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Hyers-Ulam-Rassias stability of first order linear partial fuzzy differential equations under generalized differentiability

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Abstract

In the present paper, we establish a multivariate fuzzy chain rule under generalized differentiability by extending the corresponding chain rule under *H*-differentiability. Based on the result, we discuss the Ulam stability problems of two types of first order linear partial fuzzy differential equations under generalized differentiability.

Keywords: Hyers-Ulam-Rassias stability; *H*-difference; fuzzy chain rule; partial differentiability; partial fuzzy differential equations

1 Introduction

In 1993, Obloza [1] first initiated the study of the Ulam stability problem of differential equations. Subsequently, Alsina and Ger [2] investigated the Hyers-Ulam stability of the differential equation $y' = y$. Further, the stability results of the differential equation $y' = \lambda y$ in various abstract spaces have been established by Miura and Takahasi *et al.* [3–5]. As of now, the stability problems of many types of linear differential equations have been systematically and extensively studied by many authors [6–14]. Especially, it is worth noting that several types of partial differential equations have attracted much attention during last few years [15–17].

Generally speaking, for an n-order X-valued differential equation (here X denotes a Banach space with the norm $\|\cdot\|$)

$$F\bigl(t,y,y',\ldots,y^{(n)}\bigr) = 0, \quad t \in I,$$

where I denotes a subinterval of $\mathbb{R}$, we say that it has Hyers-Ulam stability or it is stable in the sense of Hyers-Ulam if for a given $\epsilon > 0$ and an n times strongly differentiable mapping $f : I \to X$ satisfying

$$\bigl\|F\bigl(t,f,f',\ldots,f^{(n)}\bigr)\bigr\| \le \epsilon$$

for all $t \in I$, then there exists an exact solution $g : I \to X$ of the preceding differential equation such that $\|f(t) - g(t)\| \le K(\epsilon)$ for all $t \in I$, where $K(\epsilon)$ depends only on ϵ and $\lim_{\epsilon\to 0} K(\epsilon) = 0$. More generally, if the ϵ and $K(\epsilon)$ are replaced by two control functions

φ and Φ in t, respectively, then we say that the differential equation mentioned above has the Hyers-Ulam-Rassias stability or it is stable in the sense of Hyers-Ulam-Rassias.

At present, most of the studies associated with fuzzy differential equations are based on generalized differentiability, because there are some defects in the original fuzzy differential equations defined by H-differentiability, *i.e.*, the length of the support or the diameter of the solution is increasing as the time increases. Specifically, Bede and Gal [18, 19] proposed the concept of strongly generalized differentiability of a fuzzy number-valued function by using unilateral derivatives, which enlarged the class of H-differentiable fuzzy number-valued functions introduced by Puri and Ralescu [20]. To a certain extent, such improved differentiability overcomes some shortcomings in the sense of H-differentiability. In addition, this improvement brings about many new problems, which are never encountered in the study of classical differential equations. These problems indicate that, under the generalized differentiability, fuzzy differential equations must be considered by means of new ideas and methods which are different from classical differential equations. Very often, it is difficult to extend some important results in classical differential equations to fuzzy environment. Therefore, it is still necessary to study some similar problems in fuzzy differential equations. Inspired by the study of the Ulam stability problems of classical differential equations, the author has recently studied the Ulam stability of three types of first order linear fuzzy differential equations under generalized differentiability [21]. To our knowledge, it is the first paper reported on the Ulam stability of fuzzy differential equations. As a continuation of our previous work, the aim of this paper is to discuss the Ulam stability problems of the following first order linear partial fuzzy differential equations:

$$b \odot u_y(x,y) \oplus \delta(y) \odot u(x,y) = a \odot u_x(x,y) \oplus \sigma(y) \quad (a,b > 0), \tag{1}$$

$$a \odot u_x(x,y) \oplus \delta(x) \odot u(x,y) = b \odot u_y(x,y) \oplus \sigma(x) \quad (a,b > 0), \tag{2}$$

where u and σ are two fuzzy number-valued functions, δ is a real-valued function and the symbol '=' means identity of membership functions on both sides.

2 Preliminaries

Let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_-$ denote the set of all natural numbers, the set of all real numbers, the set of all positive real numbers, and the set of all negative real numbers, respectively. Denote by $\mathbb{R}_{\mathcal{F}}$ the class of fuzzy sets $u : \mathbb{R} \to [0,1]$ with the following properties:

(i) u is normal, *i.e.*, there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, that is, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$;

(iii) u is upper semicontinuous;

(iv) $cl\{x \in \mathbb{R} : u(x) > 0\}$ is compact, where cl denotes the closure of a set.

Usually, the set $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers. If every real number is equivalently represented by its characteristic function, then it is easy to know that $\mathbb{R} \subset \mathbb{R}_{\mathcal{F}}$. For $0 < \alpha \leq 1$, we denote $[u]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$ and $[u]^0 = cl\{x \in \mathbb{R} : u(x) > 0\}$. Then it follows from the conditions (i)-(iv) that the α-level set $[u]^\alpha$ is a nonempty compact interval for all $\alpha \in [0,1]$ and each $u \in \mathbb{R}_{\mathcal{F}}$.

For $u, v \in \mathbb{R}_{\mathcal{F}}$, $\lambda \in \mathbb{R}$, the addition $u \oplus v$ and scalar multiplication $\lambda \odot u$ can be defined, levelwise, by

$$[u \oplus v]^\alpha = [u]^\alpha + [v]^\alpha \quad \text{and} \quad [\lambda \odot u]^\alpha = \lambda [u]^\alpha$$

for all $\alpha \in [0,1]$.

The supremum metric between two fuzzy numbers u and v is defined by

$$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\},$$
$$D(u,v) = \sup_{\alpha \in [0,1]} d_H\big([u]^\alpha, [v]^\alpha\big),$$

where d_H is the Hausdorff metric. It is well known that the metric space $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space and the following properties for the metric D are satisfied:

(P1) $D(u \oplus w, v \oplus w) = D(u,v)$, $\forall u, v, w \in \mathbb{R}_{\mathcal{F}}$;

(P2) $D(\lambda \odot u, \lambda \odot v) = |\lambda| D(u,v)$, $\forall \lambda \in \mathbb{R}$, $u, v \in \mathbb{R}_{\mathcal{F}}$;

(P3) $D(u \oplus v, w \oplus e) \le D(u,w) + D(v,e)$, $\forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}$.

Let $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v \oplus w$, then w is called the H-difference of u and v, and it is denoted by $u \ominus v$.

Throughout this paper, the symbol '$\ominus$' always stands for the H-difference. In general, $u \ominus v \neq u \oplus (-1) \odot v$, $(-1) \odot v = -v$.

The concept of strongly generalized differentiability was introduced by Bede and Gal [19] and further studied by Chalco-Cano and Román-Flores [22]. Here we shall extend this concept to a bivariate fuzzy number-valued function.

Definition 2.1 Let $\mathfrak{D} = (a,b) \times (c,d)$ be an open domain, where $a, b, c, d \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, $c < d$, and let $F : \mathfrak{D} \to \mathbb{R}_{\mathcal{F}}$ be a bivariate fuzzy number-valued mapping and $(x_0, y_0) \in \mathfrak{D}$. We say that F is partially differentiable at (x_0, y_0) with respect to the variable x if there exists an element $F_x(x_0, y_0) \in \mathbb{R}_{\mathcal{F}}$ (or $\frac{\partial F}{\partial x}|_{(x_0,y_0)} \in \mathbb{R}_{\mathcal{F}}$) such that either:

(i) for all $\Delta x > 0$ sufficiently small, the H-differences $F(x_0 + \Delta x, y_0) \ominus F(x_0, y_0)$, $F(x_0, y_0) \ominus F(x_0 - \Delta x, y_0)$ exist and the limits (in the metric D)

$$\begin{aligned} \lim_{\Delta x \to 0+} \frac{F(x_0 + \Delta x, y_0) \ominus F(x_0, y_0)}{\Delta x} &= \lim_{\Delta x \to 0+} \frac{F(x_0, y_0) \ominus F(x_0 - \Delta x, y_0)}{\Delta x} \\ &= F_x(x_0, y_0) = \frac{\partial F}{\partial x}\bigg|_{(x_0,y_0)} \end{aligned}$$

or

(ii) for all $\Delta x > 0$ sufficiently small, the H-differences $F(x_0, y_0) \ominus F(x_0 + \Delta x, y_0)$, $F(x_0 - \Delta x, y_0) \ominus F(x_0, y_0)$ exist and the limits (in the metric D)

$$\begin{aligned} \lim_{\Delta x \to 0+} \frac{F(x_0, y_0) \ominus F(x_0 + \Delta x, y_0)}{-\Delta x} &= \lim_{\Delta x \to 0+} \frac{F(x_0 - \Delta x, y_0) \ominus F(x_0, y_0)}{-\Delta x} \\ &= F_x(x_0, y_0) = \frac{\partial F}{\partial x}\bigg|_{(x_0,y_0)}, \end{aligned}$$

where Δx and $-\Delta x$ in the denominators denote $\frac{1}{\Delta x}\cdot$ and $-\frac{1}{\Delta x}\cdot$, respectively.

