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NONLINEAR VARIATIONAL EVOLUTION INEQUALITIES IN HILBERT SPACES

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ABSTRACT. The regular problem for solutions of the nonlinear functional differential equations with a nonlinear hemicontinuous and coercive operator A and a nonlinear term $f(\cdot,\cdot)$: $x'(t)+Ax(t)+\partial\phi(x(t)) \ni f(t,x(t))+h(t)$ is studied. The existence, uniqueness, and a variation of solutions of the equation are given.

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1. Introduction. Let H and V be two real separable Hilbert spaces such that V is a dense subspace of H. Let the operator A be given a single-valued operator, which is hemicontinuous and coercive from V to V^*. Here V^* stands for the dual space of V. Let $\phi : V \to (-\infty,+\infty]$ be a lower semicontinuous, proper convex function. Then the subdifferential operator $\partial\phi : V \to V^*$ of ϕ is defined by

$$\partial\phi(x) = \{x^* \in V^*; \phi(x) \le \phi(y)+(x^*,x-y),\ y \in V\}, \tag{1.1}$$

where $(\cdot,\cdot)$ denotes the duality pairing between V^* and V. We are interested in the following nonlinear functional differential equation on H:

$$\begin{gathered}\frac{dx(t)}{dt}+Ax(t)+\partial\phi(x(t)) \ni f(t,x(t))+h(t), \quad 0<t\le T,\\ x(0)=x_0,\end{gathered} \tag{1.2}$$

where the nonlinear mapping f is a Lipschitz continuous from $\mathbb{R}\times V$ into H. Equation (1.2) is caused by the following nonlinear variational inequality problem:

$$\begin{gathered}\left(\frac{dx(t)}{dt}+Ax(t),x(t)-z\right)+\phi(x(t))-\phi(z)\\ \le (f(t,x(t))+h(t),x(t)-z), \quad \text{a.e., } 0<t\le T,\ z\in V,\\ x(0)=x_0.\end{gathered} \tag{1.3}$$

If A is a linear continuous symmetric operator from V into V^* and satisfies the coercive condition, then equation (1.2), which is called the linear parabolic variational inequality, is extensively studied in Barbu [5, Sec. 4.3.2] (also see [4, Sec. 4.3.1]). The existence of solutions for the semilinear equation with similar conditions for nonlinear term f have been dealt with in [1, 2, 6]. Using more general hypotheses for nonlinear term $f(\cdot,x)$, we intend to investigate the existence and the norm estimate of a solution of the above nonlinear equation on $L^2(0,T;V)\cap W^{1,2}(0,T;V^*)$, which is also applicable to optimal control problem. A typical example was given in the last section.

2. Perturbation of subdifferential operator. Let H and V be two real Hilbert spaces. Assume that V is a dense subspace in H and the injection of V into H is continuous. If H is identified with its dual space we may write $V\subset H\subset V^*$ densely and the corresponding injections are continuous. The norm on V (respectively H) will be denoted by $\|\cdot\|$ (respectively $|\cdot|$). The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1,v_2), which is the ordinary inner product in H if $v_1,v_2\in H$. For the sake of simplicity, we may consider

$$\|u\|\le|u|\le\|u\|_*,\quad u\in V, \tag{2.1}$$

where $\|\cdot\|_*$ is the norm of the element of V^*.

Remark 2.1. If an operator A_0 is bounded linear from V to V^* and generates an analytic semigroup, then it is easily seen that

$$H=\left\{x\in V^*:\int_0^T\|A_0e^{tA_0}x\|_*^2\,dt<\infty\right\}\quad\text{for the time } T>0. \tag{2.2}$$

Therefore, in terms of the intermediate theory we can see that

$$(V,V^*)_{1/2,2}=H, \tag{2.3}$$

where $(V,V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*.

