Morabito *Boundary Value Problems* (2016) 2016:124
DOI 10.1186/s13661-016-0631-6

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Asymptotically radial solutions to an elliptic problem on expanding annular domains in Riemannian manifolds with radial symmetry

Filippo Morabito*

*Correspondence:
morabito.math@gmail.com
Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon, South Korea
School of Mathematics, KIAS, 87 Hoegi-ro, Dongdaemun-gu, Seoul, South Korea

Abstract

We consider the boundary value problem

$$\begin{cases} \Delta_{\mathbf{g}} u + u^p = 0 & \text{in } \Omega_R, \\ u = 0 & \text{on } \partial\Omega_R, \end{cases}$$

Ω_R being a smooth bounded domain diffeomorphic to the expanding domain $A_R := \{x \in M, R < r(x) < R+1\}$ in a Riemannian manifold M of dimension $n \geq 2$ endowed with the metric $\mathbf{g} = dr^2 + S^2(r) g_{\mathbb{S}^{n-1}}$. After recalling a result about existence, uniqueness, and non-degeneracy of the positive radial solution when $\Omega_R = A_R$, we prove that there exists a positive non-radial solution to the aforementioned problem on the domain Ω_R. Such a solution is close to the radial solution to the corresponding problem on A_R.

MSC: 35B32; 35J60; 58J32

1 Introduction

Many authors studied the following boundary value problem:

$$\begin{cases} \Delta u + \lambda u + u^p = 0 & \text{in } A, \\ u > 0 & \text{in } A, \\ u = 0 & \text{on } \partial A, \end{cases} \tag{1}$$

where $A \subset \mathbb{R}^n$, $n \geq 2$, is an annulus, that is,

$$A = \{x \in \mathbb{R}^n : R_1 < r(x) < R_2\},$$

with $r(x)$ equal to the distance to the origin. The radial solution always exists for any $p > 1$, it is unique and radially non-degenerate. This result is shown in [1] by Ni and Nussbaum.

We would like also to mention the work [2] by Kabeya, Yanagida, and Yotsutani where general structure theorems about positive radial solutions to semilinear elliptic equations of the form $Lu + h(|x|, u) = 0$ on radially symmetric domains $(a, b) \times \mathbb{S}^{n-1}$, $-\infty \leq a < b \leq +\infty$, with various boundary conditions are shown. Precisely, if $u = u(r)$ then $Lu = (g(r)u'(r))'$, with $r = |x|$. A classification result for positive radial solutions to the scalar field equation $\Delta u + K(r)u^p = 0$ on $\mathbb{R}^n$ according to their behavior as $r \to +\infty$ has been shown by Yanagida and Yotsutani in [3]. Furthermore in [4] the same authors proved some existence results for positive radial solutions to $\Delta u + h(r, u) = 0$ on radially symmetric domains for different non-linearities.

The invariance of the annulus with respect to different symmetry groups has been exploited by several authors to show the existence of non-radial positive solutions in expanding annuli with R_1, R_2 big enough.

In the recent work [5] Gladiali *et al.* considered the problem (1) on expanding annuli,

$$A_R := \{x \in \mathbb{R}^n : R < r(x) < R + 1\},$$

$\lambda < \lambda_{1,A_R}$, λ_{1,A_R} being the first eigenvalue of $-\Delta$ on A_R. They have showed the existence of non-radial solutions which arise by bifurcation from the positive radial solution.

On the other hand in recent years an increasing number of authors turned their attention to the study of elliptic partial differential equations on Riemannian manifolds. We mention only the following work: [6] by Mancini and Sandeep, where the existence and uniqueness of the positive finite energy radial solution to the equation $\Delta_{\mathbb{H}^n} u + \lambda u + u^p = 0$ in the hyperbolic space are studied; [7] by Bonforte *et al.*, which deals the study of infinite energy radial solutions to the Emden-Fowler equation in the hyperbolic space; [8] by Berchio, Ferrero, and Grillo, where stability and qualitative properties of radial solutions to the Emden-Fowler equation in radially symmetric Riemannian manifolds are investigated.

In [9], under the assumption $\lambda < 0$, the results shown in [5] have been extended to annular domains in an unbounded Riemannian manifold M of dimension $n \geq 2$ endowed with the metric $\mathbf{g} := dr^2 + S^2(r)g_{\mathbb{S}^{n-1}}$. $g_{\mathbb{S}^{n-1}}$ denotes the standard metric of the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$; $r \in [0, +\infty)$ is the geodesic distance measured from a point O. In this case Δ is replaced by the Laplace-Beltrami operator $\Delta_{\mathbf{g}}$.

Problem (1) has been studied also in the case where the expanding annulus is replaced by an expanding domain in $\mathbb{R}^n$ which is diffeomorphic to an annulus. For example in [10, 11] the existence is shown of an increasing number of solutions as the domain expands. Furthermore in [11] the authors show such solutions are not close to the radial one, indeed they exhibit a finite number of bumps.

In [12] Bartsch *et al.* show instead the existence of a positive solution to the problem (1) on an expanding annular domain Ω_R, which is close to the radial solution to the corresponding problem on the annulus A_R to which Ω_R is diffeomorphic.

In this article we extend the result of [12] to the case of an unbounded Riemannian manifold M of dimension $n \geq 2$ with metric $\mathbf{g}$ given above. The function $S(r)$ enjoys the following properties:

- $S(r) \in C^2([0, +\infty))$; $S(r) > 0$ for $r > 0$ and increasing;
- $\lim_{r\to+\infty} \frac{S'(r)}{S(r)} = l < +\infty$, $(\frac{S'(R)}{S(R)})' = o(1)$;
- $((\frac{S'(R)}{S(R)})' S^{n-1}(R))' = o(S'(R)S^{n-2}(R))$.

All L^p-norms are computed with respect to the Riemannian measure on M given by the density $dvol = S^{n-1}(r)\, dr\, d\theta$, with $\theta \in \mathbb{S}^{n-1}$.

The function $S(r)$ satisfies sufficient conditions (see Lemma 4.1 in [8]) which allow us to show that $\lambda_{1,C_{R_1}}$, the first eigenvalue of $-\Delta_{\mathbf{g}}$ on $C_{R_1} := \{x \in M : r(x) \geq R_1\}$, is non-negative. Such a lemma also provides sufficient conditions to show that $\lambda_{1,M}$, the first eigenvalue on M, is non-negative. Since the first eigenvalue on A, $\lambda_{1,A}$, is a decreasing function of R_2 and $C_{R_1} = \lim_{R_2 \to +\infty} A$, the first eigenvalue on A satisfies $\lambda_{1,A} > \lambda_{1,C_{R_1}} \geq 0$.

In this work we consider the case $\lambda = 0$ but some of the results presented here are valid also for $0 < \lambda < \lambda_{1,A}$.

First we recall the result concerning the existence, the uniqueness, and the non-degeneracy of the radial solution to the problem

$$\begin{cases} \Delta_{\mathbf{g}} u + u^p = 0 & \text{in } A, \\ u > 0 & \text{in } A, \\ u = 0 & \text{on } \partial A, \end{cases} \tag{2}$$

with $p > 1$ and $A := \{x \in M \mid R_1 < r(x) < R_2\} \subset M$. This is done in Section 2.

The existence of the radial positive solution u in an annulus suggests that a positive solution exists also on a domain which is diffeomorphic to an annulus and is close to it, and such a solution is a small deformation of u.

Let $g : \mathbb{S}^{n-1} \to \mathbb{R}$ be a positive C^∞-function and $\Omega_R \subset M$ be the set

$$\Omega_R := \left\{(r,\theta) \in \mathbb{R}^+ \times \mathbb{S}^{n-1} : R + g(\theta)S^{-\delta}(R) < r < R + 1 + g(\theta)S^{-\delta}(R)\right\}$$

for $R > 0$,

$$\max\left\{0, \frac{1}{2}(n-5)\right\} < \delta \leq \frac{1}{2}(n-1). \tag{3}$$

In [12] δ is chosen to be equal to 0 if $2 \leq n \leq 4$. The reason why we make a different choice is explained in Remark 6.5. The upper bound is used in Section 7.2.

Then the following map is a diffeomorphism between Ω_R and the annulus $A_R = \{x \in M : R < r(x) < R + 1\}$:

$$T(r,\theta) = \left(r - g(\theta)S^{-\delta}(R), \theta\right).$$

Clearly if $R \gg 1$ then Ω_R is a small deformation of A_R.

If $w_R \in H_0^1(A_R)$ denotes the positive radial solution to

$$\begin{cases} \Delta_{\mathbf{g}} u + u^p = 0 & \text{in } A_R, \\ u > 0 & \text{in } A_R, \\ u = 0 & \text{on } \partial A_R, \end{cases} \tag{4}$$

then we define

$$\begin{aligned} \tilde{u}_R &:= w_R \circ T \in H_0^1(\Omega_R), \\ \tilde{u}_R(r,\theta) &= w_R\left(r - g(\theta)S^{-\delta}(R), \theta\right). \end{aligned} \tag{5}$$

The main result of this article shown in Section 7 is the following.

Theorem 1.1 *There exists a sequence of radii* $\{R_k\}_k$ *divergent to* $+\infty$ *with the property that for every* $\delta > 0$ *there exists* $k_\delta \in \mathbb{N}$ *such that for any* $k \geq k_\delta$ *and for* $R \in [R_k + \delta, R_{k+1} - \delta]$, *the problem*

$$\begin{cases} \Delta_{\mathbf{g}} u + u^p = 0 & \text{in } \Omega_R, \\ u > 0 & \text{in } \Omega_R, \\ u = 0 & \text{on } \partial\Omega_R, \end{cases} \tag{6}$$

admits a positive solution

$$u_R = \tilde{u}_R + \phi_R$$

for some $\phi_R \in H_0^1(\Omega_R)$. *Moreover, the difference* $S(R_{k+1}) - S(R_k)$ *is bounded away from zero by a constant independent of k and* $\phi_R \to 0$ *in* $H_0^1(\Omega_R)$ *for* $R \in [R_k + \delta, R_{k+1} - \delta]$ *as* $k \to +\infty$.

Two examples of radially symmetric metrics whose function $S(r)$ satisfies the hypotheses given above are $S(r) = \frac{1}{\sqrt{-c}} \sinh(\sqrt{-c}r)$, $c < 0$ and $S(r) = r$. The corresponding ambient manifold is the space form with constant curvature equal to c (hyperbolic space) and to 0 ($\mathbb{R}^n$), respectively.

2 Existence, uniqueness and radial non-degeneracy of the radial solution

The existence of a positive radial solution to problem (2) for any $p > 1$ easily follows from a standard variational approach.

The uniqueness of the positive radial solution and the radial non-degeneracy can be shown following [9], where we considered $f(u) = \lambda u + u^p$, $\lambda < 0$, and $n - 1$ was replaced by a constant $\omega \geq 0$.

We consider the problem

$$\begin{cases} u''(r) + \omega \frac{S'(r)}{S(r)} u' + u^p = 0 & \text{in } (R_1, R_2), \\ u > 0 & \text{in } (R_1, R_2), \\ u(R_1) = u(R_2) = 0. \end{cases} \tag{7}$$

We define

$$G(r) := \alpha S^{\beta-2}(r)\big[(\alpha + 1 - \omega)\big(S'(r)\big)^2 - S''(r)S(r)\big],$$

where $\alpha = 2\frac{\omega}{p+3}$, $\beta = \alpha(p - 1)$.

Theorem 2.1 *Let* $\omega \geq 0$, $p \in (1, +\infty)$. *Suppose that* G' *satisfies the following*:

1. $G'(r)$ *is of constant sign on* (R_1, R_2) *or*
2. $G'(R_1) > 0$ *and* $G'(r)$ *changes sign only once on* (R_1, R_2).

Then the problem (7) *admits at most one solution. In other terms the problem* (2) *admits at most one radially symmetric solution. Moreover, the solution is non-degenerate in the space of* H^1*-radially symmetric functions.*

Remark 2.2 By Proposition 2.8 in [8] the hypotheses of Theorem 2.1 are satisfied provided $\frac{2n+1}{2n-3} \leq p < \frac{n+2}{n-2}$, the function $S(r)$ is four times differentiable, $S'''(r) > 0$, and $(\frac{S'(r)}{S'''(r)})' \leq 0$ for $r \in (R_1, R_2)$.

