x'')$.

3. Main Results

In this paper, we always make the following assumptions.

(H_1) There exist nonnegative constants θ, ϱ such that $0 < \theta \leq t_k - t_{k-1} \leq \varrho$, for $k \in N$.

(H_2) $[f(t, \varphi)]^+ \leq p([\varphi]^+_\tau)$ for $t \geq t_0$ and $\varphi \in$ PC, where the vector function $p(\cdot) : R^n_+ \to R^n_+$ is continuous and monotonically nondecreasing in R^n_+.

(H_3) $[I_k(t, x)]^+ \leq q([x]^+)$, for $t \geq t_0$, $k \in N$ and $x \in R^n$, where the vector function $q(\cdot) : R^n_+ \to R^n_+$ is continuous and monotonically nondecreasing in R^n_+.

To obtain attractivity, we first give the quasi-invariant properties of (2).

Theorem 4. *Assume that in addition to* (H_1)–(H_3)*, there is a vector* $z^* \geq 0$ *such that*

$$p(Mz^*) + W\left[I - e^{-W\theta}\right]^{-1} q(Mz^*) - Wz^* < 0, \tag{5}$$

where $W = \mathrm{diag}\{w_1, \dots, w_n\}$, $M = \mathrm{diag}\{m_1, \dots, m_n\}$, $w_i > 0, m_i \geq 1, i = 1, 2, \dots, n$, *are defined by*

$$w_i = \begin{cases} -a_i - \dfrac{\ln|1 + b_i|}{\varrho}, & \text{if } 0 < |1 + b_i| < 1, \\ -a_i - \dfrac{\ln|1 + b_i|}{\theta}, & \text{if } |1 + b_i| \geq 1, \end{cases}$$
$$m_i = \begin{cases} \dfrac{1}{|1 + b_i|}, & \text{if } 0 < |1 + b_i| < 1, \\ 1, & \text{if } |1 + b_i| \geq 1. \end{cases} \tag{6}$$

Then, the set $D = \{\phi \in PC \mid [\phi]^+_\tau \leq z^*\}$ *is a positive quasi-invariant set of* (2)*. When* $M = I$ *especially,* D *is a positive invariant set of* (2)*.*

Proof. Let $x(t) = x(t, t_0, \phi)$ be a solution of (2) through (t_0, ϕ). It is easily verified that the following formula for the variation of parameters is valid:

$$\begin{aligned} x(t) = {} & K(t, t_0)\phi(0) + \int_{t_0}^{t} K(t, s) f(s, x_s)\, ds \\ & + \sum_{t_0<t_k\leq t} K(t, t_k) I_k(t_k^-, x(t_k^-)), \quad t \geq t_0, \end{aligned} \tag{7}$$

where $K(t, s)$ is the Cauchy matrix of linear impulsive system

$$\begin{cases} \dot{y}(t) = Ay(t), & t \neq t_k, \\ \Delta y(t_k^+) = By(t_k^-), & k \in N. \end{cases} \tag{8}$$

According to the representation of the Cauchy matrix (see page 74 [2]),

$$K(t, s) = e^{A(t-s)} \prod_{s<t_k\leq t} (I + B), \quad t \geq s \geq t_0. \tag{9}$$

Since $0 < \theta \le t_k - t_{k-1} \le \varrho$, for $k \in N$, we obtain the following estimate:

$$\prod_{s<t_k\le t} |1+b_i| \le \begin{cases} |1+b_i|^{((t-s)/\varrho)-1} = \dfrac{1}{|1+b_i|} e^{(\ln|1+b_i|/\varrho)(t-s)}, & \text{if } 0 < |1+b_i| < 1, \\ |1+b_i|^{(t-s)/\theta} = e^{(\ln|1+b_i|/\theta)(t-s)}, & \text{if } |1+b_i| \ge 1. \end{cases} \tag{10}$$

In terms of the definition of M and W,

$$[K(t,s)]^+ \le Me^{-W(t-s)}, \quad t \ge s \ge t_0. \tag{11}$$

By (7) and (11) and the assumptions (H_2) and (H_3), then

$$\begin{aligned} [x(t)]^+ \le{}& Me^{-W(t-t_0)}[\phi]_\tau^+ + M\int_{t_0}^t e^{-W(t-s)} p\left([x_s]_\tau^+\right) ds \\ &+ M\sum_{t_0<t_k\le t} e^{-W(t-t_k)} q\left([x(t_k^-)]^+\right), \quad t \ge t_0. \end{aligned} \tag{12}$$