We note that nonlinear operator A is said to be hemicontinuous on V if

$$\operatorname*{w-lim}_{t\to0}A(x+ty)=Ax\quad\text{for every } x,y\in V, \tag{2.4}$$

where "w-lim" indicates the weak convergence on V. Let $A:V\to V^*$ be given a single valued and hemicontinuous from V to V^* such that

$$\begin{gathered}A(0)=0,\qquad (Au-Av,u-v)\ge\omega_1\|u-v\|^2-\omega_2|u-v|^2,\\ \|Au\|_*\le\omega_3(\|u\|+1)\end{gathered} \tag{2.5}$$

for every $u,v\in V$, where $\omega_2\in\Re$ and ω_1,ω_3 are some positive constants. Here, we note that if $A(0)\ne0$ we need the following assumption:

$$(Au,u)\ge\omega_1\|u\|^2-\omega_2|u|^2\quad\text{for every } u\in V. \tag{2.6}$$

It is also known that $A+\omega_2I$ is maximal monotone and $R(A+\omega_2I)=V^*$ where $R(A+\omega_2I)$ is the range of $A+\omega_2I$ and I is the identity operator.

First, let us be concerned with the following perturbation of subdifferential operator:

$$\frac{dx(t)}{dt}+Ax(t)+\partial\phi(x(t))\ni h(t),\quad 0<t\le T,\qquad x(0)=x_0. \tag{2.7}$$

To prove the regularity for the nonlinear equation (1.2) without the nonlinear term $f(\cdot,x)$ we apply the method in [5, Sec. 4.3.2].

Proposition 2.1. *Let $h \in L^2(0,T;V^*)$ and $x_0 \in V$ satisfying that $\phi(x_0) < \infty$. Then (2.7) has a unique solution*

$$x \in L^2(0,T;V) \cap C\Big([0,T];H\Big) \tag{2.8}$$

which satisfies

$$\|x\|_{L^2\cap C} \le C_1\Big(1+\|x_0\|+\|h\|_{L^2(0,T;V^*)}\Big), \tag{2.9}$$

where C_1 is a constant and $L^2 \cap C = L^2(0,T;V) \cap C([0,T];H)$.

Proof. Substituting $v(t) = e^{\omega_2 t}x(t)$ we can rewrite (2.7) as follows:

$$\begin{gathered} \frac{dv(t)}{dt} + (A+\omega_2 I)v(t) + e^{-\omega_2 t}\partial\phi(v(t)) \ni e^{-\omega_2 t}h(t), \quad 0 < t \le T, \\ v(0) = e^{\omega_2 t}x_0. \end{gathered} \tag{2.10}$$

Then the regular problem for (2.7) is equivalent to that for (2.10). Consider the operator $L : D(L) \subset H \to H$

$$\begin{gathered} Lv = \{Av + e^{-\omega_2 t}\,\partial\phi(v) + \omega_2 v\} \cap H \quad \forall v \in D(L), \\ D(L) = \Big\{v \in V; \{Av + e^{-\omega_2 t}\,\partial\phi(v) + \omega_2 v\} \cap H \ne 0\Big\}. \end{gathered} \tag{2.11}$$

Since $A + \omega_2 I$ is a monotone, hemicontinuous and bounded operator from V into V^* and $e^{-\omega_2 t}\partial\phi$ is maximal monotone, we infer in [4, Cor. 1.1 of Ch. 2] that L is maximal monotone. Then in [5, Thm. 1.4] (also see [4, Thm. 2.3, Cor. 2.1]), for every $x_0 \in D(L)$ and $h \in W^{1.1}([0,T];H)$, the Cauchy problem (2.10) has a unique solution $v \in W^{1,\infty}([0,T];H)$. Let us assume that $x_0 \in D(L)$ and $h \in W^{1,2}(0,T;H)$. Multiplying (2.7) by $x - x_0$ and using (2.5) and the maximal monotonicity of $\partial\phi$ it holds

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|x(t)-x_0|^2 + \omega_1\|x(t)-x_0\|^2 \\ &\qquad \le \omega_2|x(t)-x_0| + (h(t) - Ax_0 - \partial\phi(x_0), x(t)-x_0). \end{aligned} \tag{2.12}$$