Remark 2.3 The metric $dr^2 + (\frac{1}{\sqrt{-c}}\sinh(\sqrt{-c}r))^2 g_{\mathbb{S}^{n-1}}$ of the space form $\mathbb{H}^n(c)$, $c < 0$, that is, the space of constant curvature c, satisfies the hypotheses of Theorem 2.1 and Theorem 2.1 of [9]. In particular the positive radial solution to $\Delta_{\mathbf{g}}u + \lambda u + u^p = 0$ with the Dirichlet boundary condition, is unique for $\lambda \leq 0$. That answers the question asked by Bandle and Kabeya in Section 5, part 2, of [13] about the uniqueness of the positive radial solution on the set $(d_0, d_1) \times \mathbb{S}^{n-1} \subset \mathbb{H}^n(-1)$.

The proof of Theorem 2.1 is omitted because it is the same as the proof of Theorem 2.1 in [9] with $\lambda = 0$.

Remark 2.4 We would like to mention the fact that the uniqueness of the positive radial solution on the annulus $\{x \in M \mid R_1 < r(x) < R_2\}$, could be proved using the results contained in [2]. Precisely Theorem A, Lemma C and Lemma 4.2 therein say that the equation $(g(r)u'(r))' + h(r,u) = 0$ has a solution on an interval (a,b), if an integrability condition is satisfied. Also note that this result is established by reducing the equation above to an equation of the form $v''_{tt} + k(t,v)$ on $(0,1)$ using the change of variable $t := \frac{\int_a^r 1/g(s)\,ds}{\int_a^b 1/g(s)\,ds}$. The integrability condition is formulated in terms of the function $k(t,v) := (\int_a^b 1/g(s)\,ds)g(r(t))h(r(t),v)$. In our case $g(r) = S^{n-1}(r)$ and $h(r,u) = S^{n-1}(r)u^p$. Because of the presence of an integral in the definition of $t = t(r)$, it is difficult to determine $r = r(t)$ which appears in the formula for $k(t,v)$. Consequently this approach is more difficult than the one provided by Theorem 2.1.

In the next sections we study how of the first eigenvalue of the linearized operator associated with (4) behaves if the inner radius of $A_R := \{x \in M \mid R < r(x) < R+1\}$, varies. To that aim we make here some observations that will be useful later.

Let u_R be the unique positive radial solution of (4). It is the solution to

$$\begin{cases} u''(r) + (n-1)\frac{S'(r)}{S(r)}u'(r) + u^p(r) = 0 & \text{in } (R, R+1), \\ u > 0 & \text{in } (R, R+1), \\ u(R) = u(R+1) = 0. \end{cases} \tag{8}$$

We recall that $\lim_{r\to+\infty} \frac{S'(r)}{S(r)} = l \in [0, +\infty)$.

Exactly as in Section 4 of [9], the function $\tilde{u}(t) := u_R(t+R)$ solves

$$\begin{cases} \tilde{u}'' + (n-1)\frac{S'(t+R)}{S(t+R)}\tilde{u}' + \tilde{u}^p = 0 & \text{on } (0,1), \\ \tilde{u} > 0 & \text{on } (0,1), \\ \tilde{u}(0) = \tilde{u}(1) = 0, \end{cases} \tag{9}$$

and it satisfies

$$\int_0^1 (\tilde{u}')^2\,dt \leq C. \tag{10}$$

So the function $\tilde{u}$ is bounded in $H_0^1((0,1))$ consequently also in $C^2((0,1))$. Furthermore $\tilde{u}$ tends to a non-vanishing function $\tilde{u}_\infty$ as $R \to +\infty$ which is the solution to

$$\begin{cases} \tilde{u}''_\infty + (n-1)l\tilde{u}'_\infty + \tilde{u}^p_\infty = 0 & \text{on } (0,1), \\ \tilde{u}_\infty \geq 0 & \text{on } (0,1), \\ \tilde{u}_\infty(0) = \tilde{u}_\infty(1) = 0. \end{cases} \tag{11}$$

3 Spectrum of the linearized operator

In this section we recall some results which can be proved as in [9]. We recall that $A = \{x \in M \mid R_1 < r(x) < R_2\}$, r being the geodesic distance of x to the point O.

We introduce two operators:

$$\tilde{L}_u^\omega : H^2(A) \cap H_0^1(A) \to L^2(A),$$
$$\tilde{L}_u^\omega := S^2\bigl(r(x)\bigr)\bigl(-\Delta_{\mathbf{g}} - \omega p u^{p-1} I\bigr);$$
$$\hat{L}_u^\omega : H^2\bigl((R_1,R_2)\bigr) \cap H_0^1\bigl((R_1,R_2)\bigr) \to L^2\bigl((R_1,R_2)\bigr),$$
$$\hat{L}_u^\omega v := S^2(r)\left(-v''(r) - (n-1)\frac{S'(r)}{S(r)} v' - \omega p u^{p-1} v\right).$$

The eigenvalues of the operator $\tilde{L}_u^\omega$ are defined as follows:

$$\tilde{\lambda}_i^\omega = \inf_{W \subset H_0^1(A), \dim W = i} \max_{v \in W, v \neq 0} \frac{\int_A (|\nabla v|^2 - \omega p u^{p-1} v^2)\, dvol}{\int_A S(r(x))^{-2} v^2\, dvol}.$$

The eigenvalues $\hat{\lambda}_i^\omega$ of the operator $\hat{L}_u^\omega$ can be evaluated similarly replacing the space $H_0^1(A)$ by $H_0^1((R_1,R_2))$.

Let w_i denote the normalized eigenfunctions ($\|w_i\|_{L^\infty} = 1$) of $\hat{L}_u^\omega$ associated with the eigenvalue $\hat{\lambda}_i^\omega$.

Lemma 3.1 *Let u denote a radial solution of* (1) *which is non-degenerate in the space of radially symmetric functions in H_0^1. Then u is degenerate, that is, there exists a non-trivial solution to*

$$\begin{cases} L_u v = -\Delta_{\mathbf{g}} v - p u^{p-1} v = 0 & \text{on } A, \\ v = 0 & \text{on } \partial A, \end{cases}$$

if and only if there exists $k \geq 1$ such that $\hat{\lambda}_1^1 + \lambda_k = 0$. Here λ_k denotes the kth eigenvalue of $-\Delta_{\mathbb{S}^{n-1}}$. The solution can be written as $w_1(r(x))\phi_k(\theta(x))$, $\phi_k(\theta(x))$ being the eigenfunction associated to λ_k.

In order to study the degeneracy of u we look at the eigenvalues ω close to 1 of the problem:

$$\begin{cases} -L_u^\omega v := \Delta_{\mathbf{g}} v + \omega p u^{p-1} v = 0 & \text{on } A, \\ v = 0 & \text{on } \partial A. \end{cases} \tag{12}$$

Remark 3.2 We observe that ω is an eigenvalue of (12) if and only if zero is an eigenvalue of $\tilde{L}_u^\omega$.

Remark 3.3 The Morse index $m(u)$ of u equals the number of negative eigenvalues of $L_u = -\Delta_{\mathbf{g}} - p u^{p-1} I$ counted with their multiplicity. $m(u)$ can be computed considering the negative eigenvalues of $\tilde{L}_u^\omega$, with $\omega = 1$.

If σ denotes the spectrum of an operator, then the spectra of $\tilde{L}_u^\omega$, $\hat{L}_u^\omega$, $-\Delta_{\mathbb{S}^{n-1}}$ are related as follows (compare Lemma 3.1 of [5]).

Proposition 3.4

$$\sigma\left(\tilde{L}_u^\omega\right) = \sigma\left(\hat{L}_u^\omega\right) + \sigma\left(-\Delta_{\mathbb{S}^{n-1}}\right).$$

In other terms, the Morse index depends only on the first eigenvalue of $\hat{L}_u^\omega$.

4 Properties of the first two eigenvalues

Let us introduce the operator

$$\bar{L}_u^\omega v := -v'' - (n-1)\frac{S'(r)}{S(r)}v' - \omega p u^{p-1} v$$

acting on functions defined on the interval $I = (R_1, R_2)$. Its eigenvalues are λ_m^ω.

The following propositions are inspired by Proposition 2.1 and Proposition 2.2 of [14].

Proposition 4.1 *If $\lambda_1^1 < 0$, then there exists $\alpha > 0$ such that if $|\omega - 1| < \alpha$, then the first eigenvalue of the operator $\bar{L}_u^\omega$, satisfies $\lambda_1^\omega < 0$.*

Proof First we show that there exists $C > 0$ such that $\lambda_1^\omega \leq C$ for any ω close enough to 1.

Let $\phi \in C_0^\infty(I)$ such that $\int_I \phi^2 S^{n-1}(r)\,dr = 1$. Since

$$\begin{aligned}
\lambda_1^\omega &\leq \int_I \bigl[(\phi')^2 - \omega p u^{p-1}\phi^2\bigr] S^{n-1}(r)\,dr, \\
\lambda_1^\omega &\leq \int_I \bigl[(\phi')^2 + (1-\omega) p u^{p-1}\phi^2\bigr] S^{n-1}(r)\,dr - \int_I p u^{p-1}\phi^2 S^{n-1}(r)\,dr \\
&\leq \int_I \bigl[(\phi')^2 + \alpha p u^{p-1}\phi^2\bigr] S^{n-1}(r)\,dr + \int_I p u^{p-1}\phi^2 S^{n-1}(r)\,dr \leq C.
\end{aligned}$$

Let $\phi_1^\omega > 0$ denote the eigenfunction of $\bar{L}_u^\omega$ on I associated with the first eigenvalue and such that $\int_I ((\phi_1^\omega)')^2 S^{n-1}(r)\,dr = 1$.

Then

$$\lambda_1^\omega = \frac{1 - \int_I \omega p u^{p-1}\phi_\omega^2 S^{n-1}(r)\,dr}{\int_I \phi_\omega^2 S^{n-1}(r)\,dr}. \tag{13}$$

As $\omega \to 1$ then the function ϕ_1^ω converges weakly in $H_0^1(I)$ (which injects into $L^2(I)$) and strongly in $L^2(I)$ to $\phi_1 \in H_0^1(I)$. ϕ_1 is not identically zero; otherwise using (13) we could show that $\lim_{\omega\to 1}|\lambda_1^\omega| = +\infty$.

Furthermore there exists a constant $C > 0$ such that

$$\lambda_1^\omega \geq \frac{1 + |\omega| p \|u^{p-1}\|_{L^\infty}\|\phi_\omega^2\|_{L^1}}{\int_I \phi_\omega^2 S^{n-1}(r)\,dr} \geq C.$$

Then λ_1^ω tends to $\bar{\lambda}$ as ω tends to 1 up to a subsequence.

Since $\phi_1^\omega > 0$ converges weakly in $H_0^1(I)$ to ϕ_1, we get $\phi_1 \geq 0$ and it solves

$$\bar{L}_u^1 v := -v'' - (n-1)\frac{S'(r)}{S(r)}v' - p u^{p-1} v = \bar{\lambda} v$$

on I with Dirichlet boundary conditions. We already proved that $\phi_1 \not\equiv 0$, so by the maximum principle we get $\phi_1 > 0$ at the interior of I and hence $\bar{\lambda}$ coincides with the first eigenvalue λ_1^1. □

Proposition 4.2 *If the second eigenvalue λ_2^1 of $\bar{L}_u^1$ is positive, then there exists $\alpha > 0$ such that $\lambda_2^\omega > 0$ for any ω satisfying $|\omega - 1| < \alpha$.*

Proof By proof *ad absurdum* we assume that $\lambda_2^\omega \leq 0$. Since $\lambda_2^\omega > \lambda_1^\omega$ and λ_1^ω is bounded independently of ω, also λ_2^ω must have a limit as ω tends to 1. Let $\tilde{\lambda} \leq 0$ denote the limit.