Analogously, we can introduce the partial derivative of F (denoted by $F_y(x_0, y_0)$ or $\frac{\partial F}{\partial y}|_{(x_0,y_0)}$) at (x_0, y_0) with respect to the variable y. In general, for any $(x, y) \in \mathfrak{D}$, the partial derivatives of F with respect to x and y can be abbreviated as F_x (or $\frac{\partial F}{\partial x}$) and F_y (or $\frac{\partial F}{\partial y}$), respectively.

Remark 1 In essence, the case (i) of Definition 2.1 follows from the H-differentiability introduced by Puri and Ralescu [20]. Then the presented definition of partial differentiability can be considered as an extension of the H-differentiability of a univariate fuzzy number-valued function.

A bivariate fuzzy number-valued function $F : \mathfrak{D} \to \mathbb{R}_{\mathcal{F}}$ is said to be partially (i)-differentiable (or partially (ii)-differentiable) on $\mathfrak{D}$ if it is partially differentiable in the sense (i) (or (ii)) of Definition 2.1.

In the following, we will recall some fundamental theorems of calculus of a univariate fuzzy number-valued function. Obviously, these results can easily be extended to partial derivatives of a bivariate fuzzy number-valued function.

Theorem 2.1 (Kaleva [23], Khastan *et al.* [24]) *Let $F : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_{\mathcal{F}}$ be a differentiable fuzzy number-valued mapping and assume that the derivative F' is integrable over $\mathbb{R}_+ \cup \{0\}$. For each $t \in \mathbb{R}_+ \cup \{0\}$,*

(i) *if F is* (i)*-differentiable, then*

$$F(t) = F(0) \oplus \int_0^t F'(s)\,ds;$$

(ii) *if F is* (ii)*-differentiable, then*

$$F(t) = F(0) \ominus \int_a^t -F'(s)\,ds.$$

Theorem 2.2 (Kaleva [23]) *Let $F : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_{\mathcal{F}}$ be continuous. Then for any $t \in \mathbb{R}_+ \cup \{0\}$ the integral $H(t) = \int_0^t F(\tau)\,d\tau$ is* (i)*-differentiable and $H'(t) = F(t)$.*

Theorem 2.3 (Khastan *et al.* [24]) *Let $F : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_{\mathcal{F}}$ be continuous. Define the integral*

$$H(t) := \gamma \ominus \int_0^t -F(\tau)\,d\tau, \quad t \in \mathbb{R}_+ \cup \{0\},$$

where $\gamma \in \mathbb{R}_{\mathcal{F}}$ is such that the preceding H-difference exists on $\mathbb{R}_+ \cup \{0\}$. Then $H(t)$ is (ii)*-differentiable and $H'(t) = F(t)$.*

For a bivariate fuzzy number-valued function $F : \mathfrak{D} \to \mathbb{R}_{\mathcal{F}}$, we say that

(i) F satisfies the condition (H1) with respect to x if for a given $(x, y) \in \mathfrak{D}$, the H-differences $F(x + \Delta x, y) \ominus F(x, y)$ and $F(x, y) \ominus F(x - \Delta x, y)$ exist for sufficiently small $\Delta x > 0$;

(ii) F satisfies the condition (H2) with respect to x if for a given $(x, y) \in \mathfrak{D}$, the H-differences $F(x, y) \ominus F(x + \Delta x, y)$ and $F(x - \Delta x, y) \ominus F(x, y)$ exist for sufficiently small $\Delta x > 0$.

Obviously, F satisfies the corresponding condition (H1) and (H2) with respect to the second variable y can be defined in a similar way, respectively.

As a direct generalization of Theorem 2.19 in [25], we can formulate the following multivariate chain rule under generalized differentiability.

Theorem 2.4 *Let $x_i : [a,b] \times [c,d] \to x_i([a,b] \times [c,d]) := I_i \subseteq \mathbb{R}$, $i = 1,2,\ldots,n$, $n \in \mathbb{N}$, be strictly increasing and differentiable functions with respect to each of the variables. Consider U an open subset of $\mathbb{R}^n$ such that $\times_{i=1}^n I_i \subseteq U$. Consider $f : U \to \mathbb{R}_{\mathcal{F}}$ a fuzzy continuous function. Assume that $f_{x_i} : U \to \mathbb{R}_{\mathcal{F}}$, $i = 1,2,\ldots,n$, i.e., the partial* (i)- *or* (ii)-*derivative with respect to x_i, exist and are fuzzy continuous. Call $z : z(s,t) := f(x_1,x_2,\ldots,x_n)$. Then $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ exist, and*

$$\frac{\partial z}{\partial s} = \sum_{i=1}^{n}{}^{*} \frac{\partial z}{\partial x_i} \odot \frac{\partial x_i}{\partial s}, \quad \forall s \in [a,b],$$
$$\frac{\partial z}{\partial t} = \sum_{i=1}^{n}{}^{*} \frac{\partial z}{\partial x_i} \odot \frac{\partial x_i}{\partial t}, \quad \forall t \in [c,d],$$

where $\frac{\partial z}{\partial x_i}$, $i = 1,2,\ldots,n$, $\frac{\partial x_i}{\partial s}$ and $\frac{\partial x_i}{\partial t}$ denote partial (i)- *or* (ii)-*derivatives of z and x_i with respect to x_i, s, and t, respectively.*

Proof It is a direct consequence of Theorem 2.19 in [25] when f is partially (i)-differentiable with respect to x_i. So we just need to prove the corresponding results hold true when f is partially (ii)-differentiable. For simplicity, we only prove the first equality, *i.e.*, $\frac{\partial z}{\partial s}$, the second one can be proved in a similar way. Setting $s \in (a,b)$. Let $(x_1,x_2,\ldots,x_n) \in U$ be fixed and let $\Delta_s x_i > 0$, $i = 1,2,\ldots,n$, be small. Now, we set

$$\begin{cases} \alpha_1 = f(x_1, x_2 + \Delta_s x_2, \ldots, x_n + \Delta_s x_n) \ominus f(x_1 + \Delta_s x_1, x_2 + \Delta_s x_2, \ldots, x_n + \Delta_s x_n) \in \mathbb{R}_{\mathcal{F}}, \\ \alpha_2 = f(x_1, x_2, x_3 + \Delta_s x_3, \ldots, x_n + \Delta_s x_n) \ominus f(x_1, x_2 + \Delta_s x_2, \ldots, x_n + \Delta_s x_n) \in \mathbb{R}_{\mathcal{F}}, \\ \cdots \\ \alpha_n = f(x_1, x_2, \ldots, x_n) \ominus f(x_1, x_2, \ldots, x_n + \Delta_s x_n) \in \mathbb{R}_{\mathcal{F}}, \end{cases}$$

equivalently, we have

$$\begin{cases} f(x_1, x_2 + \Delta_s x_2, \ldots, x_n + \Delta_s x_n) = \alpha_1 + f(x_1 + \Delta_s x_1, x_2 + \Delta_s x_2, \ldots, x_n + \Delta_s x_n), \\ f(x_1, x_2, x_3 + \Delta_s x_3, \ldots, x_n + \Delta_s x_n) = \alpha_2 + f(x_1, x_2 + \Delta_s x_2, \ldots, x_n + \Delta_s x_n), \\ \cdots \\ f(x_1, x_2, \ldots, x_n) = \alpha_n + f(x_1, x_2, \ldots, x_n + \Delta_s x_n). \end{cases}$$

Summing both sides of the above equalities and using the cancellation law, we get

$$f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n}{}^{*} \alpha_i \oplus f(x_1 + \Delta_s x_1, x_2 + \Delta_s x_2, \ldots, x_n + \Delta_s x_n).$$

That is,

$$f(x_1,x_2,\dots,x_n)\ominus f(x_1+\Delta_s x_1,x_2+\Delta_s x_2,\dots,x_n+\Delta_s x_n)=\sum_{i=1}^{n}{}^{*}\alpha_i.$$

Since f is partially (ii)-differentiable with respect to x_i, the H-differences mentioned above α_i, $i=1,2,\dots,n$, exist in $\mathbb{R}_{\mathcal{F}}$ for small $\Delta_s x_i>0$. Define

$$\Delta_s x_i := \phi_i(s+\Delta s,t)-\phi_i(s,t),\qquad \Delta s>0, i=1,2,\dots,n.$$