Since

$$\begin{aligned} (h(t) - Ax_0 - \partial\phi(x_0), x(t)-x_0) &\le \|h(t) - Ax_0 - \partial\phi(x_0)\|_*\|x(t)-x_0\| \\ &\le \frac{1}{2c}\|h(t) - Ax_0 - \partial\phi(x_0)\|_*^2 + \frac{c}{2}\|x(t)-x_0\|^2 \end{aligned} \tag{2.13}$$

for every real number c, so using Gronwall's inequality, the inequality (2.12) implies that

$$|x(t)-x_0|^2 + \int_0^t \|x(s)-x_0\|^2\,ds \le C_1\left(\int_0^t \|h(s)\|_*^2\,ds + \|x_0\|^2 + 1\right) \tag{2.14}$$

for some positive constant C_1, that is,

$$\|x\|_{L^2(0,T;V)\cap C([0,T];H)} \le C_1\Big(1+\|x_0\|+\|h\|_{L^2(0,T;V^*)}\Big). \tag{2.15}$$

Hence we have proved (2.9). Let $x_0 \in V$ such that $\partial\phi(x_0) < \infty$ and $h \in L^2(0,T;V^*)$. Then there exist sequences $\{x_{0n}\} \subset D(L)$ and $\{h_n\} \subset W^{1,2}(0,T;H)$ such that $x_{0n} \to x_0$

in V and $h_n \to h$ in $L^2(0,T;V^*)$ as $n \to \infty$. Let $x_n \in W^{1,\infty}(0,T;H)$ be the solution of (2.7) with initial value x_{0n} and with h_n instead of h. Since $\partial\phi$ is monotone, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left|x_n(t)-x_m(t)\right|^2&+\omega_1\left\|x_n(t)-x_m(t)\right\|^2\\&<(h_n(t)-h_m(t),x_n(t)-x_m(t))+\omega_2\left|x_n(t)-x_m(t)\right|^2\\&\le\frac{1}{2c}\|h_n(t)-h_m(t)\|_*^2+\frac{c}{2}\|x_n(t)-x_m(t)\|^2\\&\quad+\omega_2\left|x_n(t)-x_m(t)\right|^2,\quad \text{a.e., } t\in(0,T)\end{aligned}\tag{2.16}$$

for every real number c. Therefore, if we choose $\omega_1-(c/2)$ then by integrating over $[0,T]$ and using Gronwall's inequality it follows that

$$\begin{aligned}\left|x_n(t)-x_m(t)\right|&+2\left(\omega_1-\frac{c}{2}\right)\left\|x_n(t)-x_m(t)\right\|_{L^2(0,T;V)}\\&\le e^{2\omega_2T_1}\left(\left|x_{0n}-x_{0m}\right|+c^{-1}\left\|h_n-h_m\right\|_{L^2(0,T;V^*)}\right),\end{aligned}\tag{2.17}$$

and hence, we have that $\lim_{n\to\infty}x_n(t)=x(t)$ exists in H. Furthermore, x satisfies (2.7). Indeed, for all $0\le s<t\le T$ and $y\in\partial\phi(x)$, multiplying (2.7) by $x(t)-x$ and integrating over $[s,t]$ we have

$$\begin{aligned}\frac{1}{2}\left(\left|x(t)-x\right|^2-\left|x(s)-x\right|^2\right)&\le\int_s^t(h(\tau)-Ax-y,x(\tau)-x)\,d\tau\\&\quad+\omega_2\int_s^t\left|x(\tau)-x\right|^2d\tau,\end{aligned}\tag{2.18}$$

and, therefore,

$$\begin{aligned}\left(\frac{x(t)-x(s)}{t-s},x(s)-x\right)&\le\frac{1}{t-s}\int_s^t(h(\tau)-Ax-y,x(\tau)-x)\,d\tau\\&\quad+\frac{\omega_2}{t-s}\int_s^t\left|x(\tau)-x\right|^2d\tau.\end{aligned}\tag{2.19}$$