If ϕ_2^ω is the eigenfunction associated with the eigenvalue λ_2^ω, then ϕ_2^ω converges weakly to a function $\tilde{\phi} \not\equiv 0$ and it solves

$$-v'' - (n-1)\frac{S'(r)}{S(r)}v' - pu^{p-1}v = \tilde{\lambda} v$$

in I with Dirichlet boundary conditions. Consequently $\tilde{\phi}$ is an eigenfunction and $\tilde{\lambda} \leq 0$ is the corresponding eigenvalue. Since by hypothesis $\lambda_2^1 > 0$, $\tilde{\lambda}$ must coincide with the first eigenvalue λ_1^1 of $\bar{L}_u^1$ and $\tilde{\phi}$ must be the first eigenfunction of $\bar{L}_u^1$.

Furthermore $\int_I \phi_1^\omega \phi_2^\omega S^{n-1}(r)\,dr = 0$. By Proposition 4.1 also ϕ_1^ω converges weakly to $\tilde{\phi}$, and from this we conclude $\int_I \tilde{\phi}^2 S^{n-1}(r)\,dr = 0$, which contradicts the fact that $\tilde{\phi}$ is nonvanishing.

This shows that $\lambda_2^\omega > 0$. □

It is well known that the unique positive radial solution to (8) has Morse index equal to 1 and consequently the first two eigenvalues of $\bar{L}_u^1$ satisfy $\lambda_1^1 < 0$, $\lambda_2^1 \geq 0$. Second, the non-degeneracy of the radial solution implies that any eigenvalue of $\bar{L}_u^1$ cannot be equal to zero. In conclusion the hypotheses of the previous propositions are satisfied.

5 Dependence of the eigenvalues on the inner radius *R*

We recall that $A_R = \{x \in M \mid R < r(x) < R+1\}$. We consider the following operators:

$$\begin{aligned}
&\tilde{L}_{u_R}^\omega : H^2(A_R) \cap H_0^1(A_R) \to L^2(A_R),\\
&\tilde{L}_{u_R}^\omega := S^2\big(r(x)\big)\big(-\Delta_{\mathbf{g}} - \omega p u_R^{p-1} I\big);\\
&\hat{L}_{u_R}^\omega : H^2\big((R,R+1)\big) \cap H_0^1\big((R,R+1)\big) \to L^2\big((R,R+1)\big),\\
&\hat{L}_{u_R}^\omega v := S^2(r)\left(-v''(r) - (n-1)\frac{S'(r)}{S(r)}v' - \omega p u_R^{p-1} v\right).
\end{aligned}$$

Let $\hat{\lambda}_m^\omega$ denote the mth eigenvalue of the operator $\hat{L}_{u_R}^\omega$.

In this section we study how $\hat{\lambda}_m^\omega$ varies as $R \to +\infty$ and the exponent p is fixed.

Proposition 5.1 *Let β_m^ω be the eigenvalues for the problem*

$$\begin{cases} -v'' - (n-1)lv' - \omega p \tilde{u}_\infty^{p-1} v = \beta_m^\omega v & \text{on } (0,1),\\ v(0) = v(1) = 0, \end{cases}$$

where $\tilde{u}_\infty$ solves (11)*. Then*

$$\hat{\lambda}_m^\omega(R) = \beta_m^\omega S^2(R) + o\big(S^2(R)\big) \quad \text{as } R \to +\infty.$$

Proof Let us define the operator

$$\bar{L}^{\omega}_{R}: H^2\big((0,1)\big) \cap H^1_0\big((0,1)\big) \to L^2\big((0,1)\big),$$
$$\bar{L}^{\omega}_{R} v := \frac{S^2(t+R)}{S^2(R)} \left(-v'' - (n-1)\frac{S'(t+R)}{S(t+R)} v' - \omega p \tilde{u}_R^{p-1} v \right).$$

If w_m is the mth eigenfunction of $\hat{L}^{\omega}_{u_R}$, then the function $\tilde{w}_{m,R}(t) = w_m(t+R)$ satisfies

$$\bar{L}^{\omega}_{R} \tilde{w}_{m,R} = \frac{\hat{\lambda}^{\omega}_m(R)}{S^2(R)} \tilde{w}_{m,R}, \tag{14}$$

and *vice versa*. Consequently the spectra of $\bar{L}^{\omega}_{R}$ and $\hat{L}^{\omega}_{R}$ are related by

$$\sigma\big(\hat{L}^{\omega}_{u_R}\big) = S^2(R)\sigma\big(\bar{L}^{\omega}_{R}\big).$$

Let $\bar{L}^{\omega}_{\infty}: H^2((0,1)) \cap H^1_0((0,1)) \to L^2((0,1))$ be the operator given by

$$\bar{L}^{\omega}_{\infty} v = -v'' - (n-1)lv' - \omega p \tilde{u}_{\infty}^{p-1} v. \tag{15}$$

Since the coefficients of $\bar{L}^{\omega}_{R}$ converge uniformly on $(0,1)$ to the coefficients of $\bar{L}^{\omega}_{\infty}$, as R tends to $+\infty$,

$$\sigma\big(\bar{L}^{\omega}_{R}\big) = \sigma\big(\bar{L}^{\omega}_{\infty}\big) + o(1).$$

Consequently

$$\sigma\big(\hat{L}^{\omega}_{u_R}\big) = S^2(R)\sigma\big(\bar{L}^{\omega}_{\infty}\big) + o\big(S^2(R)\big). \qquad \square$$

Corollary 5.2 *Let α be the number described by Propositions* 4.1 *and* 4.2 *and suppose that* $|\omega - 1| < \alpha$. *Then the second eigenvalue satisfies* $\hat{\lambda}^{\omega}_2(R) > 0$ *for R large enough.*

Proposition 5.3 *Let ω and α as in Corollary* 5.2. *Then there exists $R_0 > 0$ such that ω can be an eigenvalue of the problem*

$$\begin{cases} -\Delta_{\mathbf{g}} v = \omega p w_R^{p-1} v & \text{in } A_R, \\ v = 0 & \text{on } \partial A_R \end{cases} \tag{16}$$

for $R > R_0$, if and only if, for some $k \geq 1$,

$$\hat{\lambda}^{\omega}_1(R) = -\lambda_k,$$

where $\lambda_k = k(k+n-2)$ is the kth eigenvalue of $-\Delta_{\mathbb{S}^{n-1}}$.

Proof In view of Remark 3.2, ω is an eigenvalue if and only if 0 belongs to the spectrum of $\tilde{L}^{\omega}_{u_R}$. By Proposition 3.4 each eigenvalue of $\tilde{L}^{\omega}_{u_R}$ is the sum of an eigenvalue of $\hat{L}^{\omega}_{u_R}$ and an eigenvalue of $-\Delta_{\mathbb{S}^{n-1}}$. Since the first two eigenvalues $\hat{\lambda}^{\omega}_1(R)$, $\hat{\lambda}^{\omega}_2(R)$ of $\hat{L}^{\omega}_{u_R}$ are, respectively, negative and positive for ω close enough to 1 and $R > R_0$, we have $\hat{\lambda}_m^\omega(R) + \lambda_k = 0$ only for $m = 1$ and $k \geq 1$. □

We set $C(R) := ((\frac{S'(R)}{S(R)})' S^{n-1}(R))'$.

Proposition 5.4 *Suppose $C(R) = o(S'(R)S^{n-2}(R))$. The first eigenvalue $\hat{\lambda}_1^\omega(R)$ of $\hat{L}_{u_R}^\omega u$ is a differentiable function of R and*

$$\frac{\partial \hat{\lambda}_1^\omega(R)}{\partial R} = 2\beta_1^\omega S(R)S'(R) + o\big(S(R)S'(R)\big)$$

as R tends to $+\infty$.

Proof Let $w_{1,R}$ denote the first eigenfunction of $\hat{L}_{u_R}^\omega$ with eigenvalue $\hat{\lambda}_1^\omega(R)$. The function $\tilde{w}_{1,R}(t) = w_{1,R}(t+R)$ is the solution to

$$\begin{cases} -v'' - (n-1)\frac{S'(t+R)}{S(t+R)}v' - \omega p \tilde{u}_R^{p-1} v = \hat{\lambda}_1^\omega(R)\frac{v}{S^2(t+R)} & \text{on } (0,1), \\ v(0) = v(1) = 0, \end{cases} \tag{17}$$

where $\tilde{u}_R(t) = u_R(t+R)$.

Let $\phi_1 \geq 0$ be the function solving

$$\begin{cases} -\phi_1'' - (n-1)l\phi_1' - \omega p \tilde{u}_\infty^{p-1}\phi_1 = \beta_1^\omega \phi_1 & \text{on } (0,1), \\ \phi_1(0) = \phi_1(1) = 0, \end{cases}$$

where $\beta_1^\omega = \lim_{R\to+\infty} \frac{\hat{\lambda}_1^\omega(R)}{S^2(R)} < 0$ is the first eigenvalue. Then $\tilde{w}_{1,R}$ tends uniformly to ϕ_1 as $R \to +\infty$.

$\tilde{w}_{1,R}$ and the eigenvalue $\hat{\lambda}_1^\omega(R)$ are analytic functions of R by the results in [15], p.380.

Then the function $W := \frac{\partial \tilde{w}_{1,R}}{\partial R}$ is the solution of the equation that we get from $\hat{L}_{\tilde{u}_R}^\omega \tilde{w}_{1,R} = \hat{\lambda}_1^\omega(R)\tilde{w}_{1,R}$ by differentiating with respect to R. That is,

$$\begin{aligned} &-W'' - (n-1)\frac{S'(t+R)}{S(t+R)}W' - (n-1)\frac{\partial}{\partial R}\left(\frac{S'(t+R)}{S(t+R)}\right)\tilde{w}_{1,R}' \\ &\quad - \omega p(p-1)\tilde{u}_R^{p-2}\frac{\partial \tilde{u}_R}{\partial R}\tilde{w}_{1,R} - \omega p \tilde{u}_R^{p-1} W \\ &= \frac{\partial \hat{\lambda}_1^\omega(R)}{\partial R}\frac{\tilde{w}_{1,R}}{S^2(t+R)} + \frac{\hat{\lambda}_1^\omega(R)}{S^2(t+R)}W - \frac{2S'(t+R)}{S^3(t+R)}\hat{\lambda}_1^\omega(R)\tilde{w}_{1,R}. \end{aligned}$$

If we multiply this identity by $\tilde{w}_{1,R}$ and integrate on $(0,1)$ with respect to the density $S^{n-1}(t+R)\,dt$ we get

$$\begin{aligned} &\int_0^1 \left[W'\tilde{w}_{1,R}' S^{n-1}(t+R) - (n-1)\tilde{w}_{1,R}'\tilde{w}_{1,R}\frac{\partial}{\partial R}\left(\frac{S'(t+R)}{S(t+R)}\right) S^{n-1}(t+R)\right] dt \\ &\quad - \int_0^1 \left[\omega p(p-1)\tilde{u}_R^{p-2}\frac{\partial \tilde{u}_R}{\partial R}\tilde{w}_{1,R}^2 + \omega p \tilde{u}_R^{p-1}\tilde{w}_{1,R} W\right] S^{n-1}(t+R)\,dt \\ &= \frac{\partial \hat{\lambda}_1^\omega(R)}{\partial R}\int_0^1 \tilde{w}_{1,R}^2 S^{n-3}(t+R)\,dt + \hat{\lambda}_1^\omega(R)\int_0^1 W\tilde{w}_{1,R}S^{n-3}(t+R)\,dt \\ &\quad - 2\hat{\lambda}_1^\omega(R)\int_0^1 \tilde{w}_{1,R}^2 S'(t+R)S^{n-4}(t+R)\,dt. \end{aligned}$$

Multiplying equation (17) (after replacing v by $\tilde{w}_{1,R}$) by W and integrating we get

$$\int_0^1 W'\tilde{w}'_{1,R}S^{n-1}(t+R)\,dt - \int_0^1 \omega p\tilde{u}_R^{p-1}\tilde{w}_{1,R}WS^{n-1}(t+R)\,dt = \hat{\lambda}_1^{\omega}(R)\int_0^1 \tilde{w}_{1,R}WS^{n-3}(t+R)\,dt.$$

