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Optimal bounds for two Sándor-type means in terms of power means

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Abstract

In the article, we prove that the double inequalities $M_\alpha(a,b) < S_{QA}(a,b) < M_\beta(a,b)$ and $M_\lambda(a,b) < S_{AQ}(a,b) < M_\mu(a,b)$ hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq \log 2/[1 + \log 2 - \sqrt{2}\log(1 + \sqrt{2})] = 1.5517\ldots$, $\beta \geq 5/3$, $\lambda \leq 4\log 2/[4 + 2\log 2 - \pi] = 1.2351\ldots$ and $\mu \geq 4/3$, where $S_{QA}(a,b) = A(a,b)e^{Q(a,b)/M(a,b)-1}$ and $S_{AQ}(a,b) = Q(a,b)e^{A(a,b)/T(a,b)-1}$ are the Sándor-type means, $A(a,b) = (a+b)/2$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$, $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$, and $M(a,b) = (a-b)/[2\sinh^{-1}((a-b)/(a+b))]$ are, respectively, the arithmetic, quadratic, second Seiffert, and Neuman-Sándor means.

MSC: 26E60

Keywords: Schwab-Borchardt mean; arithmetic mean; quadratic mean; Neuman-Sándor mean; second Seiffert mean; Sándor-type mean; power mean

1 Introduction

For $p \in \mathbb{R}$ and $a,b > 0$ with $a \neq b$, the pth power mean $M_p(a,b)$ and Schwab-Borchardt mean $\mathrm{SB}(a,b)$ [1, 2] of a and b are, respectively, given by

$$M_p(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0 \end{cases} \tag{1.1}$$

and

$$\mathrm{SB}(a,b) = \begin{cases} \frac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)}, & a < b, \\ \frac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & a > b, \end{cases}$$

where $\cos^{-1}(x)$ and $\cosh^{-1}(x) = \log(x + \sqrt{x^2-1})$ are the inverse cosine and inverse hyperbolic cosine functions, respectively.

It is well known that the power mean $M_p(a,b)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$, the Schwab-Borchardt mean $\mathrm{SB}(a,b)$ is strictly increasing in both a and b, nonsymmetric and homogeneous of degree 1 with respect to a and b. Many symmetric bivariate means are special cases of the Schwab-Borchardt mean. For example, $P(a,b) = (a-b)/[2\arcsin((a-b)/(a+b))] =$

$\mathrm{SB}[G(a,b),A(a,b)]$ is the first Seiffert mean, $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))] = \mathrm{SB}[A(a,b),Q(a,b)]$ is the second Seiffert mean, $M(a,b) = (a-b)/[2\sinh^{-1}((a-b)/(a+b))] = \mathrm{SB}[Q(a,b),A(a,b)]$ is the Neuman-Sándor mean, $L(a,b) = (a-b)/[2\tanh^{-1}((a-b)/(a+b))] = \mathrm{SB}[A(a,b),G(a,b)]$ is the logarithmic mean, where $\sinh^{-1}(x) = \log(x+\sqrt{1+x^2})$ is the inverse hyperbolic sine function, $\tanh^{-1}(x) = \log[(1+x)/(1-x)]/2$ is the inverse hyperbolic tangent function, and $G(a,b) = \sqrt{ab}$, $A(a,b) = (a+b)/2$, and $Q(a,b) = \sqrt{(a^2+b^2)/2}$ are the geometric, arithmetic, and quadratic means of a and b, respectively.