This implies

$$\left(\frac{d}{dt}x(t),x(t)-x\right)\le\left(h(t)-Ax-y+\omega_2(x(t)-x),x(t)-x\right),\tag{2.20}$$

a.e., $t\in(0,T)$, that is,

$$\left(\frac{d}{dt}x(t)-h(t)-\omega_2x(t)+(Ax+y+\omega_2x),x(t)-x\right)\le 0.\tag{2.21}$$

Since $A+\partial\phi+\omega_2I$ is maximal monotone, we have

$$\frac{d}{dt}x(t)-h(t)-\omega_2x(t)\in-(A+\partial\phi+\omega_2I)x(t),\quad \text{a.e., } t\in(0,T).\tag{2.22}$$

Thus, the proof is complete. □

Corollary 2.1. *Assume the hypotheses as in Proposition 2.1, in addition, assume that $\partial\phi$ satisfies the growth condition as follows:*

$$\|z\|_*\le M(|x|+1),\quad \text{a.e., } x\in D(\phi),\ z\in\partial\phi(x).\tag{2.23}$$

Then equation (2.7) has a unique solution

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \cap C([0,T];H) \tag{2.24}$$

which satisfies

$$\|x\|_{L^2 \cap W^{1,2} \cap C} \le C\Big(1 + \|x_0\| + \|h\|_{L^2(0,T;V^*)}\Big). \tag{2.25}$$

Proof. From (2.7) and (2.23) it follows that

$$\left\|\frac{d}{dt}x(t)\right\|_* + \omega_1 \|x(t)\| \le \omega_2 |x(t)| + M(|x(t)| + 1) + \|h(t)\|_*. \tag{2.26}$$

Hence, by virtue of (2.15) we have that

$$\|x\|_{W^{1,2}(0,T;H)} \le C_2\Big(1 + \|x_0\| + \|h\|_{L^2(0,T;V^*)}\Big). \tag{2.27}$$

□

Remark 2.2. If V is compactly imbedded in H, the imbedding $L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset L^2(0,T;H)$ is compact in Aubin [3, Rem. 1, Thm. 2]. Hence, the mapping $h \mapsto x$ is compact from $L^2(0,T;H)$ to $L^2(0,T;H)$.

3. Nonlinear integrodifferential equation. Let $f : [0,T] \times V \to H$ be a nonlinear mapping satisfying the following variational evolution inequality:

$$|f(t,x) - f(t,y)| \le L\|x - y\|, \qquad f(t,0) = 0 \tag{3.1}$$

for a positive constant L.

Theorem 3.1. *Let (2.5) and (3.1) be satisfied. Then (1.2) has a unique solution*

$$x \in L^2(0,T;V) \cap C([0,T];H). \tag{3.2}$$

Furthermore, there exists a constant C_2 such that

$$\|x\|_{L^2 \cap C} \le C_2\Big(1 + \|x_0\| + \|h\|_{L^2(0,T;V^*)}\Big). \tag{3.3}$$

If $(x_0, h) \in V \times L^2(0,T;V^*)$, then $x \in L^2(0,T;V) \cap C([0,T];H)$ and the mapping

$$V \times L^2(0,T;V^*) \ni (x_0, h) \longmapsto x \in L^2(0,T;V) \cap C([0,T];H) \tag{3.4}$$

is continuous.

Proof. Let $y \in L^2(0,T;V)$. Then from (3.1), $f(\cdot, y(\cdot)) \in L^2(0,T;H)$. Thus, by virtue of Proposition 2.1 we know that the problem

$$\begin{gathered}\frac{dx(t)}{dt} + Ax(t) + \partial\phi(x(t)) \ni f(t, y(t)) + h(t), \quad 0 < t \le T, \\ x(0) = x_0\end{gathered} \tag{3.5}$$

has a unique solution $x_y \in L^2(0,T;V) \cap C([0,T];H)$, where x_y is the solution of (3.5).

