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INTEGRAL OPERATORS OF CERTAIN UNIVALENT FUNCTIONS

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ABSTRAT. A function f, analytic in the unit disc Δ, is said to be in the family $R_n(\alpha)$ if $\mathrm{Re}\{(z^n f(z))^{(n+1)}/(z^{n-1}f(z))^{(n)}\} > (n+\alpha)/(n+1)$ for some $\alpha(0 \le \alpha < 1)$ and for all z in Δ, where $n \in No$, $No = \{0,1,2,. .\}$. The The class $R_n(\alpha)$ contains the starlike functions of order α for $n \ge 0$, and the convex functions of order α for $n \ge 1$. We study a class of integral operators defined on $R_n(\alpha)$. Finally an argument theorem is proved.

KEY WORDS AND PHRASES: Univalent, convolution, starlike, convex

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1 INTRODUCTION.

Let A denote the family of functions f which are analytic in the unit disc $\Delta = \{z:|z| < 1\}$ and normalised such that $f(0) = 0 = f'(0) - 1$. The Hadamard product or convolution of two functions $f,g \in A$ is denoted by $f*g$. Let $D^n f = (z/(1-z)^{n+1})*f$, $n \in No = \{0,1,2,...\}$ which implies that

$$D^n f = z(z^{n-1}f)^{(n)}/n! \ , \ n \in No \ .$$

Denote by $S^*(\alpha)$ and $K(\alpha)$ the subfamilies of A whose members are, respectively, starlike of order α and convex of order α, $0 \le \alpha < 1$. Then

$$f \in S^*(\alpha) \iff \mathrm{Re}(D^1 f/D^0 f) > \alpha, \ z \in \Delta,$$

$$f \in K(\alpha) \iff \mathrm{Re}(D^2 f/D^1 f) > (1+\alpha)/2, z \in \Delta$$

Ruscheweyh [16] introduced the classes $\{K_n\}$ of functions $f \in A$ which satisfy the condition

$$\mathrm{Re}(D^{n+1}f/D^n f) > \frac{1}{2} \ , \ z \in \Delta \tag{1.1}$$

so that the definition of K_n is a natural extension of $S^*(1/2)$, and K(0) He proved that $K_{n+1} \subset K_n$ for each $n \in N_o$ Since $K_o = S^*(1/2)$, the elements of K_n are univalent and starlike of order 1/2.

In this paper, we consider the classes of functions $f \in A$ which satisfy the condition

$$\mathrm{Re}(z(D^n f)'/D^n f) > \alpha, \ z \in \Delta \tag{1.2}$$

for some $\alpha(0 \leq \alpha < 1)$ We denote these classes by $R_n(\alpha)$ We have $R_o(\alpha) = S^*(\alpha)$ and $R_1(\alpha) = K(\alpha)$ for $0 \leq \alpha < 1$. The classes $R_n = R_n(0)$ were considered earlier by Singh and Singh [17]. It is readily seen that for each $n \geq 0$, $R_n(\alpha) \subset R_n(0)$ and for each $n \geq 1$, $R_n(\alpha) \subset K_n$ We note that in definition (1.2), restriction $\alpha \geq 0$ can be replaced by $\alpha \geq (1-n)/2$ for each $n \geq 1$ and, further, that the negative choices of α permit us fully to partition K_n into classes $R_n(\alpha) \subset K_n$ $(n \geq 1)$ such that

$$\bigcup_{\frac{1-n}{2} \leq \alpha < 1} R_n(\alpha) = K_n .$$

It can be easily seen that $R_{n+1}(\alpha) \subset R_n(\alpha)$ for each $n \in N_o$ and for all α. These inclusion relations establish that $R_n(\alpha) \subset S^*(\alpha)$ for each $n \geq 0$ and $R_n(\alpha) \subset K(\alpha)$ for each $n \geq 1$.

An important problem in univalent functions is the following: Given a compact family F and an operator J defined on F, is $J(f) \in F$ for every $f \in F$? Libera [11] established that the operator

$$J(f) = \frac{2}{z}\int_o^z f(t)dt \tag{1.3}$$

preserves convexity, starlikeness, and close-to-convexity. Bernardi [5] greatly generalised Libera's results. Many authors [1,2,7,8,12,15,17] studied operators of the form

$$J(f) = \frac{1+\gamma}{z^\gamma}\int_o^z t^{\gamma-1} f(t)dt , \tag{1.4}$$

where γ is a real (or complex) constant and f belongs to some favoured class of univalent functions from A . Recently, operators (1 4) have been studied in more general form by Causey and White [6], Miller, Mocanu and Reade [14], Barnard and Kellogg [3], and Bajpai [2]

In this paper, we study a class of integral operators of the form (1.4) defined on our family $R_n(\alpha)$ We also obtain an argument theorem for the class $R_n(\alpha)$.