The Sándor mean $X(a,b) = A(a,b)e^{G(a,b)/P(a,b)-1}$ [3] can be rewritten as $X(a,b) = A(a,b)e^{G(a,b)/\mathrm{SB}[G(a,b),A(a,b)]-1}$. Yang [4] proved that $S(a,b) = be^{a/\mathrm{SB}(a,b)-1}$ is a mean of a and b, and introduced two Sándor-type means $S_{QA}(a,b)$ and $S_{AQ}(a,b)$ as follows:

$$\begin{aligned} S_{QA}(a,b) &\triangleq S\bigl[Q(a,b),A(a,b)\bigr] \\ &= A(a,b)e^{Q(a,b)/\mathrm{SB}[Q(a,b),A(a,b)]-1} = A(a,b)e^{Q(a,b)/M(a,b)-1}, \qquad (1.2) \\ S_{AQ}(a,b) &\triangleq S\bigl[A(a,b),Q(a,b)\bigr] \\ &= Q(a,b)e^{A(a,b)/\mathrm{SB}[A(a,b),Q(a,b)]-1} = Q(a,b)e^{A(a,b)/T(a,b)-1}. \qquad (1.3) \end{aligned}$$

Recently, the bounds involving the power mean for certain bivariate means and Gaussian hypergeometric function have attracted the attention of many researchers [5–21].

Radó [22] (see also [23–25]) proved that the double inequalities

$$M_p(a,b) < L(a,b) < M_q(a,b), \qquad M_\lambda(a,b) < I(a,b) < M_\mu(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $p\leq 0$, $q\geq 1/3$, $\lambda\leq 2/3$, and $\mu\geq\log 2$, where $I(a,b) = (b^b/a^a)^{1/(b-a)}/e$ is the identric mean of a and b.

In [26–29], the authors proved that the double inequality

$$M_p(a,b) < T^*(a,b) < M_q(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $p\leq 3/2$ and $q\geq\log 2/(\log\pi-\log 2)$, where $T^*(a,b) = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta$ is the Toader mean of a and b.

Jagers [30], Hästö [31, 32], Costin and Toader [33], and Li *et al.* [34] proved that $p_1 = \log 2/\log\pi$, $q_1 = 2/3$, $p_2 = \log 2/(\log\pi-\log 2)$, and $q_2 = 5/3$ are the best possible parameters such that the double inequalities

$$M_{p_1}(a,b) < P(a,b) < M_{q_1}(a,b), \qquad M_{p_2}(a,b) < T(a,b) < M_{q_2}(a,b)$$

hold for all $a,b>0$ with $a\neq b$.

In [35–38], the authors proved that the double inequalities

$$\begin{aligned} &M_{\lambda_1}(a,b) < M(a,b) < M_{\mu_1}(a,b), \\ &M_{\lambda_2}(a,b) < U(a,b) < M_{\mu_2}(a,b), \\ &M_{\lambda_3}(a,b) < X(a,b) < M_{\mu_3}(a,b) \end{aligned}$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\lambda_1\leq\log 2/\log[2\log(1+\sqrt{2})]$, $\mu_1\geq 4/3$, $\lambda_2\leq 2\log 2/(2\log\pi-\log 2)$, $\mu_2\geq 4/3$, $\lambda_3\leq 1/3$, and $\mu_3\geq\log 2/(1+\log 2)$, where $U(a,b) = (a-b)/[\sqrt{2}\arctan(\frac{a-b}{\sqrt{2ab}})]$ is the Yang mean of a and b.

The main purpose of this paper is to present the best possible parameters α, β, λ, and μ such that the double inequalities

$$M_\alpha(a,b) < S_{QA}(a,b) < M_\beta(a,b), \qquad M_\lambda(a,b) < S_{AQ}(a,b) < M_\mu(a,b)$$

hold for all $a, b > 0$ with $a \neq b$.

2 Lemmas

In order to prove our main results we need two lemmas, which we present in this section.

Lemma 2.1 *Let* $p \in \mathbb{R}$ *and*

$$f(x) = (p-1)x^{p+1} - 3x^p + 3x^{p-2} + (1-p)x^{p-3} + 3x^{2p-2} + x^{2p-3} - x - 3. \tag{2.1}$$

Then the following statements are true:

(1) $f(x) > 0$ *for all* $x \in (1,\infty)$ *if* $p = 5/3$;

(2) *there exists* $\sigma \in (1,\infty)$ *such that* $f(x) < 0$ *for* $x \in (1,\sigma)$ *and* $f(x) > 0$ *for* $x \in (\sigma,\infty)$ *if* $p = \log 2/[1 + \log 2 - \sqrt{2}\log(1+\sqrt{2})] = 1.5517\ldots$.