If we subtract these two equations we conclude:

$$\begin{aligned} &-(n-1)\int_0^1 \tilde{w}'_{1,R}\tilde{w}_{1,R}\frac{\partial}{\partial R}\left(\frac{S'(t+R)}{S(t+R)}\right)S^{n-1}(t+R)\,dt \\ &\quad - \omega p(p-1)\int_0^1 \tilde{u}_R^{p-2}\frac{\partial \tilde{u}_R}{\partial R}\tilde{w}_{1,R}^2 S^{n-1}(t+R)\,dt \\ &= \frac{\partial \hat{\lambda}_1^{\omega}(R)}{\partial R}\int_0^1 \tilde{w}_{1,R}^2 S^{n-3}(t+R)\,dt - 2\hat{\lambda}_1^{\omega}(R)\int_0^1 \tilde{w}_{1,R}^2 S'(t+R)S^{n-4}(t+R)\,dt. \end{aligned} \tag{18}$$

The first term in (18) can be estimated as follows:

$$\begin{aligned} &\int_0^1 \tilde{w}_{1,R}\tilde{w}'_{1,R}\frac{\partial}{\partial R}\left(\frac{S'(t+R)}{S(t+R)}\right)S^{n-1}(t+R)\,dt \\ &= -\frac{1}{2}\int_0^1 \tilde{w}_{1,R}^2\left(\frac{\partial}{\partial R}\left(\frac{S'(t+R)}{S(t+R)}\right)S^{n-1}(t+R)\right)'\,dt \\ &= o\left(\left(\left(\frac{S'(R)}{S(R)}\right)' S^{n-1}(R)\right)'\right) \\ &= o\left(\left(\left[S(R)S''(R) - \left(S'(R)\right)^2\right]S^{n-3}(R)\right)'\right) = o\left(S'(R)S(R)^{n-2}\right). \end{aligned}$$

Secondly, using Lemma 5.5, we get

$$\int_0^1 \tilde{u}_R^{p-2}\frac{\partial \tilde{u}_R}{\partial R}\tilde{w}_{1,R}^2 S^{n-1}(t+R)\,dt = \int_0^1 \tilde{u}_R^{p-2}S(R)\frac{\partial \tilde{u}_R}{\partial R}\tilde{w}_{1,R}^2\frac{S^{n-1}(t+R)}{S(R)}\,dt = o\left(S^{n-2}(R)\right).$$

After dividing (18) by $S(R)^{n-3}$, we deduce

$$\begin{aligned} &\frac{\partial \hat{\lambda}_1^{\omega}(R)}{\partial R}\int_0^1 \tilde{w}_{1,R}^2\frac{S^{n-3}(t+R)}{S^{n-3}(R)}\,dt \\ &= 2\hat{\lambda}_1^{\omega}(R)\int_0^1 \tilde{w}_{1,R}^2\frac{S'(t+R)S^{n-4}(t+R)}{S^{n-3}(R)}\,dt + o\left(S(R)S'(R)\right). \end{aligned}$$

As $\tilde{w}_{1,R}$ tends to ϕ_1, and $\hat{\lambda}_1^{\omega}(R)$ tends to $\beta_1^{\omega}S^2(R)$, we can conclude

$$\frac{\partial \hat{\lambda}_1^{\omega}(R)}{\partial R}\left(\int_0^1 \phi_1^2\,dt + o(1)\right) = 2\beta_1^{\omega}S(R)S'(R)\left(\int_0^1 \phi_1^2\,dt + o(1)\right) + o\left(S(R)S'(R)\right). \qquad \square$$

Lemma 5.5 *The radial function $\tilde{u}_R = u_R(t+R)$ which solves* (9) *is continuously differentiable with respect to R. Moreover, if $(\frac{S'(R)}{S(R)})' = o(1)$, then*

$$\lim_{R\to+\infty} S^q(R)\int_0^1 \left|\frac{\partial \tilde{u}_R}{\partial R}\right|^q dt = 0, \quad \forall q > 1.$$

Proof The differentiability with respect to R follows from the implicit function theorem applied to the function

$$F(w,R) = w'' + (n-1)\frac{S'(t+R)}{S(t+R)}w' + w^p$$

and the radial non-degeneracy of $\tilde{u}_R$.

The function $V := \frac{\partial \tilde{u}_R}{\partial R}$ is the solution to

$$\begin{cases} V'' + (n-1)\frac{S'(t+R)}{S(t+R)}V' + (n-1)(\frac{S'(t+R)}{S(t+R)})'\tilde{u}'_R + p\tilde{u}_R^{p-1}V = 0 \quad \text{on } (0,1), \\ V(0) = V(1) = 0. \end{cases}$$

We show that $S(R)\|V(\cdot,R)\|_{H_0^1((0,1))} \leq C$. If by contradiction this is not true, then there exists a divergent sequence $\{R_m\}_m$ such that $S(R_m)\|V(\cdot,R_m)\|_{H_0^1((0,1))} \to +\infty$ as $m \to +\infty$.

The function $z_m = \frac{V(\cdot,R_m)}{\|V(\cdot,R_m)\|_{H_0^1}}$ is the solution to

$$\begin{cases} z''_m + (n-1)\frac{S'(t+R)}{S(t+R)}z'_m + (n-1)(\frac{S'(t+R)}{S(t+R)})'\frac{S(R_m)\tilde{u}'_{R_m}}{S(R_m)\|V(\cdot,R_m)\|_{H_0^1}} + p\tilde{u}_{R_m}^{p-1}z_m = 0 \quad \text{on } (0,1), \\ z_m(0) = z_m(1) = 0. \end{cases}$$

We observe that $z_m \to z_0$ weakly in $H_0^1(0,1)$ and strongly in $L^q((0,1))$ for any $q > 1$. Furthermore since $\tilde{u}'_{R_m}$ is bounded as follows from (10), we can consider the limit of the equation above and see that z_0 solves

$$\begin{cases} z''_0 + (n-1)lz'_0 + p\tilde{u}_\infty^{p-1}z_0 = 0 \quad \text{on } (0,1), \\ z_0(0) = z_0(1) = 0. \end{cases} \tag{19}$$

Lemma 5.6 says that $z_0 \equiv 0$, but that contradicts $\|z_0\|_{H_0^1((0,1))} = 1$.

From the claim we now proved it follows that $S(R)V(\cdot,R)$ converges weakly in H_0^1 and strongly in L^q to a function $\bar{V}$ which solves

$$\begin{cases} \bar{V}'' + (n-1)l\bar{V}' + p\tilde{u}_\infty^{p-1}\bar{V} = 0 \quad \text{on } (0,1), \\ \bar{V}(0) = \bar{V}(1) = 0. \end{cases}$$

From that we deduce $\bar{V} \equiv 0$. □

Lemma 5.6 *The unique solution of problem* (19) *is* $z_0 \equiv 0$.

Proof The problem (19) is also the limit as R tends to $+\infty$:

$$\begin{cases} w'' + (n-1)\frac{S'(t+R)}{S(t+R)}w' + p\tilde{u}_R^{p-1}w = 0 \quad \text{on } (0,1), \\ w(0) = w(1) = 0. \end{cases}$$

Since $\tilde{u}_R$ is radially non-degenerate, the problem above and its limit (19) admit only the trivial solution. □

Finally we are able to show that there exist values of the inner radius R for which ω is an eigenvalue of (16).

Proposition 5.7 *If* $|\omega - 1| < \alpha$ *as in Propositions* 4.1 *and* 4.2, *then there exists* $\bar{R} > 0$ *such that* ω *can be an eigenvalue of the problem*

$$\begin{cases} \Delta_{\mathbf{g}} v + \omega p u_R^{p-1} v = 0 & \text{in } A_R, \\ v = 0 & \text{on } \partial A_R, \end{cases} \tag{20}$$

at most for values of R *which belong to a sequence* $\{R_k^\omega\}_k$, *with* $R_k^\omega > \bar{R}$. *Such a sequence satisfies*

$$S(R_k^\omega) = \sqrt{\frac{-k(k+n-2)}{\beta_1^\omega}} + o(1)$$

as $k \to +\infty$.

Proof Proposition 5.4 ensures that there exists $\bar{R}$ such that $\hat{\lambda}_1^\omega(R)$ is strictly decreasing for $R > \bar{R}$. Hence the equation $\hat{\lambda}_1^\omega(R) + \lambda_k = 0$ (see Proposition 5.3) has at most one solution $R = R_k^\omega$ for $k \geq 1$. From Proposition 5.1 we get

$$\hat{\lambda}_1^\omega(R_k^\omega) = (\beta_1^\omega + o(1)) S^2(R_k^\omega) = -k(k+n-2).$$

From this we easily reach our conclusion. □

When $\omega = 1$ we get the values of R for which the operator L_{u_R} (defined in Lemma 3.1) is possibly degenerate.

Corollary 5.8 *There exists* $\bar{R}$ *such that* L_{u_R} *is degenerate for* $R = R_k^1 > \bar{R}$. *Indeed* $\omega = 1$ *is an eigenvalue of* (20) *if and only if* $\hat{\lambda}_1^1(R_k^1)$ *satisfies the condition*

$$\hat{\lambda}_1^1(R_k^1) + \lambda_k = 0.$$

Furthermore the sequence $\{R_k^1\}_k$ *satisfies*

$$S(R_k^1) = \sqrt{\frac{-k(k+n-2)}{\beta_1^1}} + o(1)$$

as $k \to +\infty$ *and*

$$\tau := \lim_{k \to +\infty} \left(S(R_{k+1}^1) - S(R_k^1)\right) = \frac{1}{\sqrt{|\beta_1^1|}}.$$

From the previous proposition we also conclude that for any $R > \bar{R}$ and $R \neq R_k^1$, $k \geq 1$ the operator L_{u_R} is non-degenerate.

The next proposition easily follows from the monotonicity of $\hat{\lambda}_1^1(R)$, Lemma 3.3, and Corollary 5.8.

Proposition 5.9 *The Morse index of the radial solution* u_R *increases when* R *crosses* R_k^1 *and tends to* $+\infty$ *as* $R \to +\infty$.

The following proposition shows that for values of R such that the differences $S(R) - S(R^1_k)$, $S(R^1_{k+1}) - S(R)$ are bounded from below, then the eigenvalue ω is bounded away from 1 by a constant independent of k.

Proposition 5.10 *For $\eta > 0$ there exists $\gamma(\eta) > 0$ and $k(\eta) \in \mathbb{N}$ such that for $k \geq k(\eta)$ and $R \in (R^1_k, R^1_{k+1})$ with* $\min\{S(R) - S(R^1_k), S(R^1_{k+1}) - S(R)\} \geq \eta$ *we have*

$$|\omega_R - 1| \geq \gamma(\eta)$$

for any eigenvalue ω_R of the problem (20).

Proof Suppose by contradiction that there exists a divergent sequence $\{k_m\}_m$, a sequence of radii $R_m \in (R^1_{k_m}, R^1_{k_m+1})$ with $\min\{S(R) - S(R^1_{k_m}), S(R^1_{k_m+1}) - S(R)\} \geq \eta$ and a sequence of eigenvalues $\{\omega_m\}_m$ such that $\lim_{m\to+\infty} \omega_m = 1$.

If m is large enough, then $|\omega_m - 1| \leq \alpha$, where α has the value given by Propositions 4.1, 4.2, and consequently

$$S(R_m) = \sqrt{\frac{h_m(h_m + n - 2)}{-\beta_1^{\omega_m}}} + o(1),$$

where $\{h_m\}_m$ is a divergent sequence of natural numbers.

Since $R_m \in (R^1_{k_m}, R^1_{k_m+1})$, $S(R_m) = S(R^1_{k_m}) + \eta_1$ or $S(R_m) = S(R^1_{k_m+1}) - \eta_1$ with $\eta \leq \eta_1 \leq \frac{S(R^1_{k_m+1})-S(R^1_{k_m})}{2}$.