2. INTEGRAL OPERATORS.

Let γ be a complex number with $\mathrm{Re}\gamma \neq -1$ We define h_γ by

$$h_\gamma(z) = \sum_{j=1}^{\infty} \frac{\gamma+1}{\gamma+j} z^j \ , \ z \in \Delta \ . \tag{2.1}$$

Let the operator $J:A \to A$ be defined by $F = J(f)$, where

$$F(z) = \frac{1+\gamma}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt \tag{2.2}$$

Then the function F can also we written in the form

$$F(z) = f(z) * h_\gamma(z) \ .$$

We need the following result of Jack [9] which is also due to Suffridge [18]

LEMMA. Let w be nonconstant and analytic in $|z| < r < 1, w(0) = 0$ If $|w|$ attains its maximum value on the circle $|z| = r$ at z_o , then $z_o w'(z_o) = kw(z_o)$, where k is a real number and $k \geq 1$

We first give a condition on $f \in A$ for which the function $J(f)$ belongs to $R_n(\alpha)$

THEOREM 1. Let $0 \leq \alpha < 1$, and $\gamma \neq -1$ be a complex constant such that $\text{Re}\gamma \geq -\alpha$, $\text{Im}\gamma \geq 0$, and $|\gamma|^2 + 2\alpha(1 + \text{Re}\gamma) \geq 1$. If for a given $n \in$ No, $f \in A$ satisfies the condition

$$\text{Re} \frac{z(D^n f(z))'}{D^n f(z)} > \alpha - \frac{(1-\alpha)(\alpha+\text{Re}\gamma)}{2\{|\gamma|^2+2\alpha\text{Re}\gamma+\alpha^2+(1-\alpha)\text{Im}\gamma\}} \tag{2.3}$$

for all $z \in \Delta$, then $F(z)$ given by (2.2) belongs to $R_n(\alpha)$.

PROOF From (2.2), we obtain

$$z(D^n F(z))' + \gamma D^n F(z) = (\gamma+1) D^n f(z). \tag{2.4}$$

Define w in Δ by

$$\frac{z(D^n F(z))'}{D^n F(z)} = \frac{1+(2\alpha-1)w(z)}{1+w(z)} \ . \tag{2.5}$$

Here $w(z)$ is analytic in Δ with $w(0) = 0$ and $w(z) \neq -1$, $z \in \Delta$ We need to show that $|w(z)| < 1$ for all $z \in \Delta$. In view of (2.4), (2.5) yields

$$\frac{D^n f(z)}{D^n F(z)} = \frac{(1+\gamma)+(2\alpha-1+\gamma)w(z)}{(1+\gamma)(1+w(z))} \tag{2.6}$$

Differentiating (2.6) logarithmically and simplifying, we obtain

$$\frac{z(D^n f(z))'}{D^n f(z)} = \alpha + (1-\alpha)\frac{1-w(z)}{1+w(z)} - \frac{2(1-\alpha)zw'(z)}{(1+w(z))(1+\gamma+(2\alpha-1+\gamma)w(z))} \quad (2.7)$$

Now (2 7) should yield $|w(z)| < 1$ for all $z \in \Delta$ for otherwise, there exists a point $z_o \in \Delta$ at which $|w(z_o)| = 1$ and by Lemma, we have $z_o w'(z) = kw(z_o)$, $k \geq 1$. For this value of $z = z_o$, we find that (2.7) yields

$$\mathrm{Re}\frac{z_o(D^n f(z_o))'}{D^n f(z_o)} = \alpha - \frac{2k(1-\alpha)(\alpha+\mathrm{Re}\gamma)}{|(1+\gamma)+(2\alpha-1+\gamma)w(z_o)|^2} \quad (2.8)$$

$$\leq \alpha - \frac{(1-\alpha)(\alpha+\mathrm{Re}\gamma)}{2\{|\gamma|^2+2\alpha\mathrm{Re}\gamma+\alpha^2+(1-\alpha)\mathrm{Im}\gamma\}}$$

which contradicts (2.3) Hence $|w(z)| < 1$ for all $z \in \Delta$ and by (2.5), it follows that $F(z) \in R_n(\alpha)$.