Since $t_k - t_{k-1} \ge \theta > 0$ and $W = \operatorname{diag}\{w_1, \dots, w_n\} > 0$, we have

$$\begin{aligned} \sum_{t_0<t_k\le t} e^{-W(t-t_k)} &= \sum_{t_0<t_k\le t} e^{-W(t-t_k)} \left[I - e^{-W\theta}\right]\left[I - e^{-W\theta}\right]^{-1} \\ &\le \sum_{t_0<t_k\le t} e^{-W(t-t_k)} \left[I - e^{-W(t_k-t_{k-1})}\right]\left[I - e^{-W\theta}\right]^{-1} \\ &= \sum_{t_0<t_k\le t} \left[e^{-W(t-t_k)} - e^{-W(t-t_{k-1})}\right]\left[I - e^{-W\theta}\right]^{-1} \\ &\le \left[I - e^{-W(t-t_0)}\right]\left[I - e^{-W\theta}\right]^{-1}. \end{aligned} \tag{13}$$

From the strict inequality (5), there is an enough small number $\varepsilon > 0$ such that

$$\begin{gathered} p(Mz) + W\left[I - e^{-W\theta}\right]^{-1} q(Mz) - Wz < 0, \\ z \stackrel{\Delta}{=} z^* + \varepsilon E > 0. \end{gathered} \tag{14}$$

In the following, we will prove that $[\phi]_\tau^+ < z$ implies

$$[x(t)]^+ = \left[x(t, t_0, \phi)\right]^+ < Mz, \quad t \ge t_0. \tag{15}$$

Otherwise, from the piecewise continuity of $x(t)$, there must be an integer i and $t^* > t_0$ such that

$$\left|x_i(t^*)\right| \ge m_i z_i, \tag{16}$$

$$[x(t)]^+ \le Mz, \quad t_0 \le t < t^*. \tag{17}$$

By using (12), (13), (14), (17), $W > 0$, and the monotonicity of $p(\cdot), q(\cdot)$, we can get

$$\begin{aligned} [x(t^*)]^+ \le{}& e^{-W(t^*-t_0)} M[\phi]_\tau^+ + M\int_{t_0}^{t^*} e^{-W(t^*-s)} p(Mz)\, ds \\ &+ M\sum_{t_0<t_k\le t^*} e^{-W(t^*-t_k)} q(Mz) \\ <{}& e^{-W(t^*-t_0)} Mz + M\left(I - e^{-W(t^*-t_0)}\right) W^{-1} p(Mz) \\ &+ M\left(I - e^{-W(t^*-t_0)}\right)\left[I - e^{-W\theta}\right]^{-1} q(Mz) \\ ={}& e^{-W(t^*-t_0)} MW^{-1} \\ &\times \left[Wz - p(Mz) - W\left[I - e^{-W\theta}\right]^{-1} q(Mz)\right] \\ &+ W^{-1}Mp(Mz) + M\left[I - e^{-W\theta}\right]^{-1} q(Mz) \\ <{}& MW^{-1}\left[Wz - p(Mz) - W\left[I - e^{-W\theta}\right]^{-1} q(Mz)\right] \\ &+ W^{-1}Mp(Mz) + M\left[I - e^{-W\theta}\right]^{-1} q(Mz) \\ ={}& Mz. \end{aligned} \tag{18}$$

This contradicts (16), and so (15) holds. Letting $\varepsilon \to 0$, from (15), we have for any $\phi \in D$ (i.e., $[\phi]_\tau^+ \le z^*$),

$$\left[x(t, t_0, \phi)\right]^+ \le Mz^*, \quad \text{that is, } \left[M^{-1}x(t, t_0, \phi)\right]^+ \le z^*, \quad t \ge t_0. \tag{19}$$

Therefore, the set $D = \{\phi \in \mathrm{PC} \mid [\phi]_\tau^+ \le z^*\}$ is a positive quasi-invariant set of (2). When $M = I$ especially, D is a positive invariant set of (2). The proof is complete. □

Based on the obtained quasi-invariant set, we have the following

Theorem 5. *Let*

$$\Delta(z) = p(z) + W\left[I - e^{-W\theta}\right]^{-1} q(z) - M^{-1}Wz, \quad z \in R_+^n. \tag{20}$$