Suppose that $S(R_m) = S(R^1_{k_m}) + \eta_1$. Since in the other case the proof is the same, it will be omitted. Then, using

$$S(R_{k_m}) = \sqrt{\frac{k_m(k_m + n - 2)}{-\beta_1^{\omega_{k_m}}}} + o(1)$$

and $\beta_1^{\omega_{k_m}} = \beta_1^1 + o(1)$, $\beta_1^{\omega_m} = \beta_1^1 + o(1)$, we get

$$\sqrt{\frac{h_m(h_m + n - 2)}{-\beta_1^1 + o(1)}} = \sqrt{\frac{k_m(k_m + n - 2)}{-\beta_1^1 + o(1)}} + \eta_1.$$

If we square this identity and we use the following Taylor formula centered at k_m:

$$\sqrt{h_m(h_m + n - 2)} = \sqrt{k_m(k_m + n - 2)} + \frac{1}{2}\frac{2k_m + n - 2}{\sqrt{k_m(k_m + n - 2)}}(h_m - k_m) + o\big((h_m - k_m)\big),$$

we get

$$(h_m - k_m)\frac{k_m + n/2 - 1}{\sqrt{k_m(k_m + n - 2)}} + o\big((h_m - k_m)\big) = \eta_1\sqrt{-\beta_1^1 + o(1)}.$$

Since

$$\frac{(k_m + n/2 - 1)}{\sqrt{k_m(k_m + n - 2)}} \to 1$$

as m tends to ∞,

$$0 < h_m - k_m = \eta_1 \sqrt{-\beta_1^1 + o(1)}\big(1 + o(1)\big) < 1$$

for m large enough. That contradicts the fact that h_m and k_m are natural numbers. □

6 Study of the approximate solutions

Lemma 6.1 *Let $\tilde{u}_R$ denote the function defined by* (5) $\tilde{u}_R(\rho,\theta) = w_R(T(\rho,\theta))$. *Then*

$$-\Delta_{\mathbf{g}} \tilde{u}_R = \tilde{u}_R^p + O\left(\frac{1}{S^{2+\delta}(R)}\right).$$

Proof Since $(\rho,\theta) = T^{-1}(r,\theta) = (r + \frac{g(\theta)}{S^\delta(R)}, \theta)$, the function $\tilde{u}_R(\rho,\theta) = w_R(T(\rho,\theta))$ satisfies the identity

$$\begin{aligned}
\Delta_{\mathbf{g}} \tilde{u}_R &= \frac{\partial^2 \tilde{u}_R}{\partial \rho^2} + (n-1)\frac{S'(\rho)}{S(\rho)}\frac{\partial \tilde{u}_R}{\partial \rho} + \frac{1}{S^2(\rho)}\Delta_{\mathbb{S}^{n-1}} \tilde{u}_R \\
&= \frac{\partial^2 \tilde{u}_R}{\partial r^2} + (n-1)\frac{S'(r + g(\theta)S^{-\delta}(R))}{S(r + g(\theta)S^{-\delta}(R))}\frac{\partial \tilde{u}_R}{\partial r} + \frac{1}{S^2(r + g(\theta)S^{-\delta}(R))}\Delta_{\mathbb{S}^{n-1}} \tilde{u}_R \\
&= \frac{\partial^2 w_R}{\partial r^2} + (n-1)\frac{S'(r)}{S(r)}\frac{\partial w_R}{\partial r} + \frac{1}{S^2(r + g(\theta)S^{-\delta}(R))}\Delta_{\mathbb{S}^{n-1}} \tilde{u}_R \\
&\quad + (n-1)\frac{S'(r + g(\theta)S^{-\delta}(R))S(r) - S(r + g(\theta)S^{-\delta}(R))S'(r)}{S(r)S(r + g(\theta)S^{-\delta}(R))}\frac{\partial w_R}{\partial r} \\
&= -w_R^p + O\left(\frac{1}{S^{2+\delta}(R)}\right).
\end{aligned}$$

This identity follows from:

- $S'(r + g(\theta)S^{-\delta}(R))S(r) - S(r + g(\theta)S^{-\delta}(R))S'(r) = O([S''(R)S(R) - (S'(R))^2]S^{-\delta}(R))$, from which we get

$$\begin{aligned}
\frac{S(r + g(\theta)S^{-\delta}(R))S'(r) - S(r)S'(r + g(\theta)S^{-\delta}(R))}{S(r)S(r + g(\theta)S^{-\delta}(R))} &= O\left(\left(\frac{S'(R)}{S(R)}\right)' S^{-\delta}(R)\right) \\
&= o\big(S^{-\delta}(R)\big).
\end{aligned}$$

  Here we used the hypothesis $(\frac{S'(R)}{S(R)})' = o(1)$.
- $|\Delta_{\mathbb{S}^{n-1}} \tilde{u}_R| = O(\frac{1}{S^\delta(R)})$, which is consequence of

$$\frac{\partial \tilde{u}_R}{\partial \theta} = -\frac{\partial w_R}{\partial r}\frac{\partial g}{\partial \theta}\frac{1}{S^\delta(R)}.$$

□

Solutions to (6) correspond to critical points of the C^2-class functional

$$I_R(u) = \frac{1}{2}\int_{\Omega_R} |\nabla u|^2 \, dvol - \frac{1}{p+1}\int_{\Omega_R} |u|^{p+1} \, dvol$$

on $H_0^1(\Omega_R)$. It is well defined for $p > 1$ if $n = 2$ and for $1 < p \leq \frac{n+2}{n-2}$ if $n \geq 3$. For any $u \in H_0^1(\Omega_R)$ we identify $I'_R(u)$ with the linear continuous operator $\operatorname{grad} I_R(u)$ from $H_0^1(\Omega_R)$ to

$H_0^1(\Omega_R)$, defined by

$$\operatorname{grad} I_R(u) := u - (-\Delta_{\mathbf{g}})^{-1}\big(|u|^{p-1}u\big). \tag{21}$$

To this aim we observe that

$$I_R'(u)[v] := \int_{\Omega_R} \big(\nabla u \nabla v - |u|^{p-1}uv\big)\, dvol.$$

If we suppose $v \in H_0^1(\Omega_R)$, then

$$\begin{aligned} I_R'(u)[v] &= \int_{\Omega_R} \big(\nabla u \nabla v - |u|^{p-1}uv\big)\, dvol = -\int_{\Omega_R} v\big(\Delta_{\mathbf{g}} u + |u|^{p-1}u\big)\, dvol \\ &= -\int_{\Omega_R} v\Delta_{\mathbf{g}}\big[u + \Delta_{\mathbf{g}}^{-1}\big(|u|^{p-1}u\big)\big]\, dvol = \int_{\Omega_R} \nabla v \nabla\big[u + \Delta_{\mathbf{g}}^{-1}\big(|u|^{p-1}u\big)\big]\, dvol. \end{aligned}$$

If $\langle w_1, w_2\rangle = \int_{\Omega_R} \nabla w_1 \nabla w_2\, dvol$ is the inner product in $H_0^1(\Omega_R)$, then by the Riesz theorem, we define $\operatorname{grad} I_R(u)$ as the operator such that

$$I_R'(u)[v] = \big\langle \operatorname{grad} I_R(u), v\big\rangle.$$

As a consequence

$$\operatorname{grad} I_R(u) = u + \Delta_{\mathbf{g}}^{-1}\big(|u|^{p-1}u\big) = u - \big(-\Delta_{\mathbf{g}}^{-1}\big)\big(|u|^{p-1}u\big).$$

Lemma 6.2 *If $p > 1$ in the case $n = 2$ and if $1 < p \leq \frac{n+2}{n-2}$ in the case $n \geq 3$, then $\|\operatorname{grad} I_R(u)\|_{H_0^1(\Omega_R)} \leq D_1 S^{-\kappa}(R)$, with $\kappa = \frac{5-n+2\delta}{2} > 0$, δ as in* (3) *and D_1 independent of R.*

Proof If we define $z_R := \operatorname{grad} I_R(\tilde{u}_R)$, then $\Delta_{\mathbf{g}}\tilde{u}_R + \tilde{u}_R^p = \Delta_{\mathbf{g}} z_R$.

From Lemma 6.1 we get

$$\begin{aligned} \int_{\Omega_R} |\nabla z_R|^2\, dvol &= \int_{\Omega_R} \big(-\Delta_{\mathbf{g}}\tilde{u}_R - \tilde{u}_R^p\big) z_R\, dvol \\ &\leq \left(\int_{\Omega_R} \big(\Delta_{\mathbf{g}}\tilde{u}_R + \tilde{u}_R^p\big)^2\, dvol\right)^{\frac{1}{2}} \left(\int_{\Omega_R} z_R^2\, dvol\right)^{\frac{1}{2}} \\ &\leq C\left(\int_{\Omega_R} \frac{1}{S^{4+2\delta}(R)}\, dvol\right)^{\frac{1}{2}} C_0 \left(\int_{\Omega_R} |\nabla z_R|^2\, dvol\right)^{\frac{1}{2}}. \end{aligned}$$

C_0 is the constant (independent of R) of the Poincaré inequality.

Since $\operatorname{meas}(\Omega_R) = O(S^{n-1}(R))$,

$$\|z_R\|_{H_0^1(\Omega_R)} \leq D_1 \frac{1}{S^{2+\delta}(R)} S^{\frac{n-1}{2}}(R) = D_1 S^{-\kappa}(R).$$ □

Lemma 6.3 *Let v be any function in $H_0^1(A_R)$, then $\tilde{v} := v \circ T \in H_0^1(\Omega_R)$ and*

$$\int_{\Omega_R} |\nabla \tilde{v}|^2\, dvol = \int_{A_R} |\nabla v|^2\, dvol + O\left(\frac{S'(R)}{S^{1+\delta}(R)} \int_{A_R} |\nabla v|^2\, dvol\right).$$

Proof We observe that $|\nabla\tilde{v}|^2 = (\frac{\partial\tilde{v}}{\partial\rho})^2 + \frac{1}{S^2(\rho)}\sum_{i=1}^{n-1} a_i^2(\theta)(\frac{\partial\tilde{v}}{\partial\theta_i})^2$, where $\theta = (\theta_1,\dots,\theta_{n-1}) \in \mathbb{S}^{n-1}$.

From the expression of T we easily deduce

$$\frac{\partial\tilde{v}}{\partial\rho} = \frac{\partial v}{\partial r}, \qquad \frac{\partial\tilde{v}}{\partial\theta_i} = \frac{\partial v}{\partial\theta_i} - S^{-\delta}(R)\frac{\partial v}{\partial r}\frac{\partial g}{\partial\theta_i}.$$

Consequently

$$\begin{aligned}
&\int_{\Omega_R} |\nabla\tilde{v}|^2\, dvol \\
&\quad= \int_{A_R}\Biggl[\left(\frac{\partial v}{\partial r}\right)^2 + \frac{1}{S^2(r+g(\theta)S^{-\delta}(R))} \\
&\qquad\times\sum_{i=1}^{n-1} a_i^2(\theta)\left(S^{-2\delta}(R)\left(\frac{\partial v}{\partial r}\frac{\partial g}{\partial\theta_i}\right)^2 + \left(\frac{\partial v}{\partial\theta_i}\right)^2 - 2S^{-\delta}(R)\frac{\partial v}{\partial r}\frac{\partial v}{\partial\theta_i}\frac{\partial g}{\partial\theta_i}\right)\Biggr]\, dvol \\
&\quad= \int_{A_R}\left[\left(\frac{\partial v}{\partial r}\right)^2 + \frac{1}{S^2(R)}\sum_{i=1}^{n-1} a_i^2(\theta)\left(\frac{\partial v}{\partial\theta_i}\right)^2\right] dvol + I_1 + I_2 = \int_{A_R} |\nabla v|^2\, dvol + I_1 + I_2,
\end{aligned}$$