COROLLARY. If for a given $n \in N_o$, $f \in A$ satisfies the condition

$$\mathrm{Re}\frac{z(D^n f(z))'}{D^n f(z)} > \frac{2\alpha(\gamma+\alpha)-(1-\alpha)}{2(\gamma+\alpha)}, \quad z \in \Delta, \quad (2.9)$$

where (α,γ) is any point in the set

$$D = \{(\alpha,\gamma) : \gamma+2\alpha \geq 1,\ 0 \leq \alpha < 1,\ \gamma > -1\},$$

then $F(z)$ given by (2.2) belongs to $R_n(\alpha)$.

PROOF. If $\gamma \neq -1$ is a real constant such that $\gamma + \alpha \geq 0$, then $|\gamma|^2+2\alpha(1+\mathrm{Re}\gamma) \geq 1$ implies $(\gamma+1)(\gamma+2\alpha-1) \geq 0$. The result follows from Theorem 1

It is easy to show that if $f \in R_n(\alpha)$, then f satisfies the condition (2 3). Thus it follows from Theorem 1 that $J(R_n(\alpha)) \subset R_n(\alpha)$ More precisely, we state the result in

THEOREM 2 If $f \in R_n(\alpha)$, then the function

$$J(f) = \frac{\gamma+1}{z^\gamma}\int_0^z f(t)\, t^{\gamma-1}dt$$

is again an element of $R_n(\alpha)$, where $\gamma \neq -1$ is a complex constant with restrictions as stated in Theorem 1.

REMARK 1 Letting $n = 0 = \gamma - 1$ and $n = 1 = \gamma$, in Theorem 1, we get $L(S^*(\beta)) \subset S^*(\alpha)$ and $L(K(\beta)) \subset K(\alpha)$ respectively, where L is the Libera transform defined in (1.3), and

$$\beta = ((2\alpha^2+3\alpha-1)/2(1+\alpha)) < \alpha.$$

These results improve the earlier results due to Libera [11] and Bernardi [5] in the sense that their results hold under much weaker conditions

In [2], Bajpai has established that $J(S^*) \subset S^*(\alpha)$ for some α. We generalize this result in

THEOREM 3. Let $J:A \to A$ be defined as in (2 2), where γ is a complex constant. If $f \in R_n$, then $J(f) \in R_n(\alpha)$, where α satisfies the inequality

$$\alpha[|1+\gamma|+|2\alpha-1+\gamma|]^2 \le 2(1-\alpha)(\alpha+\mathrm{Re}\gamma), \quad \text{and} \quad 0 \le \alpha < 1$$

PROOF Proceeding as in Theorem 1 and applying Lemma, we have

$$\mathrm{Re}\,\frac{z_0(D^nf(z_0))'}{D^nf(z_0)} \le \alpha - \frac{2(1-\alpha)(\alpha+\mathrm{Re}\gamma)}{|(1+\gamma)+(2\alpha-1+\gamma)w(z_0)|^2}$$

$$\le \alpha - \frac{2(1-\alpha)(\alpha+\mathrm{Re}\gamma)}{(|1+\gamma|+|2\alpha-1+\gamma|)^2},$$

where $\mathrm{Re}\gamma \ge -\alpha$. Since the right hand side is ≤ 0, we have a contradiction for $f \in R_n \equiv R_n(0)$. Thus we must have $|w(z)| < 1$ for all z in Δ and by (2 5), it follows that $J(f) \in R_n(\alpha)$.

REMARK 1 If we let $n=0=\gamma-1$ in the above theorem, then $L(S^*) \subset S^*(\frac{\sqrt{17}-3}{4})$, where $L(f) = (2/z)\int_0^z f(t)dt$ Thus we have recovered a result of Miller, Mocanu and Reade ([14], pp 162-163).

REMARK 2 If $n = 1$, γ is a real constant such that $\gamma+\alpha \ge 0$, and $f \in K$, then it follows from Theorem 3 that the function $F(z)$ in (2 2) is an element of $K(\alpha)$, where

$$\alpha = \frac{-(2\gamma+1) + \sqrt{(2\gamma-1)^2+8(1+\gamma)}}{4}.$$

This result was proved by Miller, Mocanu and Reade ([14], pp 165) Further, this is an improvement of an earlier result due to Bernardi [5], who proved that $f \in K$ implies $F \in K$.