Proof For part (1), if $p = 5/3$, then (2.1) leads to

$$f(x) = \frac{(x^{\frac{2}{3}}-1)(x^{\frac{1}{3}}-1)^2}{3x^{\frac{4}{3}}}\bigl(2x^{\frac{8}{3}} + 4x^{\frac{7}{3}} + 8x^2 + 3x^{\frac{5}{3}} + 9x^{\frac{4}{3}} + 3x + 8x^{\frac{2}{3}} + 4x^{\frac{1}{3}} + 2\bigr). \tag{2.2}$$

Therefore, part (1) follows from (2.2).

For part (2), let $p = \log 2/[1 + \log 2 - \sqrt{2}\log(1+\sqrt{2})] = 1.5517\ldots$, $f_1(x) = f'(x)$, $f_2(x) = x^{5-p}f_1'(x)$ and $f_3(x) = f_2'(x)$. Then simple computations lead to

$$f(1) = 0, \qquad \lim_{x\to+\infty} f(x) = +\infty, \tag{2.3}$$

$$f_1(1) = 12\left(p - \frac{5}{3}\right) < 0, \qquad \lim_{x\to+\infty} f_1(x) = +\infty, \tag{2.4}$$

$$f_2(1) = 24\left(p - \frac{3}{2}\right)\left(p - \frac{5}{3}\right) < 0, \qquad \lim_{x\to+\infty} f_2(x) = +\infty, \tag{2.5}$$

$$\begin{aligned} f_3(x) = {}& 6\bigl(p^2-1\bigr)(2p-3)x^p + 2p(p-2)(2p-3)x^{p-1} \\ &+ 4p\bigl(p^2-1\bigr)x^3 - 9p(p-1)x^2 + 3(p-2)(p-3). \end{aligned} \tag{2.6}$$

Note that

$$2p(p-2)(2p-3)x^{p-1} > 2p(p-2)(2p-3)x^p, \qquad -9p(p-1)x^2 > -9p(p-1)x^3, \tag{2.7}$$

$$p(p-1)(4p-5)x^3 > p(p-1)(4p-5) \tag{2.8}$$

for $x > 1$, and

$$16p^3 - 32p^2 + 18 > 16 \times 1.55^3 - 32 \times 1.552^2 + 18 = 0.503472 > 0, \tag{2.9}$$

$$4p^3 - 6p^2 - 10p + 18 > 4 \times 1.5^3 - 6 \times 1.6^2 - 10 \times 1.6 + 18 = 0.14 > 0. \tag{2.10}$$

It follows from (2.6)-(2.10) that

$$\begin{aligned} f_3(x) &> 6(p^2-1)(2p-3)x^p + 2p(p-2)(2p-3)x^p \\ &\quad + 4p(p^2-1)x^3 - 9p(p-1)x^3 + 3(p-2)(p-3) \\ &= (16p^3-32p^2+18)x^p + p(p-1)(4p-5)x^3 + 3(p-2)(p-3) \\ &> (16p^3-32p^2+18)x^p + p(p-1)(4p-5) + 3(p-2)(p-3) \\ &= (16p^3-32p^2+18)x^p + (4p^3-6p^2-10p+18) > 0 \end{aligned} \tag{2.11}$$

for $x > 1$.

Inequality (2.11) implies that $f_2(x)$ is strictly increasing on $(1,\infty)$. Then from (2.5) we know that there exists $\sigma_1 > 1$ such that $f_1(x)$ is strictly decreasing on $(1,\sigma_1]$ and strictly increasing on $[\sigma_1,\infty)$.