with

$$\begin{aligned}
I_1 &= \int_{A_R}\frac{1}{S^2(R+g(\theta)S^{-\delta}(R))} \\
&\qquad\times\sum_{i=1}^{n-1} a_i^2(\theta)\left[S^{-2\delta}(R)\left(\frac{\partial v}{\partial r}\right)^2\left(\frac{\partial g}{\partial\theta_i}\right)^2 - 2S^{-\delta}(R)\frac{\partial v}{\partial r}\frac{\partial v}{\partial\theta_i}\frac{\partial g}{\partial\theta_i}\right] dvol, \\
I_2 &= \int_{A_R}\left(S^{-2}\left(r+g(\theta)S^{-\delta}(R)\right) - S^{-2}(r)\right)\sum_{i=1}^{n-1} a_i^2(\theta)\left(\frac{\partial v}{\partial\theta_i}\right)^2 dvol, \\
|I_2| &\le \int_{A_R}\frac{|S^2(r) - S^2(r+g(\theta)S^{-\delta}(R))|}{S^2(r+g(\theta)S^{-\delta}(R))S^2(r)}\sum_{i=1}^{n-1} a_i^2(\theta)\left(\frac{\partial v}{\partial\theta_i}\right)^2 dvol \\
&\le CS'(R)S^{-1-\delta}(R)\int_{A_R} S^{-2}(r)\sum_{i=1}^{n-1}\left(\frac{\partial v}{\partial\theta_i}\right)^2 dvol \\
&\le CS'(R)S^{-1-\delta}(R)\int_{A_R} |\nabla v|^2\, dvol, \\
|I_1| &\le \frac{C}{S^{2+2\delta}(R)}\int_{A_R}\left(\frac{\partial v}{\partial r}\right)^2 dvol \\
&\quad+ \int_{A_R}\frac{C}{S^2(r)S^{\delta}(R)}\sum_{i=1}^{n-1}\left[S(r)\left(\frac{\partial v}{\partial r}\right)^2 + \frac{1}{S(r)}\left(\frac{\partial v}{\partial\theta_i}\right)^2\right] dvol \\
&\le \frac{C}{S^{2+2\delta}(R)}\int_{A_R}\left(\frac{\partial v}{\partial r}\right)^2 dvol + \frac{C}{S^{1+\delta}(R)}\int_{A_R}\left(\frac{\partial v}{\partial r}\right)^2 dvol \\
&\quad+ \frac{C}{S^{1+\delta}(R)}\int_{A_R}\sum_{i=1}^{n-1}\frac{1}{S^2(r)}\left(\frac{\partial v}{\partial\theta_i}\right)^2 dvol \\
&\le CS^{-1-\delta}(R)\int_{A_R} |\nabla v|^2\, dvol.
\end{aligned}$$

□

We consider the eigenvalue problems

$$\begin{cases} \Delta_{\mathbf{g}} v + \tilde{\lambda} p \tilde{u}_R^{p-1} v = 0 & \text{in } \Omega_R, \\ v = 0 & \text{on } \partial\Omega_R, \end{cases} \tag{22}$$

$$\begin{cases} \Delta_{\mathbf{g}} v + \lambda p w_R^{p-1} v = 0 & \text{in } A_R, \\ v = 0 & \text{on } \partial A_R. \end{cases} \tag{23}$$

$\tilde{\psi}_{R,1}, \dots, \tilde{\psi}_{R,k}$ denote the unit L^2-eigenfunctions of (22) and $\tilde{\lambda}_{R,1}, \dots, \tilde{\lambda}_{R,k}$ are the corresponding eigenvalues. $\phi_{R,1}, \dots, \phi_{R,k}$ denote the eigenfunctions of (23) and $\lambda_{R,1}, \dots, \lambda_{R,k}$ are the corresponding eigenvalues. Let us consider the functionals

$$\tilde{Q}_R(u) = \frac{\int_{\Omega_R} |\nabla u|^2 \, dvol}{\int_{\Omega_R} p \tilde{u}_R^{p-1} u^2 \, dvol}, \quad u \in H_0^1(\Omega_R), u \not\equiv 0,$$

$$Q_R(v) = \frac{\int_{A_R} |\nabla v|^2 \, dvol}{\int_{A_R} p w_R^{p-1} v^2 \, dvol}, \quad v \in H_0^1(A_R), v \not\equiv 0.$$

Lemma 6.4 *Let $\tilde{V}_{R,k}$ denote the subspace of $H_0^1(\Omega_R)$ spanned by $\tilde{\phi}_{R,1}, \dots, \tilde{\phi}_{R,k}$ with $\tilde{\phi}_{R,i} = \phi_{R,i} \circ T$ with $i = 1, \dots, k$, then*

$$\tilde{Q}_R(\tilde{v}) \le \lambda_{R,k} + O\big(S'(R) S^{-1-\delta}(R)\big) \lambda_{R,k} + O\big(S^{-1}(R)\big) \lambda_{R,k} \quad \text{as } R \to +\infty$$

for any $\tilde{v} \in \tilde{V}_{R,k}$.

Remark 6.5 The reason of our choice for the lower bound for the value of δ (see (3)) is that the term $O(S'(R) S^{-1-\delta}(R))$, which appears in Lemmas 6.4 and 6.6, must tend to 0 as $R \to +\infty$, also when $S'(R)$ is unbounded.

Proof The function $\tilde{v}$ can be expressed as $\tilde{v} = \sum_{i=1}^k \alpha_i \tilde{\phi}_{R,i}$. Consequently

$$\tilde{Q}_R(\tilde{v}) = \frac{\sum_{i,j=1}^k \alpha_i \alpha_j \int_{\Omega_R} \nabla \tilde{\phi}_{R,i} \nabla \tilde{\phi}_{R,j} \, dvol}{\sum_{i,j=1}^k \alpha_i \alpha_j \int_{\Omega_R} p \tilde{u}_R^{p-1} \tilde{\phi}_{R,i} \tilde{\phi}_{R,j} \, dvol},$$

$$\nabla \tilde{\phi}_{R,i} \nabla \tilde{\phi}_{R,j} = \frac{\partial \tilde{\phi}_{R,i}}{\partial \rho} \frac{\partial \tilde{\phi}_{R,j}}{\partial \rho} + \frac{1}{S^2(\rho)} \sum_{l=1}^k a_l^2(\theta) \frac{\partial \tilde{\phi}_{R,i}}{\partial \theta_l} \frac{\partial \tilde{\phi}_{R,j}}{\partial \theta_l},$$

$$\nabla \phi_{R,i} \nabla \phi_{R,j} = \frac{\partial \phi_{R,i}}{\partial r} \frac{\partial \phi_{R,j}}{\partial r} + \frac{1}{S^2(r)} \sum_{l=1}^k a_l^2(\theta) \frac{\partial \phi_{R,i}}{\partial \theta_l} \frac{\partial \phi_{R,j}}{\partial \theta_l}.$$

Now we will express $\nabla \tilde{\phi}_{R,i} \nabla \tilde{\phi}_{R,j}$ in terms of $\nabla \phi_{R,i} \nabla \phi_{R,j}$:

$$\nabla \tilde{\phi}_{R,i} \nabla \tilde{\phi}_{R,j} = \frac{\partial \phi_{R,i}}{\partial r} \frac{\partial \phi_{R,j}}{\partial r} + \frac{1}{S^2(r + g(\theta) S^{-\delta}(R))} \times \sum_{l=1}^k a_l^2(\theta) \left(\frac{\partial \phi_{R,i}}{\partial \theta_l} - S^{-\delta}(R) \frac{\partial \phi_{R,i}}{\partial r} \frac{\partial g}{\partial \theta_l} \right) \left(\frac{\partial \phi_{R,j}}{\partial \theta_l} - S^{-\delta}(R) \frac{\partial \phi_{R,j}}{\partial r} \frac{\partial g}{\partial \theta_l} \right).$$

Observe that

$$\frac{1}{S^2(r+g(\theta)S^{-\delta}(R))} = \frac{1}{S^2(r)(1+O(S'(r)S^{-1}(r)S^{-\delta}(R)))} = \frac{1}{S^2(r)} + O\big(S'(r)S^{-3}(r)S^{-\delta}(R)\big).$$

Then $\nabla\tilde{\phi}_{R,i}\nabla\tilde{\phi}_{R,j}$ equals

$$\begin{aligned}
&\frac{\partial\phi_{R,i}}{\partial r}\frac{\partial\phi_{R,j}}{\partial r} + \left[\frac{1}{S^2(r)} + O\big(S'(R)S^{-3-\delta}(R)\big)\right] \\
&\qquad \times \sum_{l=1}^{k} a_l^2(\theta)\left(\frac{\partial\phi_{R,i}}{\partial\theta_l} - S^{-\delta}(R)\frac{\partial\phi_{R,i}}{\partial r}\frac{\partial g}{\partial\theta_l}\right)\left(\frac{\partial\phi_{R,j}}{\partial\theta_l} - S^{-\delta}(R)\frac{\partial\phi_{R,j}}{\partial r}\frac{\partial g}{\partial\theta_l}\right) \\
&\quad \le \nabla\phi_{R,i}\nabla\phi_{R,j} + O\big(S'(R)S^{-3-\delta}(R)\big)\sum_{l=1}^{k}\frac{\partial\phi_{R,i}}{\partial\theta_l}\frac{\partial\phi_{R,j}}{\partial\theta_l} \\
&\qquad + C\left[\frac{1}{S^2(r)} + O\big(S'(R)S^{-3-\delta}(R)\big)\right] \\
&\qquad \times \sum_{l=1}^{k}\left[S(r)\left(\frac{\partial\phi_{R,i}}{\partial r}\right)^2 + S(r)\left(\frac{\partial\phi_{R,j}}{\partial r}\right)^2 + \frac{1}{S(r)}\left(\frac{\partial\phi_{R,i}}{\partial\theta_l}\right)^2 + \frac{1}{S(r)}\left(\frac{\partial\phi_{R,j}}{\partial\theta_l}\right)^2\right] \\
&\quad = \nabla\phi_{R,i}\nabla\phi_{R,j} + O\big(S'(R)S^{-1-\delta}(R)\big)\nabla\phi_{R,i}\nabla\phi_{R,j} \\
&\qquad + C\left[\frac{1}{S^2(r)} + O\big(S'(R)S^{-3-\delta}(R)\big)\right]\cdot S(r)\big[|\nabla\phi_{R,i}|^2 + |\nabla\phi_{R,j}|^2\big] \\
&\quad = \big(1 + O\big(S'(R)S^{-1-\delta}(R)\big)\big)\nabla\phi_{R,i}\nabla\phi_{R,j} + O\big(S^{-1}(r)\big)\big[|\nabla\phi_{R,i}|^2 + |\nabla\phi_{R,j}|^2\big].
\end{aligned}$$

We used the inequality

$$\frac{\partial\phi_{R,j}}{\partial r}\frac{\partial\phi_{R,i}}{\partial\theta_l} \le S(r)\left(\frac{\partial\phi_{R,j}}{\partial r}\right)^2 + \frac{1}{S(r)}\left(\frac{\partial\phi_{R,i}}{\partial\theta_l}\right)^2.$$

Now we will express $p\int_{\Omega_R}\tilde{u}_R^{p-1}\tilde{\phi}_{R,i}\tilde{\phi}_{R,j}\,dvol$ in terms of $p\int_{A_R}w_R^{p-1}\phi_{R,i}\phi_{R,j}\,dvol$:

$$\begin{aligned}
&p\int_{\Omega_R}\tilde{u}_R^{p-1}\tilde{\phi}_{R,i}\tilde{\phi}_{R,j}\,dvol \\
&\quad = p\int_{\mathbb{S}^{n-1}}\int_{R+g(\theta)S^{-\delta}(R)}^{R+1+g(\theta)S^{-\delta}(R)}\tilde{u}_R^{p-1}\tilde{\phi}_{R,i}\tilde{\phi}_{R,j}S^{n-1}(\rho)\,d\rho\,d\theta \\
&\quad = p\int_{\mathbb{S}^{n-1}}\int_{R}^{R+1}w_R^{p-1}\phi_{R,i}\phi_{R,j}S^{n-1}\big(r+g(\theta)S^{-\delta}(R)\big)\,dr\,d\theta \\
&\quad = p\int_{\mathbb{S}^{n-1}}\int_{R}^{R+1}w_R^{p-1}\phi_{R,i}\phi_{R,j}S^{n-1}(r)\,dr\,d\theta \\
&\qquad + p\int_{\mathbb{S}^{n-1}}\int_{R}^{R+1}w_R^{p-1}\phi_{R,i}\phi_{R,j}\big(S^{n-1}\big(r+g(\theta)S^{-\delta}(R)\big) - S^{n-1}(r)\big)\,dr\,d\theta \\
&\quad \le p\int_{A_R}w_R^{p-1}\phi_{R,i}\phi_{R,j}\,dvol + O\big(S'(R)S^{-1-\delta}(R)\big)\int_{A_R}w_R^{p-1}\phi_{R,i}\phi_{R,j}\,dvol \\
&\quad = p\int_{A_R}w_R^{p-1}\phi_{R,i}\phi_{R,j}\,dvol\big(1 + O\big(S'(R)S^{-1-\delta}(R)\big)\big),
\end{aligned}$$

with $O(S'(R)S^{-1-\delta}(R)) > 0$.