That is,

$$\phi_i(s+\Delta s,t)=x_i+\Delta_s x_i,\qquad x_i:=\phi_i(s,t).$$

Since $\phi_i(s,\cdot)$, $i=1,2,\dots,n$, is strictly increasing with respect to s, we know that $\Delta_s x_i>0$. The continuity of $\phi_i(s,\cdot)$ implies that $\Delta_s x_i\to 0$ as $\Delta s\to 0$. Then, by (ii) of Theorem 2.1, we can infer that

$$\begin{aligned}
&\lim_{\Delta s\to 0+} D\Bigg(\frac{f(\phi_1(s,t),\dots,\phi_n(s,t))\ominus f(\phi_1(s+\Delta s,t),\dots,\phi_n(s+\Delta s,t))}{-\Delta s},\\
&\quad\sum_{i=1}^{n}{}^{*} f_{x_i}(x_1,\dots,x_n)\odot\frac{\partial x_i(s,t)}{\partial s}\Bigg)\\
&=\lim_{\Delta s\to 0+} D\Bigg(\frac{f(x_1,\dots,x_n)\ominus f(x_1+\Delta_s x_1,\dots,x_n+\Delta_s x_n)}{-\Delta s},\sum_{i=1}^{n}{}^{*} f_{x_i}(x_1,\dots,x_n)\odot\frac{\partial x_i(s,t)}{\partial s}\Bigg)\\
&=\lim_{\Delta s\to 0+} D\Bigg(\frac{\sum_{i=1}^{n}{}^{*}\alpha_i}{-\Delta s},\sum_{i=1}^{n}{}^{*} f_{x_i}(x_1,\dots,x_n)\odot\frac{\partial x_i(s,t)}{\partial s}\Bigg)\\
&\le\lim_{\Delta s\to 0+} D\Bigg(\frac{f(x_1,x_2+\Delta_s x_2,\dots,x_n+\Delta_s x_n)\ominus f(x_1+\Delta_s x_1,x_2+\Delta_s x_2,\dots,x_n+\Delta_s x_n)}{-\Delta_s x_1}\\
&\quad\odot\frac{\Delta_s x_1}{\Delta s},f_{x_1}(x_1,\dots,x_n)\odot\frac{\partial x_1(s,t)}{\partial s}\Bigg)\\
&\quad+\lim_{\Delta s\to 0+} D\Bigg(\frac{f(x_1,x_2,x_3+\Delta_s x_3,\dots,x_n+\Delta_s x_n)\ominus f(x_1,x_2+\Delta_s x_2,\dots,x_n+\Delta_s x_n)}{-\Delta_s x_2}\\
&\quad\odot\frac{\Delta_s x_2}{\Delta s},f_{x_2}(x_1,\dots,x_n)\odot\frac{\partial x_2(s,t)}{\partial s}\Bigg)+\cdots\\
&\quad+\lim_{\Delta s\to 0+} D\Bigg(\frac{f(x_1,x_2,\dots,x_{n-1},x_n)\ominus f(x_1,x_2,\dots,x_{n-1},x_n+\Delta_s x_n)}{-\Delta_s x_n}\odot\frac{\Delta_s x_n}{\Delta s},\\
&\quad f_{x_n}(x_1,\dots,x_n)\odot\frac{\partial x_n(s,t)}{\partial s}\Bigg)\\
&=\lim_{\Delta s\to 0+} D\Bigg(\frac{-\int_{x_1}^{x_1+\Delta_s x_1} f_{x_1}(\tau,x_2+\Delta_s x_2,\dots,x_n+\Delta_s x_n)\,d\tau}{-\Delta_s x_1}\odot\frac{\Delta_s x_1}{\Delta s},\\
&\quad f_{x_1}(x_1,\dots,x_n)\odot\frac{\partial x_1(s,t)}{\partial s}\Bigg)
\end{aligned}$$

$$+\lim_{\Delta s\to 0+} D\biggl(\frac{-\int_{x_2}^{x_2+\Delta_s x_2} f_{x_2}(x_1,\tau,x_3+\Delta_s x_3,\ldots,x_n+\Delta_s x_n)\,d\tau}{-\Delta_s x_2}\odot\frac{\Delta_s x_2}{\Delta s},$$
$$f_{x_2}(x_1,\ldots,x_n)\odot\frac{\partial x_2(s,t)}{\partial s}\biggr)+\cdots$$
$$+\lim_{\Delta s\to 0+} D\biggl(\frac{-\int_{x_{n-1}}^{x_{n-1}+\Delta_s x_{n-1}} f_{x_{n-1}}(x_1,x_2,\ldots,x_{n-2},\tau,x_n+\Delta_s x_n)\,d\tau}{-\Delta_s x_{n-1}}\odot\frac{\Delta_s x_{n-1}}{\Delta s},$$
$$f_{x_{n-1}}(x_1,\ldots,x_n)\odot\frac{\partial x_{n-1}(s,t)}{\partial s}\biggr)$$
$$+\lim_{\Delta s\to 0+} D\biggl(f_{x_n}(x_1,x_2,\ldots,x_n)\odot\frac{\Delta_s x_n}{\Delta s},f_{x_n}(x_1,\ldots,x_n)\odot\frac{\partial x_n(s,t)}{\partial s}\biggr):=\Lambda.$$

Further, by Lemmas 2.16 and 2.17 in [25], we get

$$\Lambda=\frac{\partial x_1(s,t)}{\partial s}\lim_{\Delta s\to 0+}\frac{1}{\Delta_s x_1}D\biggl(\int_{x_1}^{x_1+\Delta_s x_1} f_{x_1}(\tau,x_2+\Delta_s x_2,\ldots,x_n+\Delta_s x_n)\,d\tau,$$
$$\Delta_s x_1\odot f_{x_1}(x_1,\ldots,x_n)\biggr)$$
$$+\frac{\partial x_2(s,t)}{\partial s}\lim_{\Delta s\to 0+}\frac{1}{\Delta_s x_2}D\biggl(\int_{x_2}^{x_2+\Delta_s x_2} f_{x_2}(x_1,\tau,x_3+\Delta_s x_3,\ldots,x_n+\Delta_s x_n)\,d\tau,$$
$$\Delta_s x_2\odot f_{x_2}(x_1,\ldots,x_n)\biggr)+\cdots$$
$$+\frac{\partial x_{n-1}(s,t)}{\partial s}\lim_{\Delta s\to 0+}\frac{1}{\Delta_s x_{n-1}}D\biggl(\int_{x_{n-1}}^{x_{n-1}+\Delta_s x_{n-1}} f_{x_{n-1}}(x_1,x_2,\ldots,x_{n-2},\tau,x_n+\Delta_s x_n)\,d\tau,$$
$$\Delta_s x_{n-1}\odot f_{x_{n-1}}(x_1,\ldots,x_n)\biggr)$$
$$=\sum_{i=1}^{n-1}\frac{\partial x_i(s,t)}{\partial s}\lim_{\Delta s\to 0+}\frac{1}{\Delta_s x_i}D\biggl(\int_{x_i}^{x_i+\Delta_s x_i} f_{x_i}(x_1,\ldots,x_{i-1},\tau,x_{i+1}$$
$$+\Delta_s x_{i+1},\ldots,x_n+\Delta_s x_n)\,d\tau,\int_{x_i}^{x_i+\Delta_s x_i} f_{x_i}(x_1,\ldots,x_n)\,d\tau\biggr)$$
$$\le\sum_{i=1}^{n-1}\frac{\partial x_i(s,t)}{\partial s}\lim_{\Delta s\to 0+}\frac{1}{\Delta_s x_i}\int_{x_i}^{x_i+\Delta_s x_i} D\bigl(f_{x_i}(x_1,\ldots,x_{i-1},\tau,x_{i+1}+\Delta_s x_{i+1},\ldots,x_n+\Delta_s x_n),$$
$$f_{x_i}(x_1,\ldots,x_n)\bigr)\,d\tau$$
$$\le\sum_{i=1}^{n-1}\frac{\partial x_i(s,t)}{\partial s}\lim_{\Delta s\to 0+}\frac{1}{\Delta_s x_i}\Bigl(\sup_{\tau_i\in[x_i,x_i+\Delta_s x_i]}\bigl(D\bigl(f_{x_i}(x_1,\ldots,x_{i-1},\tau_i,x_{i+1}$$
$$+\Delta_s x_{i+1},\ldots,x_n+\Delta_s x_n),f_{x_i}(x_1,\ldots,x_n)\bigr)\bigr)\Bigr)\Delta_s x_i$$
$$\le\sum_{i=1}^{n-1}\frac{\partial x_i(s,t)}{\partial s}\lim_{\Delta s\to 0+}D\bigl(f_{x_i}\bigl(x_1,\ldots,x_{i-1},\tau_i^*,x_{i+1}+\Delta_s x_{i+1},\ldots,x_n+\Delta_s x_n\bigr),f_{x_i}(x_1,\ldots,x_n)\bigr)$$
$$\bigl(\text{for some }\tau_i^*\in[x_i,x_i+\Delta_s x_i]\bigr)$$
$$=\Theta.$$

When $\Delta s \to 0+$, $\Delta_s x_i \to 0$ for every $i = 1, 2, \ldots, n$. Then $\tau_i^* \to x_i$. By the continuity of f_{x_i}, $i = 1, 2, \ldots, n$, we conclude that

$$\begin{aligned}\Theta &= \sum_{i=1}^{n-1} \frac{\partial x_i(s,t)}{\partial s} \lim_{\Delta s \to 0+} D\bigl(f_{x_i}(x_1, \ldots, x_n), f_{x_i}(x_1, \ldots, x_n)\bigr) \\ &= \sum_{i=1}^{n-1} \frac{\partial x_i(s,t)}{\partial s} \cdot 0 = 0,\end{aligned}$$

which implies that the first equality holds. The proof of the theorem is now completed. □

3 Hyers-Ulam-Rassias stability of linear partial fuzzy differential equation (1)

In this section, we shall establish the stability results of the linear partial fuzzy differential equation (1) under different differentiability.