Let us choose a constant $c > 0$ such that

$$\omega_1 - \frac{c}{2} > 0, \tag{3.6}$$

and let us fix $T_0 > 0$ so that

$$(2c\omega_1 - c^2)^{-1} e^{2\omega_2 T_0} L < 1. \tag{3.7}$$

Let $x_i, i = 1, 2$, be the solution of (3.5) corresponding to y_i. Then, by the monotonicity of $\partial\phi$, it follows that

$$\begin{aligned}(\dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)) + (Ax_1(t) - Ax_2(t), x_1(t) - x_2(t))& \\ \le (f(t, y_1(t)) - f(t, y_2(t)), x_1(t) - x_2(t)),&\end{aligned} \tag{3.8}$$

and hence, using the assumption (2.5), we have that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|x_1(t) - x_2(t)|^2 + \omega_1 ||x_1(t) - x_2(t)||^2& \\ \le \omega_2 |x_1(t) - x_2(t)|^2 + ||f(t, y_1(t)) - f(t, y_2(t))||_* ||x_1(t) - x_2(t)||.&\end{aligned} \tag{3.9}$$

Since

$$\begin{aligned}||f(t, y_1(t)) - f(t, y_2(t))||_* ||x_1(t) - x_2(t)||& \\ \le \frac{1}{2c} \| f(t, y_1(t)) - f(t, y_2(t)) \|_*^2 + \frac{c}{2} ||x_1(t) - x_2(t)||^2&\end{aligned} \tag{3.10}$$

for every $c > 0$ and by integrating (3.9) over $(0, T_0)$ we have

$$\begin{aligned}|x_1(T_0) - x_2(T_0)|^2 + (2\omega_1 - c)\int_0^{T_0} ||x_1(t) - x_2(t)||^2 dt& \\ \le \frac{1}{c} \left|\left| f(t, y_1) - f(t, y_2)\right|\right|_{L^2(0,T_0;V^*)} + 2\omega_2 \int_0^{T_0} |x_1(t) - x_2(t)|^2 dt,&\end{aligned} \tag{3.11}$$

and by Gronwall's inequality,

$$||x_1 - x_2||^2_{L^2(0,T_0;V)} \le (2c\omega_1 - c^2)^{-1} e^{2\omega_2 T_0} ||f(t, y_1) - f(t, y_2)||^2_{L^2(0,T_0;V^*)}. \tag{3.12}$$

Thus, from (3.1) it follows that

$$\|x_1 - x_2\|_{L^2} \le (2c\omega_1 - c^2)^{-1} e^{2\omega_2 T_0} L ||y_1 - y_2||_{L^2(0,T_0;V)}. \tag{3.13}$$

Hence we have proved that $y \mapsto x$ is a strictly contraction from $L^2(0, T_0; V)$ to itself if condition (3.7) is satisfied. It shows that (1.2) has a unique solution in $[0, T_0]$.

Let y be the solution of

$$\frac{dy(t)}{dt} + Ay(t) + \partial\phi(y(t)) \ni 0, \quad 0 < t \le T_0, \qquad y(0) = x_0. \tag{3.14}$$

Then, since

$$\frac{d}{dt}(x(t) - y(t)) + (Ax(t) - Ay(t)) + (\partial\phi(x(t)) - \partial\phi(y(t))) \ni f(t, x(t)) + h(t), \tag{3.15}$$

multiplying by $x(t) - y(t)$ and using the monotonicity of $\partial\phi$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|x(t) - y(t)|^2 + \omega_1 ||x(t) - y(t)||^2& \\ \le \omega_2 |x(t) - y(t)|^2 + ||f(t, x(t)) + h(t)||_* ||x(t) - y(t)||.&\end{aligned} \tag{3.16}$$