Assume that all conditions in Theorem 4 hold. Define

$$\begin{aligned} \Omega_1 &= \left\{z \in R_+^n \mid \Delta(Mz) < 0\right\}, \\ \Omega_2 &= \left\{z \in R_+^n \mid \Delta(z) < 0\right\}, \\ \Omega_3 &= \left\{z \in R_+^n \mid \Delta(z) \ge 0\right\}, \\ \Omega_1^* &= \bigcup_{z^*\in\Omega_1} \left\{z \in R_+^n \mid z \le z^*\right\}, \\ \Omega_2^* &= \bigcup_{z^*\in\Omega_2} \left\{z \in R_+^n \mid z \le z^*\right\}. \end{aligned} \tag{21}$$

Then, $S = \{\phi \in PC \mid [\phi]_\tau^+ \in \Omega_2^* \cap \Omega_3\}$ *is an attracting set of* (2) *and* $D = \{\phi \in PC \mid [\phi]_\tau^+ \in \Omega_1^*\}$ *is the attracting basin of* S.

Proof. From (5) and the definitions of the above sets, then $z^* \in \Omega_1$, $Mz^* \in \Omega_2$, $0 \in \Omega_1^*$, $0 \in \Omega_2^*$, $0 \in \Omega_3$. Obviously, $\Omega_1, \Omega_2, \Omega_1^*, \Omega_2^*, \Omega_3$ and $\Omega_2^* \cap \Omega_3$ are nonempty, and so the definitions of the sets of S and D are valid. For any $\phi \in D$, there is a $z^* \in \Omega_1$ satisfying $[\phi]_\tau^+ \le z^*$. According to Theorem 4, we obtain

$$[x(t)]^+ = [x(t, t_0, \phi)]^+ \le Mz^* \in \Omega_2, \quad \forall t \ge t_0. \tag{22}$$

That is,

$$\sigma \overset{\Delta}{=} \limsup_{t\to\infty} [x(t)]^+ \in \Omega_2^*. \tag{23}$$

Then, for any given $\varepsilon > 0$, there is a $T_1 > t_0$ such that

$$[x(t)]^+ \le \varepsilon E + \sigma, \quad t \ge T_1. \tag{24}$$

In light of $W = \operatorname{diag}\{w_i\} > 0$, for the above $\varepsilon > 0$ and T_1, we can find an enough large $T_2 > 0$ such that

$$\begin{gathered}\int_{T_2}^{\infty} e^{-Ws} ds \le \varepsilon I,\\ \sum_{t_0<t_k\le T_1} e^{-W(t-t_k)} \le \varepsilon I, \quad t > T_2.\end{gathered} \tag{25}$$

Using (12), (13), (22), (24), and (25), we have for $t \ge \tau + T_1 + T_2$,

$$\begin{aligned}
[x(t)]^+ &\le e^{-W(t-t_0)} M[\phi]_\tau^+ + \int_{t_0}^{t} e^{-W(t-s)} Mp\left([x_s]_\tau^+\right) ds \\
&\quad + \sum_{t_0<t_k\le t} Me^{-W(t-t_k)} q\left(\left[x\left(t_k^-\right)\right]^+\right) \\
&\le e^{-W(t-t_0)} M[\phi]_\tau^+ \\
&\quad + \left\{\int_{t_0}^{t-T_2} + \int_{t-T_2}^{t}\right\} e^{-W(t-s)} Mp\left([x_s]_\tau^+\right) ds \\
&\quad + \left\{\sum_{t_0<t_k\le T_1} + \sum_{T_1<t_k\le t}\right\} Me^{-W(t-t_k)} q\left(\left[x\left(t_k^-\right)\right]^+\right) \\
&\le e^{-W(t-t_0)} Mz^* + \int_{T_2}^{\infty} e^{-Ws} Mp(Mz^*) ds \\
&\quad + \int_{t-T_2}^{t} e^{-W(t-s)} Mp(\varepsilon E + \sigma) ds \\
&\quad + M\sum_{t_0<t_k\le T_1} e^{-W(t-t_k)} q(Mz^*) \\
&\quad + M\sum_{T_1<t_k\le t} e^{-W(t-t_k)} q(\varepsilon E + \sigma) \\
&\le e^{-W(t-t_0)} Mz^* + \varepsilon Mp(Mz^*) \\
&\quad + \left(I - e^{-WT_2}\right) W^{-1} Mp(\varepsilon E + \sigma) + \varepsilon Mq(Mz^*) \\
&\quad + M\left(I - e^{-W(t-T_1)}\right)\left[I - e^{-W\theta}\right]^{-1} q(\varepsilon E + \sigma) \\
&\le e^{-W(t-t_0)} Mz^* + \varepsilon M\left[p(Mz^*) + q(Mz^*)\right] \\
&\quad + W^{-1} Mp(\varepsilon E + \sigma) + M\left[I - e^{-W\theta}\right]^{-1} q(\varepsilon E + \sigma).
\end{aligned} \tag{26}$$