It follows from (2.4) and the piecewise monotonicity of $f_1(x)$ that there exists $\sigma_2 > 1$ such that $f(x)$ is strictly decreasing on $(1,\sigma_2]$ and strictly increasing on $[\sigma_2,\infty)$.

Therefore, part (2) follows from (2.3) and the piecewise monotonicity of $f(x)$. □

Lemma 2.2 *Let* $p \in \mathbb{R}$, *and*

$$g(x) = (p-1)x^{p+1} - (p+1)x^p + (p+1)x^{p-1} + (1-p)x^{p-2} + x^{2p-1} + x^{2p-2} - x - 1. \tag{2.12}$$

Then the following statements are true:

(1) $g(x) > 0$ *for all* $x \in (1,\infty)$ *if* $p = 4/3$;

(2) *there exists* $\tau \in (1,\infty)$ *such that* $g(x) < 0$ *for* $x \in (1,\tau)$ *and* $g(x) > 0$ *for* $x \in (\tau,\infty)$ *if* $p = 4\log 2/[4+2\log 2-\pi] = 1.2351\ldots$.

Proof For part (1), if $p = 4/3$, then (2.12) becomes

$$g(x) = \frac{(x^{1/3}-1)^3}{3x^{2/3}}\left(x^2 + 3x^{5/3} + 9x^{4/3} + 12x + 9x^{2/3} + 3x^{1/3} + 1\right). \tag{2.13}$$

Therefore, part (1) follows from (2.13).

For part (2), let $p = 4\log 2/[4+2\log 2-\pi] = 1.2351\ldots$, $g_1(x) = g'(x)$, $g_2(x) = x^{4-p}g_1'(x)/(p-1)$, and $g_3(x) = g_2'(x)$. Then simple computations lead to

$$g(1) = 0, \qquad \lim_{x\to+\infty} g(x) = +\infty, \tag{2.14}$$

$$g_1(1) = 6\left(p - \frac{4}{3}\right) < 0, \qquad \lim_{x\to+\infty} g_1(x) = +\infty, \tag{2.15}$$

$$g_2(1) = 12\left(p - \frac{4}{3}\right) < 0, \qquad \lim_{x\to+\infty} g_2(x) = +\infty, \tag{2.16}$$

$$\begin{aligned} g_3(x) &= 2(p+1)(2p-1)x^p + 2p(2p-3)x^{p-1} \\ &\quad + 3p(p+1)x^2 - 2p(p+1)x + (p+1)(p-2). \end{aligned} \tag{2.17}$$

Note that

$$2p(2p-3)x^{p-1} > 2p(2p-3)x^p,$$

$$\begin{aligned} &2p(p+1)x < 2p(p+1)x^2, \\ &(p+1)(p-2) > (p+1)(p-2)x^2 \end{aligned} \tag{2.18}$$

for $x > 1$.

It follows from (2.17) and (2.18) that

$$\begin{aligned} g_3(x) &> 2(p+1)(2p-1)x^p + 2p(2p-3)x^p + 3p(p+1)x^2 \\ &\quad - 2p(p+1)x^2 + (p+1)(p-2)x^2 \\ &= 2\left(4p^2 - 2p - 1\right)x^p + 2\left(p^2 - 1\right)x^2 > 0 \end{aligned} \tag{2.19}$$

for $x > 1$.

Inequality (2.19) implies that $g_2(x)$ is strictly increasing on $(1, \infty)$. Then from (2.16) we know that there exists $\tau_1 \in (1, \infty)$ such that $g_1(x)$ is strictly decreasing on $(1, \tau_1]$ and strictly increasing on $[\tau_1, \infty)$.

It follows from (2.15) and the piecewise monotonicity of $g_1(x)$ that there exists $\tau_2 \in (1, \infty)$ such that $g(x)$ is strictly decreasing on $(1, \tau_2]$ and strictly increasing on $[\tau_2, \infty)$.