Therefore, putting

$$N = (2c\omega_1 - c^2)^{-1} e^{2\omega_2 T_0}, \tag{3.17}$$

from (3.1), it follows that

$$\begin{aligned} \|x - y\|_{L^2(0,T_0;V)} &\le N\|f(\cdot,x) + h\|_{L^2(0,T_0;V^*)} \\ &\le NL\|x\|_{L^2(0,T_0;V)} + N\|h\|_{L^2(0,T_0;V^*)}, \end{aligned} \tag{3.18}$$

and hence

$$\begin{aligned} \|x\|_{L^2(0,T_0;V)} &\le \frac{1}{1-NL}\|y\|_{L^2(0,T_0;V)} + N\|h\|_{L^2(0,T_0;V^*)} \\ &\le \frac{C_1}{1-NL}\Big(1 + \|x_0\| + N\|h\|_{L^2(0,T_0;V^*)}\Big) \\ &\le C_2\Big(1 + \|x_0\| + \|h\|_{L^2(0,T_0;V^*)}\Big) \end{aligned} \tag{3.19}$$

for some positive constant C_2. Since condition (3.7) is independent of the initial values, the solution of (1.2) can be extended to the interval $[0, nT_0]$ for natural number n, i.e., for the initial value $x(nT_0)$ in the interval $[nT_0, (n+1)T_0]$, as analogous estimate (3.19) holds for the solution in $[0, (n+1)T_0]$. Furthermore, similar to (2.12) and (2.15) in Section 2, the estimate (3.3) is easily obtained.

Now we prove the last result.If $(x_0, h) \in V \times L^2(0,T;V^*)$ then x belongs to $L^2(0,T;V)$. Let $(x_{0i}, h_i) \in V \times L^2(0,T;V^*)$ and x_i be the solution of (1.2) with (x_{0i}, h_i) in place of (x_0, u) for $i = 1,2$. Multiplying (1.2) by $x_1(t) - x_2(t)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|x_1(t) - x_2(t)|^2 + \omega_1\|x_1(t) - x_2(t)\|^2 \\ &\quad \le \omega_2|x_1(t) - x_2(t)|^2 + \|f(t,x_1(t)) - f(t,x_2(t))\|_*\|x_1(t) - x_2(t)\| \\ &\qquad + \|h_1(t) - h_2(t)\|_*\|x_1(t) - x_2(t)\|. \end{aligned} \tag{3.20}$$

If $\omega_1 - c/2 > 0$, we can choose a constant $c_1 > 0$ so that

$$\begin{gathered} \omega_1 - \frac{c}{2} - \frac{c_1}{2} > 0, \\ \|h_1(t) - h_2(t)\|_*\|x_1(t) - x_2(t)\| \le \frac{1}{2c_1}\|h_1(t) - h_2(t)\|_*^2 + \frac{c_1}{2}\|x_1(t) - x_2(t)\|^2. \end{gathered} \tag{3.21}$$

Let $T_1 < T$ be such that

$$2\omega_1 - c - c_1 - c^{-1}e^{2\omega_2 T_1}L > 0. \tag{3.22}$$

Integrating (3.20) over $[0, T_1]$, where $T_1 < T$ and as seen in the first part of the proof, it follows that

$$
\begin{aligned}
&(2\omega_1 - c - c_1)||x_1 - x_2||^2_{L^2(0,T_0;V)} \\
&\quad \le e^{2\omega_2 t_1}\left\{||x_{01} - x_{02}|| + \frac{1}{c}||f(t,x_1) - f(t,x_2)||^2_{L^2(0,T_0;V^*)} + \frac{1}{c_1}||h_1 - h_2||_{L^2(0,T_0;V^*)}\right\} \\
&\quad \le e^{2\omega_2 T_1}\left\{\|x_{01} - x_{02}\| + \frac{1}{c}L\|x_1 - x_2\|^2_{L^2(0,T_0;V)} + \frac{1}{c_1}\|h_1 - h_2\|_{L^2(0,T_0;V^*)}\right\}.
\end{aligned}
\tag{3.23}
$$