This implies that

$$\begin{aligned}\sigma = \limsup_{t\to+\infty} [x(t)]^+ &\le \varepsilon M\left[p(Mz^*) + q(Mz^*)\right] \\ &\quad + W^{-1} Mp(\varepsilon E + \sigma) + M\left[I - e^{-W\theta}\right]^{-1} q(\varepsilon E + \sigma).\end{aligned} \tag{27}$$

Letting $\epsilon \to 0^+$, then

$$\sigma \le W^{-1} Mp(\sigma) + M\left[I - e^{-W\theta}\right]^{-1} q(\sigma). \tag{28}$$

That is, $\Delta(\sigma) \ge 0$ and $\sigma \in \Omega_3$. Thus,

$$\sigma \in \Omega_2^* \cap \Omega_3. \tag{29}$$

From the definition of σ and S, $\operatorname{dist}(x_t(t_0, \phi), S) \to 0$ as $t \to +\infty$. The proof is complete. □

From the above theorems, we can obtain sufficient conditions ensuring global attractivity and stability in the following corollaries.

Corollary 6. *Assume that* (H_1)*–*(H_3) *hold with*

$$\begin{gathered}p\left([\varphi]_\tau^+\right) = P[\varphi]_\tau^+ + \mu,\\ P = \left(p_{ij}\right)_{n\times n} \ge 0, \qquad \mu = (\mu_1, \ldots, \mu_n)^T \ge 0,\\ q\left([x]^+\right) = Q[x]^+ + \nu,\\ Q = \left(q_{ij}\right)_{n\times n} \ge 0, \qquad \nu = (\nu_1, \ldots, \nu_n)^T \ge 0.\end{gathered} \tag{30}$$

If the spectral radius

$$\rho(\Lambda) < 1, \quad \text{where } \Lambda = W^{-1}MP + M\left[I - e^{-\theta W}\right]^{-1} Q, \tag{31}$$

then $D = \{\phi \in PC \mid [\phi]_\tau^+ \le Z \overset{\Delta}{=} (I - \Lambda)^{-1} W^{-1}(\mu + W[I - e^{-\theta W}]^{-1}\nu)\}$ *is a positive quasi-invariant set of* (2), *and* $S = \{\phi \in PC \mid [\phi]_\tau^+ \le (I - \Lambda)^{-1} W^{-1} M(\mu + W[I - e^{-\theta W}]^{-1}\nu)\}$ *is a global attracting set of* (2).

Proof. Since $p(z) = Pz + \mu$ and $q(z) = Qz + \nu$, we directly calculate

$$\Delta(z) = M^{-1}W(\Lambda - I)z + \mu + W\left[I - e^{-\theta W}\right]^{-1}\nu,$$
$$\Delta(Mz) = W(\Lambda - I)z + \mu + W\left[I - e^{-\theta W}\right]^{-1}\nu. \tag{32}$$

Without loss of generality, we assume that $\mu, \nu > 0$. Since $\rho(\Lambda) < 1$, $(I - \Lambda)^{-1}$ exists and $(I - \Lambda)^{-1} \geq 0$ (see [23]), and so $Z > 0$. For any $\kappa > 0$, we take $z^* = (1+\kappa)Z > 0$ in Theorem 4 and verify the condition (5):

$$\Delta(Mz^*) = -\kappa\left(\mu + W\left[I - e^{-\theta W}\right]^{-1}\nu\right) < 0. \tag{33}$$