3.1 Stability of (1) under partial (i)-differentiability

Theorem 3.1 *Let $\sigma : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_{\mathcal{F}}$ be a continuous fuzzy number-valued function and let $u : \mathbb{R} \times \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_{\mathcal{F}}$ be a bivariate fuzzy number-valued function which has continuous partial* (i)*-derivatives with respect to each of the variables. Assume that u satisfies the following inequality*:

$$D\bigl(b \odot u_y(x,y) \oplus \delta(y) \odot u(x,y), a \odot u_x(x,y) \oplus \sigma(y)\bigr) \le \varphi(y) \tag{3}$$

for all $x \in \mathbb{R}$ and $y \in \mathbb{R}_+ \cup \{0\}$, where $a, b > 0$ are constants, $\delta : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$ is a continuous function and $\varphi : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$ is a function. Moreover, assume that the following conditions are satisfied:

(i) *$\int_0^y \delta(\tau)\,d\tau$ exists for all $y \in \mathbb{R}_+ \cup \{0\}$*;
(ii) *$\int_0^y \exp(\frac{1}{b}\int_0^\omega \delta(\tau)\,d\tau) \odot \sigma(\omega)\,d\omega$ exists for all $y \in \mathbb{R}_+ \cup \{0\}$*;
(iii) *$\int_0^\infty \varphi(\omega)\exp(\frac{1}{b}\int_0^\omega \delta(\tau)\,d\tau)\,d\omega$ exists*;
(iv) *$\lim_{x\to-\infty, y\to+\infty} u(x,y)$ exists*;
(v) *the H-difference $\exp(\int_0^y \frac{1}{b}\delta(\tau)\,d\tau)u(x,y) \ominus \frac{1}{b}\int_0^y \exp(\frac{1}{b}\int_0^\omega \delta(\tau)\,d\tau) \odot \sigma(\omega)\,d\omega$ exists for each $x \in \mathbb{R}$ and each $y \in \mathbb{R}_+ \cup \{0\}$.*

Then there exists a unique $u_0 \in \mathbb{R}_{\mathcal{F}}$ such that

$$D\bigl(u(x,y), \widehat{u}(x,y)\bigr) \le \frac{1}{b}\exp\biggl(-\frac{1}{b}\int_0^y \delta(\tau)\,d\tau\biggr)\int_y^\infty \varphi(\omega)\exp\biggl(\frac{1}{b}\int_0^\omega \delta(\tau)\,d\tau\biggr)\,d\omega \tag{4}$$

for all $x \in \mathbb{R}$ and all $y \in \mathbb{R}_+ \cup \{0\}$, where

$$\widehat{u}(x,y) = \exp\biggl(-\frac{1}{b}\int_0^y \delta(\tau)\,d\tau\biggr) \odot \biggl(u_0 \oplus \frac{1}{b}\int_0^y \exp\biggl(\frac{1}{b}\int_0^\omega \delta(\tau)\,d\tau\biggr) \odot \sigma(\omega)\,d\omega\biggr). \tag{5}$$

Proof First, we introduce a new coordinate (ξ, η) and choose the following coordinate transformation:

$$\xi = x + \frac{a}{b}y, \qquad \eta = \frac{1}{b}y. \tag{6}$$

Setting $\widetilde{u}(\xi,\eta) = u(\xi - a\eta, b\eta) = u(x,y)$. By (6), we know that the new variable ξ is strictly increasing with respect to x and y, respectively. Meanwhile, η is strictly increasing in y. According to Theorem 2.4, it follows from the previous equality that

$$\begin{aligned} u_x(x,y) &= \widetilde{u}_\xi(\xi,\eta) \odot \frac{\partial \xi}{\partial x} = \widetilde{u}_\xi(\xi,\eta), \\ u_y(x,y) &= \widetilde{u}_\xi(\xi,\eta) \odot \frac{\partial \xi}{\partial y} \oplus \widetilde{u}_\eta(\xi,\eta) \odot \frac{\partial \eta}{\partial y} = \frac{a}{b} \odot \widetilde{u}_\xi(\xi,\eta) \oplus \frac{1}{b}\widetilde{u}_\eta(\xi,\eta). \end{aligned}$$

Applying these two equalities to (3), we obtain

$$D\big(\widetilde{u}_\eta(\xi,\eta) \oplus \widetilde{\delta}(\eta) \odot \widetilde{u}(\xi,\eta), \widetilde{\sigma}(\eta)\big) \le \widetilde{\varphi}(\eta) \tag{7}$$

for all $\xi \in \mathbb{R}$ and all $\eta \in \mathbb{R}_+ \cup \{0\}$, where $\widetilde{\delta} = \delta(b\eta) = \delta(y)$ and $\widetilde{\sigma}(\eta) = \sigma(b\eta) = \sigma(y)$, $\widetilde{\varphi}(\eta) = \varphi(b\eta) = \varphi(y)$.

If we set

$$\xi = \lambda + \frac{a}{b}\tau, \qquad \mu = \frac{1}{b}\tau,$$

then $\widetilde{\delta}(\mu) = \delta(b\mu) = \delta(t)$ and it follows from (i) that

$$\int_0^y \widetilde{\delta}(\mu)\,d\mu = \frac{1}{b}\int_0^{by} \delta(\tau)\,d\tau \tag{8}$$

exists for all $y \in \mathbb{R}_+ \cup \{0\}$. Moreover, if we set

$$\xi = \nu + \frac{a}{b}\tau, \qquad \nu = \frac{1}{b}\omega,$$

then $\widetilde{\sigma}(\nu) = \sigma(b\nu) = \sigma(\omega)$ and from (8) we can infer that

$$\int_0^y \exp\left(\int_0^\nu \widetilde{\delta}(\mu)\,d\mu\right) \odot \widetilde{\sigma}(\nu)\,d\nu = \frac{1}{b}\int_0^{by} \exp\left(\frac{1}{b}\int_0^\omega \delta(\tau)\,d\tau\right) \odot \sigma(\omega)\,d\omega. \tag{9}$$

From (ii), it can easily be seen that the integral of the left side exists for all $y \in \mathbb{R}_+ \cup \{0\}$. Furthermore, it follows from (iii) that the following integral of the left side exists:

$$\int_0^\infty \widetilde{\varphi}(\nu) \exp\left(\int_0^\nu \widetilde{\delta}(\mu)\,d\mu\right) d\nu = \frac{1}{b}\int_0^\infty \varphi(\omega) \exp\left(\frac{1}{b}\int_0^\omega \delta(\tau)\,d\tau\right) d\omega. \tag{10}$$

In view of the inequality (7), the conditions (8), (9), and (10), together with Theorem 3.1 in [21], imply that, for each fixed $\xi \in \mathbb{R}$, there exists a unique $\theta(\xi) \in \mathbb{R}_{\mathcal{F}}$ such that

$$\begin{aligned} &D\left(\widetilde{u}(\xi,\eta), \exp\left(-\int_0^\eta \widetilde{\delta}(\mu)\,d\mu\right) \odot \left(\theta(\xi) \oplus \int_0^\eta \exp\left(\int_0^\nu \widetilde{\delta}(\mu)\,d\mu\right) \odot \widetilde{\sigma}(\nu)\,d\nu\right)\right) \\ &\quad \le \exp\left(-\int_0^\eta \widetilde{\delta}(\mu)\,d\mu\right) \int_\eta^\infty \widetilde{\varphi}(\nu) \exp\left(\int_0^\nu \widetilde{\delta}(\mu)\,d\mu\right) d\nu \end{aligned} \tag{11}$$

for all $\eta \in \mathbb{R}_+$. According to the proof of Theorem 3.1 in [21], we know that

$$\theta(\xi) = \lim_{\eta\to\infty}\left(\exp\left(\int_0^\eta \widetilde{\delta}(\mu)\,d\mu\right)\widetilde{u}(\xi,\eta) \ominus \int_0^\eta \exp\left(\int_0^\nu \widetilde{\delta}(\mu)\,d\mu\right) \odot \widetilde{\sigma}(\nu)\,d\nu\right). \tag{12}$$

Notice that the conditions (iv), (v) together with the equalities (8) and (9) imply that the H-difference of (12) exists, and then we conclude that

$$\begin{aligned}\theta(\xi) = \lim_{\eta\to\infty}\biggl(&\exp\biggl(\frac{1}{b}\int_0^{b\eta}\delta(\tau)\,d\tau\biggr)u(\xi - a\eta, b\eta)\\ &\ominus \frac{1}{b}\int_0^{b\eta}\exp\biggl(\frac{1}{b}\int_0^{\omega}\delta(\tau)\,d\tau\biggr)\odot\sigma(\omega)\,d\omega\biggr)\end{aligned}$$

is a constant fuzzy number; we write simply u_0.