For $\gamma = n$, where $n \in N_0$, we have an improvement over Theorem 2

THEOREM 4. Let

$$F(z) = f(z) * h_n(z) = \frac{n+1}{z^n}\int_0^z f(t)t^{n-1}dt \tag{2.10}$$

If $f \in R_n(\alpha)$, then $F \in R_{n+1}(\alpha)$

PROOF. From (2.10), we obtain

$$z(D^{n+1}F(z))' + nD^{n+1}F(z) = (n+1)D^{n+1}f(z) \tag{2.11}$$

and

$$z(D^nF(z))' + nD^nF(z) = (n+1)D^nf(z) \tag{2.12}$$

Using the identity

$$z(D^nf(z))' = (n+1)D^{n+1}f(z) - nD^nf(z) \tag{2.13}$$

in (2.11) and (2.12), we obtain

$$(n+1)D^{n+1}f(z) = (n+2)D^{n+2}F(z) - D^{n+1}F(z) \tag{2.14}$$

and

$$D^nf(z) = D^{n+1}F(z) \tag{2.15}$$

In view of the identity (2 13) and the relations (2.14) and (2 15), $f \in R_n(\alpha)$ yields

$$\text{Re}\ \{\frac{(n+2)D^{n+2}F(z) - (n+1)D^{n+1}F(z)}{D^{n+1}F(z)}\} > \alpha$$

which implies that

$$\text{Re}\ \{\frac{z(D^{n+1}F(z))'}{D^{n+1}F(z)}\} > \alpha\ , \quad z \in \Delta$$

This proves that $F \in R_{n+1}(\alpha)$.

REMARK For $n = 0$, Theorem 4 gives the well known result: $J(S^*(\alpha)) \subset K(\alpha)$, where $J(f) = \int_0^z (f(t)/t)dt$

We now investigate the converse of Theorem 2. In fact, we find the sharp radius of the disc in which $f \in R_n(\beta)$ when F, defined in (2.2), is in $R_n(\alpha)$ for $0 \le \alpha < 1$, $0 < \beta \le 1$: In [12], Libera and Livingston have solved this converse problem for the case $n = 0$, $\gamma = 1$ when $\alpha \le \beta < 1$. These authors were not able to obtain suitable results for the complementary case when $\beta < \alpha$ However, the method used in the next theorem gives results that are more general and also covers both $\beta \ge \alpha$ and $\beta < \alpha$.

THEOREM 5. If F is an element of $R_n(\alpha)$ for $n \ge 0$ and $0 \le \alpha < 1$,

$$F(z) = \frac{1+\gamma}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt \tag{2.16}$$

with $z \in \Delta$, $\text{Re}\gamma \ge -\alpha$, and $0 \le \beta < 1$, then the function f is an element of $R_n(\beta)$ for $|z| < r_o$, where r_o is the smallest positive root in (0,1) of the equation

$$(\gamma+2\alpha-1)(2\alpha-\beta-1)r^2+2((\gamma+\alpha)(\alpha-\beta)-(1-\alpha)(2-\alpha))r+(\gamma+1)(1-\beta) = 0 \tag{2.17}$$

The result is sharp

PROOF Since $F \in R_n(\alpha)$, we can write

$$\frac{z(D^nF(z))'}{D^nF(z)} = \alpha + (1-\alpha)P_n(z), \tag{2.18}$$

where $P_n(z)$ is analytic in Δ and satisfies the conditions $P_n(0) = 1$ $\mathrm{Re}P_n(z) > 0$ for $z \in \Delta$ Using the identity

$$z(D^nF(z))' = (n+1)D^{n+1}F(z) - nD^nF(z) \tag{2.19}$$

in (2.18) and then taking logarithmic derivative, we obtain

$$z(D^{n+1}F(z))' = D^{n+1}F(z)[\alpha+(1-\alpha)P_n(z) + \frac{(1-\alpha)zP_n'(z)}{n+\alpha+(1-\alpha)P_n(z)}] \tag{2.20}$$

From (2 16) we obtain

$$z(D^{n+1}F(z))' + \gamma D^{n+1}F(z) = (\gamma+1)D^{n+1}f(z). \tag{2.21}$$

From (2 20) and (2 21) we have

$$(\gamma+1)D^{n+1}f(z) = D^{n+1}F(z)\ [\alpha+\gamma+(1-\alpha)P_n(z) + \frac{(1-\alpha)zP_n'(z)}{n+\alpha+(1-\alpha)P_n(z)}] \tag{2.22}$$