Therefore, part (2) follows from (2.14) and the piecewise monotonicity of $g(x)$. □

3 Main results

Theorem 3.1 *The double inequality*

$$M_\alpha(a,b) < S_{QA}(a,b) < M_\beta(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq \log 2/[1 + \log 2 - \log(1 + \sqrt{2})] = 1.5517\ldots$ and $\beta \geq 5/3$.

Proof Since both $S_{QA}(a,b)$ and $M_p(a,b)$ are symmetric and homogeneous of degree one, we assume that $a > b$. Let $x = a/b > 1$ and $p > 0$. Then (1.1) and (1.2) lead to

$$\begin{aligned} &\log\left[S_{QA}(a,b)\right] - \log\left[M_p(a,b)\right] \\ &\quad = \log\left(\frac{x+1}{2}\right) + \frac{\sqrt{2(x^2+1)}\sinh^{-1}(\frac{x-1}{x+1})}{x-1} - \frac{1}{p}\log\left(\frac{x^p+1}{2}\right) - 1. \end{aligned} \tag{3.1}$$

Let

$$F(x) = \log\left(\frac{x+1}{2}\right) + \frac{\sqrt{2(x^2+1)}\sinh^{-1}(\frac{x-1}{x+1})}{x-1} - \frac{1}{p}\log\left(\frac{x^p+1}{2}\right) - 1. \tag{3.2}$$

Then elaborated computations lead to

$$F\left(1^+\right) = 0, \tag{3.3}$$

$$\lim_{x \to +\infty} F(x) = \sqrt{2}\log(1+\sqrt{2}) - (1 + \log 2) + \frac{1}{p}\log 2, \tag{3.4}$$

$$F'(x) = \frac{2(x+1)}{(x-1)^2\sqrt{2(x^2+1)}}F_1(x), \tag{3.5}$$

where

$$F_1(x) = \frac{\sqrt{2(x^2+1)}(x-1)(x^{p-1}+1)}{2(x+1)(x^p+1)} - \sinh^{-1}\left(\frac{x-1}{x+1}\right),$$

$$F_1(1) = 0, \qquad \lim_{x\to\infty} F_1(x) = \frac{\sqrt{2}}{2} - \log(1+\sqrt{2}) = -0.1742\ldots < 0, \tag{3.6}$$

$$F_1'(x) = -\frac{x(x-1)}{(x+1)^2(x^p+1)^2\sqrt{2(x^2+1)}} f(x), \tag{3.7}$$

where $f(x)$ is defined by (2.1).

We divide the proof into four cases.

Case 1.1. $p = \log 2/[1+\log 2 - \log(1+\sqrt{2})]$. Then it follows from Lemma 2.1(2) and (3.7) that there exists $\sigma \in (1,\infty)$ such that $F_1(x)$ is strictly increasing on $(1,\sigma]$ and strictly decreasing on $[\sigma,\infty)$.

Equations (3.5) and (3.6) together with the piecewise monotonicity of $F_1(x)$ lead to the conclusion that there exists $\sigma_0 \in (1,\infty)$ such that $F(x)$ is strictly increasing on $(1,\sigma_0]$ and strictly decreasing on $[\sigma_0,\infty)$.

Note that (3.4) becomes

$$\lim_{x\to+\infty} F(x) = 0. \tag{3.8}$$

Therefore,

$$S_{QA}(a,b) > M_{\log 2/[1+\log 2-\log(1+\sqrt{2})]}(a,b)$$

for all $a,b>0$ with $a \neq b$ follows from (3.1)-(3.3) and (3.8) together with the piecewise monotonicity of $F(x)$.

Case 1.2. $p = 5/3$. Then it follows from Lemma 2.1(1) and (3.7) that $F_1(x)$ is strictly decreasing on $(1,\infty)$.

Therefore,

$$S_{QA}(a,b) < M_{5/3}(a,b)$$

for all $a,b>0$ with $a \neq b$ follows from (3.1)-(3.3), (3.5), (3.6), and the monotonicity of $F(x)$.