By the preceding transformations and the equality (8), we can infer that

$$\int_0^{\eta}\widetilde{\delta}(\mu)\,d\mu = \frac{1}{b}\int_0^{y}\delta(\tau)\,d\tau, \qquad \int_0^{\nu}\widetilde{\delta}(\mu)\,d\mu = \frac{1}{b}\int_0^{\omega}\delta(\tau)\,d\tau.$$

Since $\widetilde{u}(\xi,\eta) = u(x,y)$, and $\widetilde{\sigma}(\nu) = \sigma(b\nu) = \sigma(\omega)$, $\widetilde{\varphi}(\nu) = \varphi(b\nu) = \varphi(\omega)$, by (9) and (10), applying these relations to the inequality (11), we can obtain the inequality (4). □

Remark 2 Under certain additional conditions, we can show that $\widehat{u}(x,y)$ is a partially (i)-differentiable solution of the partial fuzzy differential equation (1). In fact, it can be seen from (5) that $\widehat{u}(x,y)$ depends only on the variable y. So we can easily obtain $\widehat{u}_x(x,y) = \widetilde{0} = \chi_0$. Now, we consider the partial (i)-derivative of $\widehat{u}(x,y)$ with respect to the variable y. For simplicity, we set

$$\begin{aligned}&f(y) := \exp\biggl(-\frac{1}{b}\int_0^{y}\delta(\tau)\,d\tau\biggr),\\ &g(y) := u_0 \oplus \frac{1}{b}\int_0^{y}\exp\biggl(\frac{1}{b}\int_0^{\omega}\delta(\tau)\,d\tau\biggr)\odot\sigma(\omega)\,d\omega.\end{aligned}$$

By Theorem 2.2 in [23], it follows that $g(y)$ is (i)-differentiable on $\mathbb{R}_+ \cup \{0\}$. According to (e) of Theorem 5 in [26], if $f(y)\odot g(y)$ satisfies the condition (H1) on $\mathbb{R}_+ \cup \{0\}$, then we conclude that $\widehat{u}(x,y) = f(y)\odot g(y)$ is partial (i)-differentiable with respect to y on $\mathbb{R}_+ \cup \{0\}$ due to $f(y)\cdot f'(y) < 0$ for each $y \in \mathbb{R}_+ \cup \{0\}$. Therefore, we have

$$\begin{aligned}\widehat{u}_y(x,y) &= \bigl(f(y)\odot g(y)\bigr)'\\ &= f(y)\odot g'(y) \ominus \bigl(-f'(y)\bigr)\odot g(y)\\ &= \frac{1}{b}\sigma(y) \ominus \frac{1}{b}\delta(y)\odot\widehat{u}(x,y).\end{aligned}$$

Applying the above relation to the left hand side of (1), we obtain

$$\begin{aligned}&b\odot\widehat{u}_y(x,y)\oplus\delta(y)\odot\widehat{u}(x,y)\\ &\quad= b\odot\biggl(\frac{1}{b}\sigma(y)\ominus\frac{1}{b}\delta(y)\odot\widehat{u}(x,y)\biggr)\oplus\delta(y)\odot\widehat{u}(x,y)\\ &\quad= \sigma(y) = a\odot\widehat{u}_x(x,y)\oplus\sigma(y),\end{aligned}$$

which equals the right hand side of (1). That is to say, $\widehat{u}(x,y)$ is a partial (i)-differentiable solution of the partial fuzzy differential equation (1).

Based on Theorem 3.1 and Remark 2, we can formulate the following theorem.

Theorem 3.2 *Let σ, δ, and φ be given as in Theorem* 3.1 *and let $u : \mathbb{R} \times \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_{\mathcal{F}}$ be a bivariate fuzzy number-valued function which has continuous partial* (i)*-derivatives with respect to each of the variables. Assume that u satisfies the inequality* (3) *for all $x \in \mathbb{R}$ and $y \in \mathbb{R}_+ \cup \{0\}$. If the conditions* (i)-(v) *given in Theorem* 3.1 *are satisfied, then there exists a unique $u_0 \in \mathbb{R}_{\mathcal{F}}$ such that the inequality* (4) *holds for all $x \in \mathbb{R}$ and $y \in \mathbb{R}_+ \cup \{0\}$, where $\widehat{u}(x,y)$ is given by* (5). *Furthermore, if $\widehat{u}(x,y)$ fulfills the condition* (H1) *with respect to the variable y, then $\widehat{u}(x,y)$ is the unique partial* (i)*-differentiable solution of* (1) *satisfying the inequality* (4).

In particular, as a direct consequence of Theorem 3.2, the Hyers-Ulam stability of (1) can be established as follows.

Corollary 3.3 *Let σ, δ, and φ be given as in Theorem* 3.1 *and let $u : \mathbb{R} \times \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_{\mathcal{F}}$ be a bivariate fuzzy number-valued function which has continuous partial* (i)*-derivatives with respect to each of the variables. For a given $\epsilon > 0$, assume that u satisfies the following inequality*:

$$D\big(b \odot u_y(x,y) \oplus \delta(y) \odot u(x,y), a \odot u_x(x,y) \oplus \sigma(y)\big) \leq \epsilon \tag{13}$$

for all $x \in \mathbb{R}$ and $y \in \mathbb{R}_+$. If the integral $\int_0^{\infty} \exp(\frac{1}{b}\int_0^{\omega} \delta(\tau)\,d\tau)\,d\omega$ exists and the conditions (i), (ii), (iv), *and* (v) *given in Theorem* 3.1 *are satisfied, then there exists a unique $u_0 \in \mathbb{R}_{\mathcal{F}}$ such that*

$$D\big(u(x,y), \widehat{u}(x,y)\big) \leq \frac{\epsilon}{b} \exp\left(-\frac{1}{b}\int_0^y \delta(\tau)\,d\tau\right) \int_y^{\infty} \exp\left(\frac{1}{b}\int_0^{\omega} \delta(\tau)\,d\tau\right) d\omega \tag{14}$$

for all $x \in \mathbb{R}$ and $y \in \mathbb{R}_+ \cup \{0\}$, where $\widehat{u}(x,y)$ is given by (5). *Furthermore, if $\widehat{u}(x,y)$ fulfills the condition* (H1) *with respect to the variable y, then $\widehat{u}(x,y)$ is the unique partial* (i)*-differentiable solution of* (1) *satisfying the inequality* (14).

3.2 Stability of (1) under partial (ii)-differentiability

Theorem 3.4 *Let $\sigma : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_{\mathcal{F}}$ be a continuous fuzzy number-valued function and let $u : \mathbb{R} \times \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_{\mathcal{F}}$ be a bivariate fuzzy number-valued function which has continuous partial* (ii)*-derivatives with respect to each of the variables. Assume that u satisfies the inequality* (3) *for all $x \in \mathbb{R}$ and $y \in \mathbb{R}_+ \cup \{0\}$, where $a, b > 0$ are constants, $\delta : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_-$ is a continuous function and $\varphi : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$ is a function. Moreover, assume that the following conditions are satisfied*:

(i) $\int_0^y \delta(\tau)\,d\tau$ *exists for all $y \in \mathbb{R}_+ \cup \{0\}$*;
(ii) $\int_0^y \exp(\frac{1}{b}\int_0^{\omega} \delta(\tau)\,d\tau) \odot \sigma(\omega)\,d\omega$ *exists for all $y \in \mathbb{R}_+ \cup \{0\}$*;
(iii) $\int_0^{\infty} \varphi(\omega) \exp(-\frac{1}{b}\int_0^{\omega} \delta(\tau)\,d\tau)\,d\omega$ *exists*;
(iv) $\lim_{x\to-\infty, y\to+\infty} u(x,y)$ *exists*.