We can write

$$\tilde{Q}_R(\tilde{v}) = \frac{\sum_{i,j=1}^k \alpha_i\alpha_j \int_{A_R}((1+O(S'(R)S^{-1-\delta}(R)))\nabla\phi_{R,i}\nabla\phi_{R,j} + O(S^{-1}(R))[|\nabla\phi_{R,i}|^2 + |\nabla\phi_{R,j}|^2])\,dvol}{\sum_{i,j=1}^k \alpha_i\alpha_j p\int_{A_R} w_R^{p-1}\phi_{R,i}\phi_{R,j}\,dvol(1+O(S'(R)S^{-1-\delta}(R)))}.$$

That can be simplified observing that $\phi_{R,i}$ and $\phi_{R,j}$ satisfy $\int_{A_R} pw_R^{p-1}\phi_{R,i}\phi_{R,j}\,dvol = \delta_{i,j}$. As a consequence

$$\sum_{i,j=1}^k \alpha_i\alpha_j p\int_{A_R} w_R^{p-1}\phi_{R,i}\phi_{R,j}\,dvol = \sum_{i=1}^k \alpha_i^2 p\int_{A_R} w_R^{p-1}\phi_{R,i}^2\,dvol = \sum_{i=1}^k \alpha_i^2.$$

Furthermore, by integrating by parts we can show the following identity:

$$p\int_{A_R} w_R^{p-1}\phi_{R,i}\phi_{R,j}\,dvol = \int_{A_R} \nabla\phi_{R,i}\nabla\phi_{R,j}\,dvol.$$

Consequently the formula for $\tilde{Q}_R(\tilde{v})$ can be written as follows:

$$\begin{aligned}
\tilde{Q}_R(\tilde{v}) &= \frac{\sum_{i,j=1}^k \alpha_i\alpha_j \int_{A_R}((1+O(S'(R)S^{-1-\delta}(R)))\nabla\phi_{R,i}\nabla\phi_{R,j} + O(S^{-1}(R))[|\nabla\phi_{R,i}|^2 + |\nabla\phi_{R,j}|^2])\,dvol}{\sum_{i=1}^k \alpha_i^2 p\int_{A_R} w_R^{p-1}\phi_{R,i}^2\,dvol} \\
&= \frac{\sum_{i=1}^k \alpha_i^2 \int_{A_R}((1+O(S'(R)S^{-1-\delta}(R)))|\nabla\phi_{R,i}|^2)\,dvol}{\sum_{i=1}^k \alpha_i^2} \\
&\quad + \frac{O(S^{-1}(R))[\sum_{i=1}^k \alpha_i^2 \int_{A_R}|\nabla\phi_{R,i}|^2\,dvol + \sum_{j=1}^k \alpha_j^2 \int_{A_R}|\nabla\phi_{R,j}|^2\,dvol]}{\sum_{i=1}^k \alpha_i^2} \\
&\le \lambda_{R,k}\bigl(1+O\bigl(S'(R)S^{-1-\delta}(R)\bigr)\bigr) + \lambda_{R,k}O\bigl(S^{-1}(R)\bigr).
\end{aligned}$$

Here we use the fact that $\lambda_{R,k}$ is the largest among the eigenvalues $\lambda_{R,i}$, $i = 1,\dots,k$. □

In the same way it is possible to show the following result.

Lemma 6.6 *Let $W_{R,k}$ denote the subspace of $H_0^1(A_R)$ spanned by $\psi_{R,1},\dots,\psi_{R,k}$ with $\psi_{R,i} = \tilde{\psi}_{R,i}\circ T^{-1}$ with $i = 1,\dots,k$, then*

$$Q_R(v) \le \tilde{\lambda}_{R,k} + O\bigl(S'(R)S^{-1-\delta}(R)\bigr)\tilde{\lambda}_{R,k} + O\bigl(S^{-1}(R)\bigr)\lambda_{R,k} \quad \text{as } R\to+\infty$$

for any $v \in W_{R,k}$.

The following proposition is the analog of Proposition 5.10 for the eigenvalues of the problem (22).

Proposition 6.7 *For any $\eta > 0$ let $\gamma(\eta) > 0$ and $k(\eta)\in\mathbb{N}$ be as in Proposition* 5.10. *Then there exists $\bar{k}(\eta) \ge k(\eta)$ such that for any $k \ge \bar{k}(\eta)$ and any $R \in [R_k^1 + \eta, R_{k+1}^1 - \eta]$ the following inequality holds*:

$$|\tilde{\omega}_R - 1| \ge \frac{\gamma(\eta)}{2}$$

for any eigenvalue $\tilde{\lambda}_R$ of (22).

The proof is omitted because it is exactly the same as the one of Proposition 5.5 in [12].

7 Proof of Theorem 1.1

7.1 Proof in the subcritical case

The exponent p satisfies $p > 1$ if $n = 2$ or

$$1 < p \leq \frac{n+2}{n-2} = 2^* - 1 \quad \text{if } n \geq 3.$$

We consider the C^2-class functional $I_R(u) := \frac{1}{2}\int_{\Omega_R}(|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1})\,dvol$ in $H_0^1(\Omega_R)$ and whose Frechet derivative $I'_R(u)$ is identified with the element $\operatorname{grad} I_R(u) \in H_0^1(\Omega_R)$ described by (21). Analogously the second derivative $I''_R(u)$ which satisfies

$$I''_R(u)[\phi,\psi] := \int_{\Omega_R} \left(\nabla\phi\nabla\psi - p|u|^{p-1}\phi\psi\right) dvol$$

can be identified with a linear continuous operator $D^2I_R(u)$ from $H_0^1(\Omega_R)$ to $H_0^1(\Omega_R)$.

Indeed, suppose $v \in H_0^1(\Omega_R)$, then

$$\begin{aligned} I''_R(u)[v,v] &= \int_{\Omega_R} \left(|\nabla v|^2 - p|u|^{p-1}v^2\right) dvol = -\int_{\Omega_R} v\left(\Delta_{\mathbf{g}} v + p|u|^{p-1}v\right) dvol \\ &= -\int_{\Omega_R} v\Delta_{\mathbf{g}}\left[v + \Delta_{\mathbf{g}}^{-1}\left(p|u|^{p-1}v\right)\right] dvol = \int_{\Omega_R} \nabla v\nabla\left[v + \Delta_{\mathbf{g}}^{-1}\left(p|u|^{p-1}v\right)\right] dvol. \end{aligned}$$

Here $\langle w_1, w_2\rangle = \int_{\Omega_R} \nabla w_1 \nabla w_2\, dvol$ is the inner product[a] in $H_0^1(\Omega_R)$. By the Riesz theorem, we define $D^2I_R(u)$ as the operator such that

$$I''_R(u)[v,v] = \left\langle D^2I_R(u)[v], v\right\rangle.$$

As a consequence

$$D^2I_R(u)[v] = v + \Delta_{\mathbf{g}}^{-1}\left(p|u|^{p-1}v\right) = v - \left(-\Delta_{\mathbf{g}}^{-1}\right)\left(p|u|^{p-1}v\right).$$

If $\tilde{u}_R := w_R \circ T$ is the function defined by (5), we look for a solution u in Ω_R having the form $u = \tilde{u}_R + \phi_R$, where $\phi_R \in H_0^1(\Omega_R)$ such that $\operatorname{grad} I_R(\tilde{u}_R + \phi_R) = 0$. This implies that the problem can be reformulated as a fixed point problem:

$$\phi_R = F_R(\phi_R),$$

where the operator

$$F_R : H_0^1(\Omega_R) \to H_0^1(\Omega_R)$$

is defined by

$$F_R(\phi) := -\left[D^2I_R(\tilde{u}_R)\right]^{-1}\left[\operatorname{grad} I_R(\tilde{u}_R) + G_R(\phi)\right]. \tag{24}$$

Here

$$G_R(\phi) = \operatorname{grad} I_R(\tilde{u}_R + \phi) - \operatorname{grad} I_R(\tilde{u}_R) - D^2 I_R(\tilde{u}_R)[\phi].$$

Note that in our case $G_R(\phi_R) = -\operatorname{grad} I_R(\tilde{u}_R) - D^2 I_R(\tilde{u}_R)[\phi_R]$.

If R_k^1 are the values of R for which Proposition 5.10 and Corollary 5.8 hold, then if we set $R_k := R_k^1$, the difference $S(R_{k+1}) - S(R_k)$ tends to $\tau = \frac{1}{\sqrt{-\beta_1^1}}$ as k tends to $+\infty$.

Let us choose $\eta > 0$ and $R \in [R_k + \eta, R_{k+1} - \eta]$ with k large enough and which will be determined below. We show that F_R maps the ball

$$B_{\eta,R} := \bigl\{\phi \in H_0^1(\Omega_R) : \|\phi\|_{H_0^1} \leq A(\eta) S(R)^{-\kappa}\bigr\}$$

into itself. $A(\eta) := 2C_1(\eta)\bar{C}$, where $C_1(\eta)$ is the constant which appears in Lemma 7.2 and $\bar{C} = \max\{D_1, D_2, D_3\}$, where D_1, D_2, D_3 are the constants which appear in Lemma 6.2 and Lemma 7.3. We recall that $\kappa = \frac{1}{2}(5 - n + 2\delta) > 0$ with δ as in (3).

If $k \geq \bar{k}(\eta)$ ($\bar{k}(\eta)$ is given by Lemma 7.2), then

$$\begin{aligned} \bigl\|F_R(\phi)\bigr\|_{H_0^1} &\leq C_1(\eta)\bigl[\bigl\|\operatorname{grad} I_R(\tilde{u}_R)\bigr\|_{H_0^1} + \bigl\|G_R(\phi)\bigr\|_{H_0^1}\bigr] \\ &\leq C_1(\eta)\bigl[D_1 S(R)^{-\kappa} + D_2\|\phi\|_{H_0^1}^q\bigr], \end{aligned}$$

where $q := \min\{p, 2\} > 1$. Consequently

$$\bigl\|F_R(\phi)\bigr\|_{H_0^1} \leq C_1(\eta)\bar{C}S(R)^{-\kappa} + C_1(\eta)\bar{C}A^q(\eta)S^{-q\kappa}(R) < A(\eta)S^{-\kappa}(R)$$

for R enough large. It remains to show that F_R is a contracting map. From Lemmas 7.2 and 7.3 we deduce

$$\begin{aligned} \bigl\|F_R(\phi_1) - F_R(\phi_2)\bigr\|_{H_0^1} &\leq C_1(\eta)\bigl[\bigl\|G_R(\phi_1) - G_R(\phi_2)\bigr\|_{H_0^1}\bigr] \\ &\leq 2C_1(\eta)\bar{C}A^d(\eta)S^{-\kappa d}(R)\|\phi_1 - \phi_2\|_{H_0^1} < \frac{1}{2}\|\phi_1 - \phi_2\|_{H_0^1}, \end{aligned}$$

where d is $p - 1$ or 1.

By the fixed point theorem we get there exists a solution $\phi_R \in B_{\eta,R}$ such that $\|\phi_R\|_{H_0^1} \leq A(\eta)S^{-\kappa}(R)$. The function $\tilde{u}_R + \phi_R$ is then the solution to (6). The sign of such a solution is shown to be positive in Lemma 7.1.

7.2 Proof in the supercritical case

p is assumed to be bigger than $\frac{n+2}{n-2}$ and $n \geq 3$.

The operator F_R defined above now is assumed to map the space $H_0^1(\Omega_R) \cap L^\infty(\Omega_R)$ into itself. We choose $\eta > 0$, $R \in [R_k + \eta, R_{k+1} - \eta]$, $\beta \in (0, \kappa)$ and $\kappa \leq 2$. We observe that this last condition is satisfied if $\delta \leq (n-1)/2$.