Case 1.3. $p > \log 2/[1+\log 2 - \log(1+\sqrt{2})]$. Then (3.4) leads to

$$\lim_{x\to+\infty} F(x) < 0. \tag{3.9}$$

Equations (3.1) and (3.2) together with inequality (3.9) imply that there exists large enough $C_0 > 1$ such that

$$S_{QA}(a,b) < M_p(a,b)$$

for all $a,b>0$ with $a/b \in (C_0,\infty)$.

Case 1.4. $1 < p < 5/3$. Let $x > 0$, $x \to 0$, then making use of (1.1) and (1.2) together with the Taylor expansion we get

$$\begin{aligned} &S_{QA}(1, 1+x) - M_p(1, 1+x) \\ &= \left(1 + \frac{x}{2}\right) e^{\sqrt{2(x^2+2x+2)} \sinh^{-1}[x/(2+x)]/x - 1} - \left[\frac{1 + (1+x)^p}{2}\right]^{1/p} \\ &= \frac{5 - 3p}{24} x^2 + o\left(x^2\right). \end{aligned} \tag{3.10}$$

Equation (3.10) implies that there exists small enough $\delta_0 > 0$ such that

$$S_{QA}(1, 1+x) > M_p(1, 1+x)$$

for $x \in (0, \delta_0)$.

Therefore, Theorem 3.1 follows easily from Cases 1.1-1.4 and the monotonicity of the function $p \to M_p(a, b)$. □

Theorem 3.2 *The double inequality*

$$M_\lambda(a, b) < S_{AQ}(a, b) < M_\mu(a, b)$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\lambda \leq 4 \log 2/[4 + 2 \log 2 - \pi] = 1.2351\ldots$ *and* $\beta \geq 4/3$.

Proof Since both $S_{AQ}(a, b)$ and $M_p(a, b)$ are symmetric and homogeneous of degree one, we assume that $a > b$. Let $x = a/b > 1$ and $p > 0$. Then (1.1) and (1.3) lead to

$$\begin{aligned} &\log\left[S_{AQ}(a, b)\right] - \log\left[M_p(a, b)\right] \\ &= \frac{1}{2} \log\left(\frac{x^2 + 1}{2}\right) + \frac{x + 1}{x - 1} \arctan\left(\frac{x - 1}{x + 1}\right) - \frac{1}{p} \log\left(\frac{x^p + 1}{2}\right) - 1. \end{aligned} \tag{3.11}$$

Let

$$G(x) = \frac{1}{2} \log\left(\frac{x^2 + 1}{2}\right) + \frac{x + 1}{x - 1} \arctan\left(\frac{x - 1}{x + 1}\right) - \frac{1}{p} \log\left(\frac{x^p + 1}{2}\right) - 1. \tag{3.12}$$

Then elaborated computations lead to

$$G\left(1^+\right) = 0, \tag{3.13}$$

$$\lim_{x \to +\infty} G(x) = \frac{\pi}{4} - \frac{1}{2} \log 2 - 1 + \frac{1}{p} \log 2, \tag{3.14}$$

$$G'(x) = \frac{2}{(x - 1)^2} G_1(x), \tag{3.15}$$

where

$$G_1(x) = \frac{(x - 1)(x^{p-1} + 1)}{2(x^p + 1)} - \arctan\left(\frac{x - 1}{x + 1}\right),$$

$$G_1(1) = 0, \qquad \lim_{x\to+\infty} G_1(x) = \frac{1}{2} - \frac{\pi}{4} < 0, \tag{3.16}$$

$$G_1'(x) = -\frac{x-1}{2(x^2+1)^2(x^p+1)^2} g(x), \tag{3.17}$$

where $g(x)$ is defined by (2.12).

We divide the proof into four cases.