According to Theorem 4, when $\kappa \to 0$, we deduce that D is a positive quasi-invariant set of (2). Furthermore, by (33),

$$(1+\kappa)Z \in \Omega_1, \qquad (1+\kappa)MZ \in \Omega_2. \tag{34}$$

From the arbitrariness of κ, we obtain $\Omega_1^* = \Omega_2^* = R_+^n$. Moreover,

$$\begin{aligned}\Omega_3 &= \{z \in R_+^n \mid \Delta(z) \geq 0\} \\ &= \left\{z \in R_+^n \mid M^{-1}W(\Lambda - I)z + \mu + W\left[I - e^{-\theta W}\right]^{-1}\nu \geq 0\right\} \\ &= \left\{z \in R_+^n \mid (I - \Lambda)z \leq W^{-1}M\left(\mu + W\left[I - e^{-\theta W}\right]^{-1}\nu\right)\right\} \\ &\subset \left\{z \in R_+^n \mid z \leq (I - \Lambda)^{-1}W^{-1}M \right.\\ &\qquad \left.\times\left(\mu + W\left[I - e^{-\theta W}\right]^{-1}\nu\right)\right\}.\end{aligned} \tag{35}$$

It follows from Theorem 5 that $S' = \{\phi \in \mathrm{PC} \mid [\phi]_\tau^+ \in \Omega_3\}$ is a global attracting set of (2) and S is also a global attracting set due to $S' \subset S$. The proof is complete. □

Corollary 7. *Assume that all conditions in Corollary 6 hold with $\mu = \nu = 0$. Then, the zero solution $x(t) = 0$ of (2) is globally asymptotically stable.*

4. Illustrative Examples

The following illustrative examples will demonstrate the effectiveness of our results and also show the different asymptotical behaviors between the impulsive system and the corresponding continuous system.

Example 8. Consider a scalar nonlinear impulsive delay system

$$\begin{aligned}\dot{x}(t) &= 0.2x(t) + 0.2x^2(t-1) + 0.1, \\ &\qquad t \neq t_k,\ k \in N,\ t \geq t_0 = 0, \\ \Delta x &= -0.6x(t_k^-) + 0.1x^2(t_k^-) + 0.1\sin\left(e^{t_k}\right), \\ &\qquad t_k = t_{k-1} + 0.15.\end{aligned} \tag{36}$$

According to Theorems 4 and 5, we have $\theta = \varrho = 0.15$, $A = 0.2$, $B = -0.6$, $M = 2.5$, $W \doteq 5.9086$, $p(z) = 0.2z^2 + 0.1$, $q(z) = 0.1z^2 + 0.1$, $\Delta(z) \doteq 1.2052z^2 - 2.3634z + 1.1052$, and so

$$\begin{aligned}\Omega_1 &= \{z \in R_+ \mid \Delta(Mz) < 0\} = (0.3079, 0.4765), \\ \Omega_1^* &= [0, 0.4765], \\ \Omega_2 &= \{z \in R_+ \mid \Delta(z) < 0\} = (0.7698, 1.1913), \\ \Omega_2^* &= [0, 1.1913], \\ \Omega_3 &= \{z \in R_+ \mid \Delta(z) \geq 0\} = [0, 0.7698] \cup [1.1913, +\infty), \\ \Omega_2^* \cap \Omega_3 &= [0, 0.7698].\end{aligned} \tag{37}$$

Thus, $S = \{\phi \in \mathrm{PC} \mid [\phi]_\tau^+ \leq 0.7698\}$ is an attracting set of (36), and $D = \{\phi \in \mathrm{PC} \mid [\phi]_\tau^+ \leq 0.4765\}$ is an attracting basin of S. However, solutions of the corresponding continuous system (i.e., $\Delta x = 0$ in (36)) may be unbounded. Taking the initial condition $\phi(s) = 0.2$, $s \in [-1, 0]$, Figure 1 shows the different asymptotic behavior between the solution of (36) with no impulse and one with impulses.

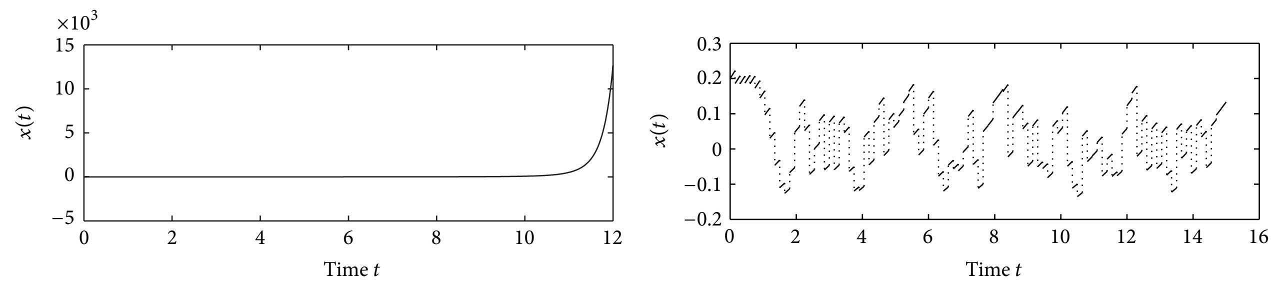
Example 9. Consider a 2-dimensional impulsive delay system