Also (2.18) together with the identity (2 4) yields

$$(1+\gamma)D^nf(z) = D^nF(z)(\alpha+\gamma+(1-\alpha)P_n(z)). \tag{2.23}$$

Now from the relations (2 22), (2 23), and (2.18) we conclude that

$$\frac{z(D^nf(z))'}{D^nf(z)} - \beta = \alpha - \beta + (1-\alpha)P_n(z) + \frac{(1-\alpha)zP_n'(z)}{\alpha+\gamma+(1-\alpha)P_n(z)}. \tag{2.24}$$

Using the well known estimates

$$|zP_n'(z)| \le (2r/(1-r^2))\mathrm{Re}\,P_n(z)$$

and

$$\mathrm{Re}P_n(z) \ge (1-r)/(1+r), \quad |z| = r$$

in (2 24), we obtain

$$\mathrm{Re}\ [\frac{z(D^nf(z))'}{D^nf(z)} - \beta] \ge (\alpha-\beta) + \frac{(1-\alpha)((1-r)(\gamma+1+(\gamma+2\alpha-1)r)-2r)}{(1-r)((\gamma+2\alpha-1)r+\gamma+1)} \tag{2.25}$$

where $\mathrm{Re}\gamma \ge -\alpha$. Therefore,

$$\mathrm{Re}\ \{\frac{z(D^nf(z))'}{D^nf(z)}\} > \beta$$

if the right side of (2.25) is positive, which is satisfied provided that $r < r_o$, where r_o is the smallest positive root in (0,1) of (2.17).

The result in the theorem is sharp with the function f defined by

$$f(z) = (1/(1+c))z^{1-c}(z^cF(z))', \tag{2.26}$$

where $c = \text{Re}\gamma \geq -\alpha$, and F is given by

$$z\frac{(D^nF(z))'}{D^nF(z)} = \frac{1-(2\alpha-1)z}{1-z} \tag{2.27}$$

REMARK. By specializing choices of α,β,γ, and n, theorem 5 gives rise to the corresponding results obtained earlier in [3,4,8,12,13,15] and by many others

3 AN ARGUMENT THEOREM.

THEOREM 6 If $f \in R_n(\alpha)$, then

$$\left|\arg \frac{D^kf(z)}{z}\right| \leq 2(1-\alpha)\sin^{-1}r + \sum_{m=0}^{k-1}\sin^{-1}\left(\frac{2(1-\alpha)r}{m+1-(m+2\alpha-1)r^2}\right)$$

for each $k(0 \leq k \leq n+1)$.

PROOF We may write

$$\frac{D^kf(z)}{z} = \frac{f(z)}{z}\prod_{m=0}^{k-1}\frac{D^{m+1}f(z)}{D^mf(z)}, \quad 0 \leq k \leq n+1,$$

which yields

$$\left|\arg \frac{D^kf(z)}{z}\right| \leq \left|\arg \frac{f(z)}{z}\right| + \sum_{m=0}^{k-1}\left|\arg \frac{D^{m+1}f(z)}{D^mf(z)}\right| . \tag{3.1}$$

Since $R_{n+1}(\alpha) \subset R_n(\alpha)$ $\forall n \in N_o$, it follows that $f \in R_m(\alpha)$ for each $m(0 \leq m \leq n)$ Setting

$$\frac{D^{m+1}f(z)}{D^mf(z)} = q_m(z), \qquad (0 \leq m \leq n), \tag{3.2}$$

we note that $\text{Re}(q_m(z)) \geq (m+\alpha)/(m+1)$

Therefore, the function

$$w(z) = \frac{\left(q_m(z) - \frac{m+\alpha}{m+1}\right) - \left(1 - \frac{m+\alpha}{m+1}\right)}{\left(q_m(z) - \frac{m+\alpha}{m+1}\right) + \left(1 - \frac{m+\alpha}{m+1}\right)} = \frac{q_m(z) - 1}{q_m(z) - \left(\frac{2(m+\alpha)}{m+1} - 1\right)}$$