Case 2.1. $p = 4\log 2/[4 + 2\log 2 - \pi]$. Then it follows from Lemma 2.2(2) and (3.17) that there exists $\tau \in (1, \infty)$ such that $G_1(x)$ is strictly increasing on $(1, \tau]$ and strictly decreasing on $[\tau, \infty)$.

Equations (3.15) and (3.16) together with the piecewise monotonicity of $G_1(x)$ lead to the conclusion that there exists $\tau_0 \in (1, \infty)$ such that $G(x)$ is strictly increasing on $(1, \tau_0]$ and strictly decreasing on $[\tau_0, \infty)$.

Note that (3.14) becomes

$$\lim_{x\to+\infty} G(x) = 0. \tag{3.18}$$

Therefore,

$$S_{AQ}(a,b) > M_{4\log 2/[4+2\log 2-\pi]}(a,b)$$

follows from (3.11)-(3.13) and (3.18) together with the piecewise monotonicity of $G(x)$.

Case 2.2. $p = 4/3$. Then Lemma 2.2(2) and (3.17) imply that $G_1(x)$ is strictly decreasing on $(1, \infty)$.

Therefore,

$$S_{AQ}(a,b) < M_{4/3}(a,b)$$

follows easily from (3.11)-(3.13), (3.15), (3.16), and the monotonicity of $G_1(x)$.

Case 2.3. $p > 4\log 2/[4 + 2\log 2 - \pi]$. Then (3.14) leads to

$$\lim_{x\to+\infty} G(x) < 0. \tag{3.19}$$

Equations (3.11) and (3.12) and inequality (3.19) imply that there exists large enough $C_1 > 1$ such that

$$S_{AQ}(a,b) < M_p(a,b)$$

for all $a, b > 0$ with $a/b \in (C_1, \infty)$.

Case 2.4. $0 < p < 4/3$. Let $x > 0$ and $x \to 0$. Then making use of (1.1) and (1.3) together with the Taylor expansion we get

$$\begin{aligned} &S_{AQ}(1,1+x) - M_p(1,1+x) \\ &\quad = \sqrt{\frac{1+(1+x)^2}{2}} e^{(2+x)\arctan[x/(2+x)]/x-1} - \left[\frac{1+(1+x)^p}{2}\right]^{1/p} \\ &\quad = \frac{4-3p}{24} x^2 + o(x^2). \end{aligned} \tag{3.20}$$

Equation (3.20) implies that there exists small enough $\delta_1 > 0$ such that

$$S_{AQ}(1, 1 + x) > M_p(1, 1 + x)$$

for $x \in (0, \delta_1)$.

Therefore, Theorem 3.2 follows easily from Cases 2.1-2.4 and the monotonicity of the function $p \to M_p(a, b)$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