We will construct an operator which maps the following set into itself:

$$C_{\eta,R} := \bigl\{\phi \in H_0^1(\Omega_R) \cap L^\infty(\Omega_R) : \|\phi\|_{H_0^1(\Omega_R)} \leq A(\eta)S(R)^{-\kappa}, \|\phi\|_{L^\infty(\Omega_R)} \leq S(R)^{-\beta}\bigr\}, \quad (25)$$

where $A(\eta)$ and η are chosen like in the subcritical case.

Suppose $N \in \mathbb{R}$ and positive. We define the function $w_N \in C^2(\mathbb{R})$ as follows:

$$w_N(s) = \begin{cases} |s|^{p+1} & \text{if } |s| \leq N, \\ N+1 & \text{if } |s| \geq N+1. \end{cases}$$

We also introduce the functional $I_{R,N}(u) : H_0^1(\Omega_R) \to H_0^1(\Omega_R)$,

$$I_{R,N}(u) = \int_{\Omega_R} \left[\frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} w_N(u) \right] dvol.$$

We set $N_0 := 2\|\tilde{u}_R\|_{L^\infty(\Omega_R)}$. We will assume $N \geq N_0$. For these values of N the operator $\operatorname{grad} I_{R,N}(\tilde{u}_R)$ coincides with $\operatorname{grad} I_R(\tilde{u}_R)$ and $D^2 I_{R,N}(\tilde{u}_R)$ coincides with $D^2 I_R(\tilde{u}_R)$.

Let $F_{R,N}$ be the operator defined like F_R (see (24)) but using $I_{R,N}$ at the place of I_R. We will show that $F_{R,N}$ is a contraction map on $C_{\eta,R}$ for R large enough.

First, reasoning like in the subcritical case, we can show that if $\|\phi\|_{H_0^1(\Omega_R)} \leq A(\eta) S^{-\kappa}(R)$ then $\|F_{R,N}(\phi)\|_{H_0^1(\Omega_R)} \leq A(\eta) S^{-\kappa}(R)$.

If we set

$$z_R(\phi) := |\tilde{u}_R + \phi|^{p-1}(\tilde{u}_R + \phi) - \tilde{u}_R^p - p\tilde{u}_R^{p-1}\phi,$$

then since $\tilde{u}_R$ is uniformly bounded, we easily conclude

$$\left| z_R(\phi) \right| \leq \begin{cases} C|\phi|^p & \text{if } 1 < p \leq 2, \\ C(|\phi|^2 + |\phi|^p) & \text{if } p > 2. \end{cases} \tag{26}$$

If $\|\phi\|_{L^\infty(\Omega_R)} \leq S^{-\beta}(R)$ then

$$\left\| z_R(\phi) \right\|_{L^\infty(\Omega_R)} \leq C\left(\|\phi\|^2_{L^\infty(\Omega_R)} + \|\phi\|^p_{L^\infty(\Omega_R)} \right) \leq C\left(S^{-2\beta}(R) + S^{-p\beta}(R) \right).$$

By Lemma 7.4 we get

$$\left\| F_{R,N}(\phi) \right\|_{L^\infty(\Omega_R)} \leq C\left(S^{-\kappa}(R) + S^{-2\beta}(R) + S^{-p\beta}(R) + S^{-2}(R) \right) \leq S^{-\beta}(R)$$

for R large enough, because $\beta \in (0, \kappa)$ and $\kappa \leq 2$.

If $R \in [R_k + \eta, R_{k+1} - \eta]$ and k is large enough, then reasoning as before we can show $F_{R,N}$ is contracting:

$$\left\| F_{R,N}(\phi_1) - F_{R,N}(\phi_2) \right\|_{H_0^1} \leq c \|\phi_1 - \phi_2\|_{H_0^1},$$

$$\left\| F_{R,N}(\phi_1) - F_{R,N}(\phi_2) \right\|_{L^\infty} \leq c' \|\phi_1 - \phi_2\|_{L^\infty},$$

with $c, c' < 1$.

By the fixed point theorem we get the existence of a function $\phi_R \in C_{\eta,R}$ (see (25)) such that $u_R = \tilde{u}_R + \phi_R$ satisfies

$$\begin{cases} \Delta_{\mathbf{g}} u_R + |u_R|^{p-1} u_R = 0 & \text{in } \Omega_R, \\ u_R = 0 & \text{on } \partial\Omega_R. \end{cases} \tag{27}$$

It remains to show that $u_R > 0$ in Ω_R. This follows from Lemma 7.1.

Lemma 7.1 *The solution $u_R = \tilde{u}_R + \phi_R$ to the problem* (6) *is positive.*

Proof We know that $\tilde{u}_R > 0$ in Ω_R and $\phi_R \to 0$ in $H_0^1(\Omega_R)$. Suppose that $u_R \leq 0$ in a regular set D_R. If $R \to +\infty$ then $\mathrm{meas}(D_R) \to 0$. We will show that such a set must be empty. If we multiply (27) by u_R^- and we integrate on D_R we get

$$\int_{\Omega_R} \left|\nabla u_R^-\right|^2 dvol = \int_{\Omega_R} |u_R|^{p-1} \left(u_R^-\right)^2 dvol \leq \|u_R\|_\infty^{p-1} \int_{\Omega_R} \left(u_R^-\right)^2 dvol.$$

Using the Poincaré inequality, if $\lambda_1(D_R)$ is the first eigenvalue of $-\Delta_{\mathbf{g}}$ on D_R, we have

$$\lambda_1(D_R) \int_{D_R} w^2 \, dvol \leq \int_{D_R} |\nabla w|^2 \, dvol.$$

From this we deduce

$$\lambda_1(D_R) \int_{\Omega_R} \left(u_R^-\right)^2 dvol \leq \|u_R\|_{L^\infty}^{p-1} \int_{\Omega_R} \left(u_R^-\right)^2 dvol;$$

this says that $\lambda_1(D_R) \leq \|u_R\|_{L^\infty}^{p-1}$, contradicting the fact that the left hand side tends to $+\infty$ as $\mathrm{meas}(D_R) \to 0$. □

The proofs of the following technical lemmas are omitted because they are exactly the same as the ones of Lemmas 6.1, 6.2, 6.3, 6.4 of [12].

The first lemma says that for R within a certain range the norm of the inverse operator $D^2 I_R(\tilde{u}_R)^{-1}$ in the space $\mathcal{L}_R := \{F : H_0^1(\Omega_R) \to H_0^1(\Omega_R) \mid F \text{ linear and continuous}\}$ is bounded.

Lemma 7.2 *If $\eta > 0$ then for any $k \geq \bar{k}(\eta) \in \mathbb{N}$, where $\bar{k}(\eta)$ is the function described by Proposition* 6.7, *and $R \in [R_k + \eta, R_{k+1} - \eta]$ the operator is invertible and*

$$\left\| \left[D^2 I_R(\tilde{u}_R)\right]^{-1} \right\|_{\mathcal{L}_R} \leq C_1(\eta),$$

where $C_1(\eta) > 0$ and independent of k.

Lemma 7.3 *The map $G_R : H_0^1(\Omega_R) \to H_0^1(\Omega_R)$ defined by*

$$G_R(\phi) := \operatorname{grad} I_R(\tilde{u}_R + \phi) - \operatorname{grad} I_R(\tilde{u}_R) - D^2 I_R(\tilde{u}_R)[\phi]$$

satisfies

$$\left\| G_R(\phi) \right\|_{H_0^1} \leq \begin{cases} D_2 \|\phi\|_{H_0^1}^p & \text{if } 1 < p \leq 2, \\ D_2 \|\phi\|_{H_0^1}^2 & \text{if } p > 2, \end{cases} \tag{28}$$

where the constant D_2 does not depend on R, provided $\|\phi\|_{H_0^1} \leq 1$.

Furthermore if $\|\phi_1\|_{H_0^1} \leq 1$, $\|\phi_2\|_{H_0^1} \leq 1$, then

$$\left\| G_R(\phi_1) - G_R(\phi_2) \right\|_{H_0^1} \leq \begin{cases} D_3 (\|\phi_1\|_{H_0^1}^{p-1} - \|\phi_2\|_{H_0^1}^{p-1}) \|\phi_1 - \phi_2\|_{H_0^1}^p & \text{if } 1 < p \leq 2, \\ D_3 (\|\phi_1\|_{H_0^1} - \|\phi_2\|_{H_0^1}) \|\phi_1 - \phi_2\|_{H_0^1} & \text{if } p > 2. \end{cases} \tag{29}$$

Lemma 7.4 *There exists $C > 0$ independent of R, such that for R large enough the following estimate holds*:

$$\|F_R(\phi)\|_{L^\infty(\Omega_R)} \leq C\bigl(\|F_R(\phi)\|_{L^2(\Omega_R)} + \|z_R\|_{L^\infty(\Omega_R)} + S^{-2}(R)\bigr).$$

Competing interests
The author declares that he has no competing interests.

Acknowledgements
This research was supported by Basic Science Research Program through the National Research Foundation of South Korea (NRF) funded by the Ministry of Education, Grant NRF-2013R1A1A1013521.

Endnote

[a] In [12] $\langle I''_R(u), v\rangle$ is not an inner product but it represents the image of v by the operator $I''_R(u)$. In our work $\langle I''_R(u), v\rangle$ is replaced by $D^2 I_R(u)[v]$.

Received: 25 March 2016 Accepted: 24 June 2016 Published online: 07 July 2016

References
1. Ni, WM, Nussbaum, RD: Uniqueness and nonuniqueness for positive radial solutions of $\Delta u + f(u, r) = 0$. Commun. Pure Appl. Math. **38**, 67-108 (1985)
2. Kabeya, Y, Yanagida, E, Yotsutani, S: Canonical forms and structure theorems for radial semi-linear elliptic problems. Commun. Pure Appl. Anal. **1**, 85-102 (2002)
3. Yanagida, E, Yotsutani, S: Pohozaev identity and its applications. RIMS Kokyuroku **834**, 80-90 (1993)
4. Yanagida, E, Yotsutani, S: A unified approach to the structure of radial solutions for semi-linear elliptic problems. Jpn. J. Ind. Appl. Math. **18**, 503-519 (2001)
5. Gladiali, F, Grossi, M, Pacella, F, Srikanth, PN: Bifurcation and symmetry breaking for a class of semilinear elliptic equations in an annulus. Calc. Var. **40**, 295-317 (2011)
6. Mancini, G, Sandeep, K: On a semilinear equation in $\mathbb{H}^n$. Ann. Sc. Norm. Super. Pisa **7**(5), 635-671 (2008)
7. Bonforte, M, Gazzola, F, Grillo, G, Vazquez, JL: Classification of radial solutions to the Emden-Fowler equation on the hyperbolic space. Calc. Var. **46**, 375-401 (2013)
8. Berchio, E, Ferrero, A, Grillo, G: Stability and qualitative properties of radial solutions of the Lane-Emden-Fowler equation on Riemannian models. J. Math. Pures Appl. **102**(1), 1-35 (2014)
9. Morabito, F: Radial and non-radial solutions to an elliptic problem on annular domains in Riemannian manifolds with radial symmetry. J. Differ. Equ. **258**(5), 1461-1493 (2015)
10. Catrina, F, Wang, ZQ: Nonlinear elliptic equations on expanding symmetric domains. J. Differ. Equ. **156**, 153-181 (1999)
11. Dancer, EN, Yan, S: Multibump solutions for an elliptic problem in expanding domains. Commun. Partial Differ. Equ. **27**, 23-55 (2002)
12. Bartsch, T, Clapp, M, Grossi, M, Pacella, F: Asymptotically radial solutions in expanding annular domains. Math. Ann. **352**, 485-515 (2012)
13. Bandle, C, Kabeya, Y: On the positive, 'radial' solutions of a semilinear elliptic equation in $\mathbb{H}^N$. Adv. Nonlinear Anal. **1**(1), 1-25 (2012)
14. Grossi, M, Pacella, F, Yadava, SL: Symmetry results for perturbed problems and related questions. Topol. Methods Nonlinear Anal. **21**, 211-226 (2003)
15. Kato, T: Perturbation Theory for Linear Operators. Springer, Berlin (1987)