Then there exists a unique $u_0 \in \mathbb{R}_{\mathcal{F}}$ such that

$$\exp\left(\frac{1}{b}\int_0^y \delta(\tau)\,d\tau\right) u(x,y) \oplus \frac{1}{b}\int_0^y -\exp\left(\int_0^{\omega} \delta(\tau)\,d\tau\right) \odot \sigma(\omega)\,d\omega \to u_0 \tag{15}$$

as $y \to \infty$. Moreover, if the H-difference

$$u_0 \ominus \frac{1}{b}\int_0^y -\exp\left(\frac{1}{b}\int_0^\omega \delta(\tau)\,d\tau\right) \odot \sigma(\omega)\,d\omega$$

exists for each $x \in \mathbb{R}$ and each $y \in \mathbb{R}_+ \cup \{0\}$, then u_0 corresponds to a unique $\widehat{u}(x,y)$ such that

$$D\big(u(x,y),\widehat{u}(x,y)\big) \le \frac{1}{b}\exp\left(-\frac{1}{b}\int_0^y \delta(\tau)\,d\tau\right)\int_y^\infty \varphi(\omega)\exp\left(-\frac{1}{b}\int_0^\omega \delta(\tau)\,d\tau\right)d\omega \quad (16)$$

for all $x \in \mathbb{R}$ and all $y \in \mathbb{R}_+ \cup \{0\}$, where

$$\widehat{u}(x,y) = \exp\left(-\frac{1}{b}\int_0^y \delta(\tau)\,d\tau\right) \odot \left(u_0 \ominus \frac{1}{b}\int_0^y -\exp\left(\frac{1}{b}\int_0^\omega \widetilde{\delta}(\tau)\,d\tau\right) \odot \sigma(\omega)\,d\omega\right). \quad (17)$$

Proof Using the same coordinate transformations as in Theorem 3.1, by Theorem 2.1, we can obtain the inequality (7). Therefore, according to Theorem 3.5 in [21], for each fixed $\xi \in \mathbb{R}$, there exists a unique $\theta(\xi) \in \mathbb{R}_{\mathcal{F}}$ such that

$$\exp\left(\int_0^\eta \widetilde{\delta}(\mu)\,d\mu\right)\widetilde{u}(\xi,\eta) \oplus \int_0^\eta -\exp\left(\int_0^\nu \widetilde{\delta}(\mu)\,d\mu\right) \odot \widetilde{\sigma}(\nu)\,d\nu \to \theta(\xi)$$

as $\eta \to \infty$, where $\widetilde{\delta}$, $\widetilde{u}$, and $\widetilde{\sigma}$ are given as in Theorem 3.1. Moreover, we have

$$\begin{aligned}&\exp\left(\frac{1}{b}\int_0^{b\eta} \delta(\tau)\,d\tau\right)u(\xi - a\eta, b\eta)\\ &\quad\oplus \frac{1}{b}\int_0^{b\eta} -\exp\left(\frac{1}{b}\int_0^\omega \delta(\tau)\,d\tau\right) \odot \sigma(\omega)\,d\omega \to \theta(\xi)\end{aligned} \quad (18)$$

as $\eta \to \infty$. Notice that every integral of (18) exists, together with the condition (iv) implying that $\theta(\xi)$ is a constant fuzzy number; we write simply u_0. This means that (15) holds.

Furthermore, if the H-difference $u_0 \ominus \frac{1}{b}\int_0^y -\exp(\frac{1}{b}\int_0^\omega \delta(\tau)\,d\tau)\sigma(\omega)\,d\omega$ exists, by Theorem 3.5 in [21], then we can infer that

$$\begin{aligned}&D\left(\widetilde{u}(\xi,\eta), \exp\left(-\int_0^\eta \widetilde{\delta}(\mu)\,d\mu\right) \odot \left(u_0 \ominus \int_0^\eta -\exp\left(\int_0^\nu \widetilde{\delta}(\mu)\,d\mu\right) \odot \widetilde{\sigma}(\nu)\,d\nu\right)\right)\\ &\quad\le \exp\left(-\int_0^\eta \widetilde{\delta}(\mu)\,d\mu\right)\int_\eta^\infty \widetilde{\varphi}(\nu)\exp\left(-\int_0^\nu \widetilde{\delta}(\mu)\,d\mu\right)d\nu\end{aligned} \quad (19)$$

for all $\eta \in \mathbb{R}_+$, where $\widetilde{\varphi}(\nu) = \varphi(b\nu) = \varphi(\omega)$. Using the foregoing relations of coordinate transformations, it follows from (19) that the inequality (16) holds. □

Remark 3 Under an additional condition, we claim that $\widehat{u}(x,y)$ is a partial (ii)-differentiable solution of the partial fuzzy differential equation (1). In fact, it can be seen from (17) that $\widehat{u}(x,y)$ depends only on the variable y. So it follows that $\widehat{u}_x(x,y) = \widetilde{0} = \chi_0$. Next, we shall check the partial (ii)-differentiability of $\widehat{u}(x,y)$ with respect to the variable y. For convenience, we set

$$f(y) := \exp\left(-\frac{1}{b}\int_0^y \delta(\tau)\,d\tau\right),$$

$$g(y) := u_0 \ominus \frac{1}{b}\int_0^y -\exp\left(\frac{1}{b}\int_0^\omega \delta(\tau)\,d\tau\right) \odot \sigma(\omega)\,d\omega.$$

According to Theorem 2.3, it follows that $g(y)$ is (ii)-differentiable on $\mathbb{R}_+ \cup \{0\}$. By (d) of Theorem 5 in [26], since $f(y)\cdot f'(y) > 0$, if $f(y)\odot g(y)$ satisfies the condition (H2) on $\mathbb{R}_+\cup\{0\}$, then we conclude that $\widehat{u}(x,y) = f(y) \odot g(y)$ is partial (ii)-differentiable with respect to y for each $y \in \mathbb{R}_+ \cup \{0\}$. Hence, we have

$$\begin{aligned}\widehat{u}_y(x,y) &= \bigl(f(y) \odot g(y)\bigr)' \\ &= f(y) \odot g'(y) \ominus \bigl(-f'(y)\bigr) \odot g(y) \\ &= \frac{1}{b}\sigma(y) \ominus \frac{1}{b}\delta(y) \odot \widehat{u}(x,y).\end{aligned}$$

Applying the above relation to (1), we get

$$\begin{aligned}&b \odot \widehat{u}_y(x,y) \oplus \delta(y) \odot \widehat{u}(x,y) \\ &\quad= b \odot \left(\frac{1}{b}\sigma(y) \ominus \frac{1}{b}\delta(y) \odot \widehat{u}(x,y)\right) \oplus \delta(y) \odot \widehat{u}(x,y) \\ &\quad= \sigma(y) = a \odot \widehat{u}_x(x,y) \oplus \sigma(y),\end{aligned}$$

which implies that $\widehat{u}(x,y)$ is a partial (ii)-differentiable solution of the partial fuzzy differential equation (1).

In view of Theorem 3.4 and Remark 3, a further result of the stability of (1) can be formulated as follows:

Theorem 3.5 *Let σ, δ, and φ be given as in Theorem* 3.4 *and let $u : \mathbb{R} \times \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_\mathcal{F}$ be a bivariate fuzzy number-valued function which has continuous partial* (ii)*-derivatives with respect to each of the variables. Assume that u satisfies the inequality* (3) *for all $x \in \mathbb{R}$ and $y \in \mathbb{R}_+ \cup \{0\}$. If the conditions* (i)-(iv) *given in Theorem* 3.4 *are satisfied, then there exists an $u_0 \in \mathbb{R}_\mathcal{F}$ such that* (15) *holds for all $x \in \mathbb{R}$ and $y \in \mathbb{R}_+ \cup \{0\}$. Furthermore, if the H-difference $u_0 \ominus \frac{1}{b}\int_0^y -\exp(\frac{1}{b}\int_0^\omega \delta(\tau)\,d\tau)\sigma(\omega)\,d\omega$ exists and $\widehat{u}(x,y)$ defined by* (17) *fulfills the condition* (H2) *with respect to the variable y, then $\widehat{u}(x,y)$ is the unique partial* (ii)*-differentiable solution of* (1) *satisfying the inequality* (16).

Based on Theorem 3.5, the following Hyers-Ulam stability result of (1) as a particular case can be obtained.