is analytic with $w(0) = 0$ and $|w(z)| < 1$ in Δ Hence by Schwarz's Lemma,

$$\left| \frac{q_m(z) - 1}{q_m(z) + 1 - 2(m+\alpha)/(m+1)} \right| < |z|$$

for z in Δ Now it is easy to see that the values of $q_m(z)$ are contained in the circle of Appolonius whose centre is at the point $(m+1-(m+2\alpha-1)r^2)/((1+m)(1-r^2))$ and has radius $2(1-\alpha)r/((m+1)(1-r^2))$ Thus $\max_{z\in\Delta}|\arg q_m(z)|$ is attained at the points where

$$\arg q_m(z) = \pm \sin^{-1}\left(\frac{2(1-\alpha)r}{m+1-(m+2\alpha-1)r}\right)$$

which gives

$$\left|\arg \frac{D^{m+1}f(z)}{D^m f(z)}\right| \le \sin^{-1}\left(\frac{2(1-\alpha)r}{m+1-(m+2\alpha-1)r}\right) , \tag{3.3}$$

for $0 \le m \le n$, and $z \in \Delta$

Next, note that $R_n(\alpha) \subset S^*(\alpha)$, $n \ge 0$, and $f \in S^*(\alpha)$ if and only if $F(z) = \int (f(z)/z)dz$ is in $K(\alpha)$ But for $F \in K(\alpha)$, we have

$$|\arg F'(z)| \le 2(1-\alpha)\sin^{-1} r \qquad (|z| = r)$$

Thus $f \in R_n(\alpha)$ implies

$$\left|\arg \frac{f(z)}{z}\right| \le 2(1-\alpha)\sin^{-1} r \tag{3.4}$$

Applying (3 3) and (3.4) to (3.1) we obtain the result.

For $n = 0$, we obtain

COROLLARY If $f \in S^*(\alpha)$, then (3.4)

and

$$|\arg f'(z)| \le 2(1-\alpha)\sin^{-1} r + \sin^{-1}\left(\frac{2(1-\alpha)r}{1-(2\alpha-1)r^2}\right)$$

REMARK The case $n = 0$, $\alpha = 0$ way proved by Krzyz [10].

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REFERENCES

1. AL-AMIRI, H.S.: Certain generalizations of pre-starlike functions, J. Australian Math. Soc (Serie A) 28(1979), 325-334

2. BAJPAI, S.K : Spirallike integral operators, Internat. J. Math. & Math. Sci., Vol 2 (1981), 337-351.

3 BARNARD, R W and KELLOGG C.: Applications of convolution operators to problems in univalent function theory, Mich. Math J. 27 (1980), no 1, 81-94

4. BERNARDI, S D.: The radius of univalence of certain analytic functions, Proc Amer Math. Soc 24 (1970), 312-318.

5 BERNARDI, S.D.: Convex and starlike univalent functions, Trans. Amer. Math. Soc. 135 (1969), 429-446

6 CAUSEY, W.M and WHITE W.L.: Starlikeness of certain functions with integral representations, J. Math. Anal. Appl. 64 (1978), 458-466.

7. GOODMAN, A.W : Univalent functions, Vol II, Mariner Publishing Company, Inc, 1983.

8. GUPTA, V P. and JAIN, P K.: On starlike functions, Rendiconti di Mat 9 (1976), 433-437.

9 JACK, I S.: Functions starlike and convex of order α, J. London Math Soc., 3(1971), 469-474.

10 KRZYZ, J.: On the derivative of close-to-convex of order α, J. London Math. Soc., 3 (1971), 469-474.

11. LIBERA, J.R.: Some classes of regular univalent functions, Proc. Amer Math. Soc. 16 (1965), 755-758.

12 LIBERA, R.J. and LIVINGSTON, A.E.: On the univalence of some classes of regular functions, Proc. Amer. Math Soc. 30 (1971), 327-336

13 LIVINGSTON, A.E.: On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 17 (1966), 352-357.

14 MILLER, S.S., MOCANU, P.T. and READE, MAXWELL O.: Starlike integral operators, Pacific J. Math. 79 (1978), 157-168

15. PADMANABHAN, K.S.: On the radius of univalence of certain classes of analytic functions, J. London Math. Soc. (2) 1 (1969), 225-231

16 RUSCHEWEYH, S.: New criteria for univalent functions, Proc. Amer. Math. Soc. 49 (1975), 109-115.

17. SINGH, R. and SINGH, S.: Integrals of certain univalent functions, Proc. Amer Math. Soc. 77 (1979), 336-340.

18 SUFFRIDGE, T.J.: Some remarks on convex maps of the unit disk, Duke Math J. 37 (1970), 775-777