Putting

$$
N_1 = 2\omega_1 - c - c_1 - c^{-1}e^{2\omega_2 T_1}L, \tag{3.24}
$$

we have

$$
||x_1 - x_2||_{L^2} \le \frac{e^{2\omega_2 T_1}}{N_1}\left(||x_{01} - x_{02}|| + \frac{1}{c_1}\|h_1 - h_2\|\right). \tag{3.25}
$$

Suppose $(x_{0n}, h_n) \to (x_0, h)$ in $V \times L^2(0,T;V^*)$, and let x_n and x be the solutions of (1.2) with (x_{0n}, h_n) and (x_0, h), respectively. Then, by virtue of (3.25) and (3.20), we see that $x_n \to x$ in $L^2(0,T_1,V) \cap C([0,T_1];H)$. This implies that $x_n(T_1) \to x(T_1)$ in V. Therefore the same argument shows that $x_n \to x$ in

$$
L^2\Big(T_1, \min\{2T_1, T\}; V\Big) \cap C\Big([T_1, \min\{2T_1, T\}]; H\Big). \tag{3.26}
$$

Repeating this process, we conclude that $x_n \to x$ in $L^2(0,T;V) \cap C([0,T];H)$.

If $\partial\phi$ satisfies the growth condition (2.23) as is seen in Corollary 2.1, we can obtain the following result. $\square$

Corollary 3.1. *Let (2.5), (3.1), and the growth condition (2.23) be satisfied. Then (1.2) has a unique solution*

$$
x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H). \tag{3.27}
$$

Furthermore, there exists a constant C_2 such that

$$
\|x\|_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_2\Big(1 + \|x_0\| + \|h\|_{L^2(0,T;V^*)}\Big). \tag{3.28}
$$

If $(x_0, h) \in V \times L^2(0,T;V^*)$, then $x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$ and the mapping

$$
V \times L^2(0,T;V^*) \ni (x_0, h) \longmapsto x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \tag{3.29}
$$

is continuous.

Example. Let Ω be a region in an n-dimensional Euclidean space $\mathbb{R}^n$ with boundary $\partial\Omega$ and closure $\overline{\Omega}$. For an integer $m \ge 0$, $C^m(\Omega)$ is the set of all m-times continuously differentiable functions in Ω, and $C_0^m(\Omega)$ is its subspace consisting of functions with compact supports in Ω. If $m \ge 0$ is an integer and $1 \le p \le \infty$, $W^{m,p}(\Omega)$ is the set of all functions f whose derivative $D^\alpha f$ up to degree m in the distribution sense belong to $L^p(\Omega)$. As usual, the norm of $W^{m,p}(\Omega)$ is given by

$$
\|f\|_{m,p} = \left(\sum_{|\alpha|\le m}\|D^\alpha f\|_p^p\right)^{1/p} = \left\{\sum_{|\alpha|\le m}\int_\Omega |D^\alpha f(x)|^p\,dx\right\}^{1/p}, \tag{3.30}
$$

where $1 \le p < \infty$ and $D^0 f = f$. In particular, $W^{0,p}(\Omega) = L^p(\Omega)$ with the norm $\|\cdot\|_{0,p}$.

$W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. For $p' = p/(p-1)$, $1 < p < \infty$, $W^{-m,p}(\Omega)$ stands the dual space $W_0^{m,p'}(\Omega)$ of $W_0^{m,p'}(\Omega)$ whose norm is denoted by $\|\cdot\|_{-m,p}$.

We take $V = W_0^{m,2}(\Omega)$, $H = L^2(\Omega)$ and $V^* = W^{-m,2}(\Omega)$ and consider a nonlinear differential operator of the form

$$Ax = \sum_{|\alpha|\le m} (-D)^\alpha A_\alpha(u,x,\dots,D^m x), \tag{3.31}$$

where $A_\alpha(u,\xi)$ are real functions defined on $\Omega\times\mathbb{R}^N$ and satisfy the following conditions:

(1) A_α are measurable in u and continuous in ξ. There exists $k\in L^2(\Omega)$ and a positive constant C such that

$$A_\alpha(u,0) = 0, \qquad |A_\alpha(u,\xi) \le C(|\xi| + k(u))|, \quad \text{a.e., } u\in\Omega, \tag{3.32}$$

where $\xi = (\xi_\alpha; |\alpha| \le m)$.