Corollary 3.6 *Let σ, δ, and φ be given as in Theorem* 3.4 *and let $u : \mathbb{R} \times \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_\mathcal{F}$ be a bivariate fuzzy number-valued function which has continuous partial* (ii)*-derivatives with respect to each of the variables. For a given $\epsilon > 0$, assume that u satisfies the inequality* (13) *for all $x \in \mathbb{R}$ and $y \in \mathbb{R}_+\cup\{0\}$. If the conditions* (i)-(iv) *given in Theorem* 3.4 *are satisfied, then there exists an $u_0 \in \mathbb{R}_\mathcal{F}$ such that* (15) *holds for all $x \in \mathbb{R}$ and $y \in \mathbb{R}_+ \cup \{0\}$. Furthermore, if the H-difference $u_0 \ominus \frac{1}{b}\int_0^y -\exp(\frac{1}{b}\int_0^\omega \delta(\tau)\,d\tau)\sigma(\omega)\,d\omega$ exists and $\widehat{u}(x,y)$ defined by* (17) *fulfills the condition* (H2) *with respect to the variable y, then $\widehat{u}(x,y)$ is the unique* *partial* (ii)*-differentiable solution of* (1) *satisfying the following inequality*:

$$D\big(u(x,y),\widehat{u}(x,y)\big) \le \frac{\epsilon}{b}\exp\left(-\frac{1}{b}\int_0^y \delta(\tau)\,d\tau\right)\int_y^\infty \exp\left(-\frac{1}{b}\int_0^\omega \delta(\tau)\,d\tau\right)d\omega$$

for all $x \in \mathbb{R}$ *and* $y \in \mathbb{R}_+ \cup \{0\}$.

4 Hyers-Ulam-Rassias stability of linear partial fuzzy differential equation (2)

Under some suitable conditions, in this section, we shall discuss the stability of the linear partial fuzzy differential equation (2).

4.1 Stability of (2) under partial (i)-differentiability

Theorem 4.1 *Let* $\sigma : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_{\mathcal{F}}$ *be a continuous fuzzy number-valued function and let* $u : \mathbb{R}_+ \cup \{0\} \times \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ *be a bivariate fuzzy number-valued function which has continuous partial* (i)*-derivatives with respect to each of the variables. Assume that u satisfies the following inequality*:

$$D\big(a \odot u_x(x,y) \oplus \delta(x) \odot u(x,y), b \odot u_y(x,y) \oplus \sigma(x)\big) \le \varphi(x) \tag{20}$$

for all $x \in \mathbb{R}_+ \cup \{0\}$ *and* $y \in \mathbb{R}$, *where* $a, b > 0$ *are constants*, $\delta : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$ *is a continuous function and* $\varphi : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$ *is a function. Moreover, assume that the following conditions are satisfied*:

(i) $\int_0^x \delta(\tau)\,d\tau$ *exists for all* $x \in \mathbb{R}_+ \cup \{0\}$;

(ii) $\int_0^x \exp(\frac{1}{a}\int_0^\omega \delta(\tau)\,d\tau) \odot \sigma(\omega)\,d\omega$ *exists for all* $x \in \mathbb{R}_+ \cup \{0\}$;

(iii) $\int_0^\infty \varphi(\omega)\exp(\frac{1}{a}\int_0^\omega \delta(\tau)\,d\tau)\,d\omega$ *exists*;

(iv) $\lim_{x\to+\infty, y\to-\infty} u(x,y)$ *exists*;

(v) *the H-difference* $\exp(\int_0^x \frac{1}{a}\delta(\tau)\,d\tau)u(x,y) \ominus \frac{1}{a}\int_0^x \exp(\frac{1}{a}\int_0^\omega \delta(\tau)\,d\tau) \odot \sigma(\omega)\,d\omega$ *exists for each* $x \in \mathbb{R}_+ \cup \{0\}$ *and each* $y \in \mathbb{R}$.

Then there exists a unique $u_0 \in \mathbb{R}_{\mathcal{F}}$ *such that*

$$D\big(u(x,y),\widehat{u}(x,y)\big) \le \frac{1}{a}\exp\left(-\frac{1}{a}\int_0^x \delta(\tau)\,d\tau\right)\int_x^\infty \varphi(\omega)\exp\left(\frac{1}{a}\int_0^\omega \delta(\tau)\,d\tau\right)d\omega \tag{21}$$

for all $x \in \mathbb{R}$ *and all* $y \in \mathbb{R}_+$, *where*

$$\widehat{u}(x,y) = \exp\left(-\frac{1}{a}\int_0^x \delta(\tau)\,d\tau\right) \odot \left(u_0 \oplus \frac{1}{a}\int_0^x \exp\left(\frac{1}{a}\int_0^\omega \widetilde{\delta}(\tau)\,d\tau\right) \odot \sigma(\omega)\,d\omega\right). \tag{22}$$

Proof According to Definition 2.1, if we set $u(x,y) = v(y,x)$ for all $x \in \mathbb{R}_+ \cup \{0\}$, $y \in \mathbb{R}$, then we get

$$\begin{aligned} u_x(x,y) &= \lim_{\Delta x\to 0+}\frac{u(x+\Delta x,y)\ominus u(x,y)}{\Delta x} \\ &= \lim_{\Delta x\to 0+}\frac{v(y,x+\Delta x)\ominus v(y,x)}{\Delta x} = v_y(y,x), \\ u_y(x,y) &= \lim_{\Delta y\to 0+}\frac{u(x,y+\Delta y)\ominus u(x,y)}{\Delta y} \\ &= \lim_{\Delta y\to 0+}\frac{v(y+\Delta y,x)\ominus v(y,x)}{\Delta x} = v_x(y,x). \end{aligned}$$

Therefore, the inequality (20) changes into the following form:

$$D\big(a \odot v_y(y,x) \oplus \delta(x) \odot v(y,x), b \odot v_x(y,x) \oplus \sigma(x)\big) \le \varphi(x)$$

for all $x \in \mathbb{R}_+ \cup \{0\}$, $y \in \mathbb{R}$. Now, we exchange the roles of the variables x and y in the preceding inequality, and we obtain

$$D\big(a \odot v_y(x,y) \oplus \delta(y) \odot v(x,y), b \odot v_y(x,y) \oplus \sigma(y)\big) \le \varphi(y)$$

for all $x \in \mathbb{R}$, $y \in \mathbb{R}_+ \cup \{0\}$.

In view of the conditions (i)-(v) and Theorem 3.1, we know that there exists a unique $u_0 \in \mathbb{R}_\mathcal{F}$ such that

$$\begin{aligned} &D\left(v(x,y), \exp\left(-\frac{1}{a}\int_0^y \delta(\tau)\,d\tau\right) \odot \left(u_0 \oplus \frac{1}{a}\int_0^y \exp\left(\frac{1}{a}\int_0^\omega \delta(\tau)\,d\tau\right) \odot \sigma(\omega)\,d\omega\right)\right) \\ &\quad \le \frac{1}{a}\exp\left(-\frac{1}{a}\int_0^y \delta(\tau)\,d\tau\right)\int_y^\infty \varphi(\omega)\exp\left(\frac{1}{a}\int_0^\omega \delta(\tau)\,d\tau\right)d\omega \end{aligned}$$

for all $x \in \mathbb{R}$, $y \in \mathbb{R}_+ \cup \{0\}$. By exchanging the roles of the variables x and y in the above inequality again, we can infer that the inequality (21) holds for all $x \in \mathbb{R}_+ \cup \{0\}$ and all $y \in \mathbb{R}$. □

Remark 4 Similar to Remark 2, by a tedious calculation, it can be verified that $\widehat{u}(x,y)$ defined by (22) is a partial (i)-differentiable solution of the partial fuzzy differential equation (2).

Based on Theorems 3.2 and 4.1, by adding an additional condition, we can obtain the following result.

Theorem 4.2 *Let σ, δ, and φ be given as in Theorem* 4.1 *and let $u : \mathbb{R}_+ \cup \{0\} \times \mathbb{R} \to \mathbb{R}_\mathcal{F}$ be a bivariate fuzzy number-valued function which has continuous partial* (i)*-derivatives with respect to each of the variables. Assume that u satisfies the inequality* (20) *for all $x \in \mathbb{R}_+ \cup \{0\}$ and $y \in \mathbb{R}$. If the conditions* (i)-(v) *given in Theorem* 4.1 *are satisfied, then there exists a unique $u_0 \in \mathbb{R}_\mathcal{F}$ such that the inequality* (21) *holds for all $x \in \mathbb{R}_+ \cup \{0\}$ and $y \in \mathbb{R}$, where $\widehat{u}(x,y)$ is given by* (22)*. Furthermore, if $\widehat{u}(x,y)$ fulfills the condition* (H1) *with respect to the variable x, then $\widehat{u}(x,y)$ is the unique partial* (i)*-differentiable solution of* (2) *satisfying the inequality* (21).

Especially, the Hyers-Ulam stability of (1) under partial (i)-differentiability can be induced by Theorem 4.2.