$$\begin{aligned}\dot{x}_1(t) &= x_1(t) + 0.5\sin(x_1(t-1)) - 0.4x_2(t-1) - 0.5, \\ &\qquad t \geq 0, \\ \dot{x}_2(t) &= -4x_2(t) - 0.5x_1(t-1) + 0.4\cos(x_2(t-1)) + 0.5, \\ &\qquad t \neq t_k, \\ \Delta x_1 &= -0.5x_1(t_k^-) + 0.1\cos(x_1(t_k^-)) + 0.5\sin\left(e^{t_k}\right), \\ &\qquad t_k = 0.1k, \\ \Delta x_2 &= 0.1x_2(t_k^-) + 0.2\sin(x_2(t_k^-)) - 0.5\cos\left(e^{t_k}\right), \\ &\qquad k \in N.\end{aligned} \tag{38}$$

According to Corollary 6, we have $\theta = \varrho = 0.1$, $A = \mathrm{diag}\{1, -4\}$, $B = \mathrm{diag}\{-0.5, 0.1\}$, $M = \mathrm{diag}\{2, 1\}$, $W = \mathrm{diag}\{5.9315, 3.0469\}$, $p(z) = Pz + \mu$, $q(z) = Qz + \nu$, $\Lambda = W^{-1}MP + M[I - e^{-\theta W}]^{-1}Q$, where

$$P = \begin{pmatrix} 0.5 & 0.4 \\ 0.5 & 0.4 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix},$$
$$\mu = \nu = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 0.6156 & 0.1349 \\ 0.1641 & 0.8928 \end{pmatrix}, \tag{39}$$

and so $\rho(\Lambda) = 0.9575 < 1$. Therefore, $D = \{\phi \in \mathrm{PC} \mid [\phi]_\tau^+ \leq (1.7105, 2.5228)^T\}$ is a positive quasi-invariant set of (38), and $S = \{\phi \in \mathrm{PC} \mid [\phi]_\tau^+ \leq (1.8637, 2.8720)^T\}$ is a global attracting set of (38). Figure 2 shows the asymptotic properties of solutions of (38) under the different initial conditions.



(a) (b)

Figure 1: The trajectory of (36) with: (a) no impulse (i.e., $\Delta x = 0$) and (b) impulses.

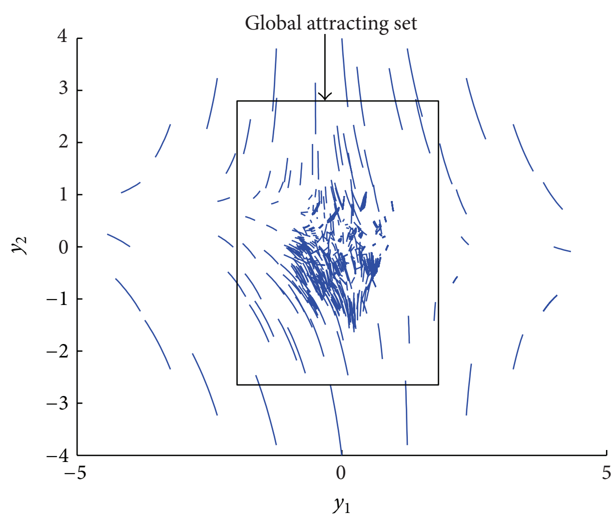


Figure 2: Global attracting set of (38).

Acknowledgments

This work is supported partially by the National Natural Science Foundation of China under Grant nos. 10971240, 61263020, and 61004042, the Key Project of Chinese Education Ministry under Grant no. 212138, the Natural Science Foundation of Chongqing under Grant CQ CSTC 2011BB0117, and the Foundation of Science and Technology project of Chongqing Education Commission under Grant KJ120630.

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