(2) For every $(\xi,\eta) \in \mathbb{R}^N\times\mathbb{R}^N$ and for almost every $u\in\Omega$ the following condition holds:

$$\sum_{|\alpha|\le m} (A_\alpha(u,\xi) - A_\alpha(u,\eta))(\xi_\alpha - \eta_\alpha) \ge \omega_1 ||\xi-\eta||_{m,2} - \omega_2 ||\xi-\eta||_{0,2}, \tag{3.33}$$

where $\omega_2\in\mathbb{R}$ and ω_1 is a positive constant.

Let the sesquilinear form $a: V\times V \to \mathbb{R}$ be defined by

$$a(x,y) = \sum_{|\alpha|\le m} \int_\Omega A_\alpha(u,x,\dots,D^m x) D^\alpha y\, du. \tag{3.34}$$

Then by Lax-Milgram theorem we know that the associated operator $A: V\to V^*$, defined by

$$(Ax,y) = a(x,y), \quad x,y\in V, \tag{3.35}$$

is monotone and semicontinuous and satisfies conditions (2.5) in Section 2.

Let $g(t,u,x,p)$, $p\in\mathbb{R}^m$, be assumed that there is a continuous $\rho(t,r): \mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$ and a real constant $1\le\gamma$ such that

$$\begin{gathered} g(t,u,0,0) = 0, \\ |g(t,u,x,p) - g(t,u,x,q)| \le \rho(t,|x|)(1 + |p|^{\gamma-1} + |q|^{\gamma-1})|p-q|, \\ |g(t,u,x,p) - g(t,u,y,p)| \le \rho(t,|x|+|y|)(1+|p|^\gamma)|x-y|. \end{gathered} \tag{3.36}$$

Let

$$f(t,x)(u) = g(t,u,x,Dx,D^2x,\dots,D^m x). \tag{3.37}$$

Then noting that

$$\begin{aligned} ||f(t,x) - f(t,y)||_{0,2}^2 \le 2\int_\Omega |g(t,u,x,p) - g(t,u,x,q)|^2 du \\ + 2\int_\Omega |g(t,u,x,q) - g(t,u,y,q)|^2 du, \end{aligned} \tag{3.38}$$

where $p = (Dx,\dots,D^m x)$ and $q = (Dy,\dots,D^m y)$, it follows from (3.36) that

$$||f(t,x)-f(t,y)||_{0,2}^2 \le L\Big(\|x\|_{m,2},\|y\|_{m,2}\Big)\|x-y\|_{m,2}, \tag{3.39}$$

where $L(\|x\|_{m,2},\|y\|_{m,2})$ is a constant depending on $\|x\|_{m,2}$ and $\|y\|_{m,2}$.

Let $\phi : V \to (-\infty,+\infty]$ be a lower semicontinuous, proper convex function. Then for $x_0 \in W_0^{m,2}(\Omega)$ satisfying that $\phi(x_0) < \infty$ and $h \in L^2(0,T;W^{-m,2}(\Omega))$, (1.2) is caused by the following nonlinear variational inequality problem:

$$\begin{gathered}\left(\frac{dx(t)}{dt}+Ax(t),\, x(t)-z\right)+\phi(x(t))-\phi(z) \\ \le (f(t,x(t))+h(t), x(t)-z), \quad a.e., 0<t\le T,\ z\in W_0^{m,2}(\Omega), \\ x(0)=x_0\end{gathered} \tag{3.40}$$

has a unique solution

$$x \in L^2\Big(0,T;W_0^{m,2}(\Omega)\Big)\cap C\Big([0,T];L^2(\Omega)\Big). \tag{3.41}$$

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