Corollary 4.3 *Let σ, δ, and φ be given as in Theorem* 4.1 *and let $u : \mathbb{R}_+ \cup \{0\} \times \mathbb{R} \to \mathbb{R}_\mathcal{F}$ be a bivariate fuzzy number-valued function which has continuous partial* (i)*-derivatives with respect to each of the variables. For a given $\epsilon > 0$, assume that u satisfies the following inequality*:

$$D\big(a \odot u_x(x,y) \oplus \delta(x) \odot u(x,y), b \odot u_y(x,y) \oplus \sigma(x)\big) \le \epsilon \tag{23}$$

for all $x \in \mathbb{R}_+ \cup \{0\}$ *and* $y \in \mathbb{R}$. *If the integral* $\int_0^\infty \exp(\frac{1}{a}\int_0^\omega \delta(\tau)\,d\tau)\,d\omega$ *exists and the conditions* (i), (ii), (iv), *and* (v) *given in Theorem* 4.1 *are satisfied, then there exists a unique* $u_0 \in \mathbb{R}_\mathcal{F}$ *such that*

$$D\big(u(x,y),\widehat{u}(x,y)\big) \le \frac{\epsilon}{a}\exp\left(-\frac{1}{a}\int_0^x \delta(\tau)\,d\tau\right)\int_x^\infty \exp\left(\frac{1}{a}\int_0^\omega \delta(\tau)\,d\tau\right)d\omega \tag{24}$$

for all $x \in \mathbb{R}_+ \cup \{0\}$ *and* $y \in \mathbb{R}$, *where* $\widehat{u}(x,y)$ *is given by* (22). *Furthermore, if* $\widehat{u}(x,y)$ *fulfills the condition* (H1) *with respect to the variable* x, *then* $\widehat{u}(x,y)$ *is the unique partial* (i)-*differentiable solution of* (2) *satisfying the inequality* (21).

4.2 Stability of (2) under partial (ii)-differentiability

Theorem 4.4 *Let* $\sigma : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_\mathcal{F}$ *be a continuous fuzzy number-valued function and let* $u : \mathbb{R}_+ \cup \{0\} \times \mathbb{R} \to \mathbb{R}_\mathcal{F}$ *be a bivariate fuzzy number-valued function which has continuous partial* (ii)-*derivatives with respect to each of the variables. Assume that u satisfies the inequality* (20) *for all* $x \in \mathbb{R}_+ \cup \{0\}$ *and* $y \in \mathbb{R}$, *where* $a, b > 0$ *are constants*, $\delta : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_-$ *is a continuous function and* $\varphi : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$ *is a function. Moreover, assume that the following conditions are satisfied*:

(i) $\int_0^x \delta(\tau)\,d\tau$ *exists for all* $x \in \mathbb{R}_+ \cup \{0\}$;
(ii) $\int_0^x \exp(\frac{1}{a}\int_0^\omega \delta(\tau)\,d\tau) \odot \sigma(\omega)\,d\omega$ *exists for all* $x \in \mathbb{R}_+ \cup \{0\}$;
(iii) $\int_0^\infty \varphi(\omega)\exp(-\frac{1}{a}\int_0^\omega \delta(\tau)\,d\tau)\,d\omega$ *exists*;
(iv) $\lim_{x\to+\infty, y\to-\infty} u(x,y)$ *exists*.

Then there exists a unique $u_0 \in \mathbb{R}_\mathcal{F}$ *such that*

$$\exp\left(\frac{1}{a}\int_0^x \delta(\tau)\,d\tau\right)u(x,y) \oplus \frac{1}{a}\int_0^x -\exp\left(\int_0^\omega \delta(\tau)\,d\tau\right)\sigma(\omega)\,d\omega \to u_0 \tag{25}$$

as $x \to \infty$. *Moreover, if the H-difference*

$$u_0 \ominus \frac{1}{a}\int_0^x -\exp\left(\frac{1}{a}\int_0^\omega \delta(\tau)\,d\tau\right)\sigma(\omega)\,d\omega$$

exists for each $x \in \mathbb{R}_+ \cup \{0\}$ *and each* $y \in \mathbb{R}$, *then* u_0 *corresponds to a unique* $\widehat{u}(x,y)$ *such that*

$$D\big(u(x,y),\widehat{u}(x,y)\big) \le \frac{1}{a}\exp\left(-\frac{1}{a}\int_0^x \delta(\tau)\,d\tau\right)\int_x^\infty \varphi(\omega)\exp\left(-\frac{1}{a}\int_0^\omega \delta(\tau)\,d\tau\right)d\omega \tag{26}$$

for all $x \in \mathbb{R}_+ \cup \{0\}$ *and all* $y \in \mathbb{R}$, *where*

$$\widehat{u}(x,y) = \exp\left(-\frac{1}{a}\int_0^x \delta(\tau)\,d\tau\right) \odot \left(u_0 \ominus \frac{1}{a}\int_0^x -\exp\left(\frac{1}{a}\int_0^\omega \widetilde{\delta}(\tau)\,d\tau\right) \odot \sigma(\omega)\,d\omega\right). \tag{27}$$

Proof By Theorem 3.4 and using the same argument as in the proof of Theorem 4.1, we can easily carry out the proof of this theorem. □

Remark 5 By a tedious calculation, it can also be checked that $\widehat{u}(x,y)$ defined by (27) is a partial (ii)-differentiable solution of the partial fuzzy differential equation (2).

Using an additional condition and Theorem 4.4, we can obtain the following theorem.

Theorem 4.5 *Let σ, δ, and φ be given as in Theorem* 4.4 *and let $u : \mathbb{R}_+ \cup \{0\} \times \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be a bivariate fuzzy number-valued function which has continuous partial* (ii)*-derivatives with respect to each of the variables. Assume that u satisfies the inequality* (20) *for all $x \in \mathbb{R}_+ \cup \{0\}$ and $y \in \mathbb{R}$. If the conditions* (i)-(iv) *given in Theorem* 4.4 *are satisfied, then there exists an $u_0 \in \mathbb{R}_{\mathcal{F}}$ such that* (25) *holds for all $x \in \mathbb{R}_+ \cup \{0\}$ and $y \in \mathbb{R}$. Furthermore, if the H-difference $u_0 \ominus \frac{1}{a}\int_0^x - \exp(\frac{1}{a}\int_0^\omega \delta(\tau)\,d\tau)\sigma(\omega)\,d\omega$ exists and $\widehat{u}(x,y)$ defined by* (27) *fulfills the condition* (H2) *with respect to the variable x, then $\widehat{u}(x,y)$ is the unique partial* (ii)*-differentiable solution of* (2) *satisfying the inequality* (26).

In particular, as a direct consequence of Theorem 4.5, we can obtain the Hyers-Ulam stability of (2) under partial (ii)-differentiability.

Corollary 4.6 *Let σ, δ, and φ be given as in Theorem* 4.4 *and let $u : \mathbb{R}_+ \cup \{0\} \times \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be a bivariate fuzzy number-valued function which has continuous partial* (ii)*-derivatives with respect to each of the variables. For a given $\epsilon > 0$, assume that u satisfies the inequality* (23) *for all $x \in \mathbb{R}_+ \cup \{0\}$ and $y \in \mathbb{R}$. If the conditions* (i)-(iv) *given in Theorem* 4.4 *are satisfied, then there exists an $u_0 \in \mathbb{R}_{\mathcal{F}}$ such that* (25) *holds for all $x \in \mathbb{R}_+ \cup \{0\}$ and $y \in \mathbb{R}$. Furthermore, if the H-difference $u_0 \ominus \frac{1}{a}\int_0^x - \exp(\frac{1}{a}\int_0^\omega \delta(\tau)\,d\tau)\sigma(\omega)\,d\omega$ exists and $\widehat{u}(x,y)$ defined by* (27) *fulfills the condition* (H2) *with respect to the variable x, then $\widehat{u}(x,y)$ is the unique partial* (ii)*-differentiable solution of* (2) *satisfying the following inequality*:

$$D\big(u(x,y), \widehat{u}(x,y)\big) \le \frac{\epsilon}{a}\exp\left(-\frac{1}{a}\int_0^x \delta(\tau)\,d\tau\right)\int_x^\infty \exp\left(-\frac{1}{a}\int_0^\omega \delta(\tau)\,d\tau\right)d\omega$$

for all $x \in \mathbb{R}_+ \cup \{0\}$ and $y \in \mathbb{R}$.

5 Conclusions

As a continuation of our previous work [21], in the present paper, we investigate the Hyers-Ulam-Rassias stability of two types of first order linear partial fuzzy differential equations (see (1) and (2)) under generalized differentiability. These results show that, under some appropriate conditions, if an approximate solution to the given equation satisfying the specific error is obtained, then the unique exact solution to the corresponding equation can be formally constructed, and the error can be accurately estimated. In addition, we also established a multivariate fuzzy chain rule under generalized differentiability in order to study the stability problems of (1) and (2) in the sense of the same differentiability.

In these results obtained in this paper, the coefficient functions δ and σ in (1) and (2) are assumed to be a univariate positive (negative) real-valued function and a univariate fuzzy number-valued function, respectively. Therefore, it is an open question whether the corresponding stability result is still true if the coefficient functions δ and σ are the functions of two variables. Moreover, it would be interesting to discuss the stability problems of (1) and (2) in which the coefficient function δ contains a finite number of zeros.

Competing interests
The author declares to have no competing interests.

Author's contributions
YS drafted the manuscript and completed all proofs of the results in this paper.

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