1. Neuman, E, Sándor, J: On the Schwab-Borchardt mean. Math. Pannon. **14**(2), 253-266 (2003)
2. Neuman, E, Sándor, J: On the Schwab-Borchardt mean II. Math. Pannon. **17**(1), 49-59 (2006)
3. Sándor, J: Two sharp inequalities for trigonometric and hyperbolic functions. Math. Inequal. Appl. **15**(2), 409-413 (2012)
4. Yang, Z-H: Three families of two-parameter means constructed by trigonometric functions. J. Inequal. Appl. **2013**, Article ID 541 (2013)
5. Zhang, X-H, Wang, G-D, Chu, Y-M: Convexity with respect to Hölder mean involving zero-balanced hypergeometric functions. J. Math. Anal. Appl. **353**(1), 256-259 (2009)
6. Chu, Y-M, Xia, W-F: Two sharp inequalities for power mean, geometric mean, and harmonic mean. J. Inequal. Appl. **2009**, Article ID 741923 (2009)
7. Chu, Y-M, Qiu, Y-F, Wang, M-K: Sharp power mean bounds for the combination of Seiffert and geometric means. Abstr. Appl. Anal. **2010**, Article ID 108920 (2010)
8. Chu, Y-M, Xia, W-F: Two optimal double inequalities between power mean and logarithmic mean. Comput. Math. Appl. **60**(1), 83-89 (2010)
9. Wang, M-K, Qiu, Y-F, Chu, Y-M, Qiu, S-L: An optimal power mean inequality for the complete elliptic integrals. Appl. Math. Lett. **24**(6), 887-890 (2011)
10. Li, Y-M, Long, B-Y, Chu, Y-M: Sharp bounds by the power mean for the generalized Heronian mean. J. Inequal. Appl. **2012**, Article ID 129 (2012)
11. Čižmešija, A: The optimal power mean bounds for two convex combinations of A-G-H means. J. Math. Inequal. **6**(1), 33-41 (2012)
12. Chu, Y-M, Shi, M-Y, Jiang, Y-P: Optimal inequalities for the power, harmonic and logarithmic means. Bull. Iran. Math. Soc. **38**(3), 597-606 (2012)
13. Li, Y-M, Long, B-Y, Chu, Y-M: A best possible double inequality for power mean. J. Appl. Math. **2012**, Article ID 379785 (2012)
14. Čižmešija, A: A new sharp double inequality for generalized Heronian, harmonic and power means. Comput. Math. Appl. **64**(4), 664-671 (2012)
15. Wang, M-K, Chu, Y-M, Qiu, S-L, Jiang, Y-P: Convexity of the complete elliptic integrals of the first kind with respect to Hölder means. J. Math. Anal. Appl. **388**(2), 1141-1146 (2012)
16. Qiu, S-L, Qiu, Y-F, Wang, M-K, Chu, Y-M: Hölder mean inequalities for the generalized Grötzsch ring and Hersch-Pfluger distortion functions. Math. Inequal. Appl. **15**(1), 237-245 (2012)
17. Chu, Y-M, Wang, M-K, Jiang, Y-P, Qiu, S-L: Concavity of the complete elliptic integrals of the second kind with respect to Hölder means. J. Math. Anal. Appl. **395**(2), 637-642 (2012)
18. Chu, Y-M, Qiu, Y-F, Wang, M-K: Hölder mean inequalities for the complete elliptic integrals. Integral Transforms Spec. Funct. **23**(7), 521-527 (2012)
19. Chu, Y-M, Qiu, S-L, Wang, M-K: Sharp inequalities involving the power mean and complete elliptic integral of the first kind. Rocky Mt. J. Math. **43**(5), 1489-1496 (2013)
20. Wang, G-D, Zhang, X-H, Chu, Y-M: A power mean inequality involving the complete elliptic integrals. Rocky Mt. J. Math. **44**(5), 1661-1667 (2014)
21. Chu, Y-M, Wu, L-M, Song, Y-Q: Sharp power mean bounds for the one-parameter harmonic mean. J. Funct. Spaces **2015**, Article ID 517647 (2015)

22. Radó, T: On convex functions. Trans. Am. Math. Soc. **37**(2), 266-285 (1935)
23. Lin, TP: The power mean and the logarithmic mean. Am. Math. Mon. **81**, 879-883 (1974)
24. Stolarsky, KB: The power and generalized logarithmic means. Am. Math. Mon. **87**(7), 545-548 (1980)
25. Pittenger, AO: Inequalities between arithmetic and logarithmic means. Publ. Elektroteh. Fak. Univ. Beogr., Ser. Mat. Fiz. **678-715**, 15-18 (1980)
26. Qiu, S-L, Shen, J-M: On two problems concerning means. J. Hangzhou Inst. Electron. Eng. **17**(3), 1-7 (1997) (in Chinese)
27. Qiu, S-L: The Muir mean and the complete elliptic integral of the second kind. J. Hangzhou Inst. Electron. Eng. **20**(1), 28-33 (2000) (in Chinese)
28. Barnard, RW, Pearce, K, Richards, KC: An inequality involving the generalized hypergeometric function and the arc length of an ellipse. SIAM J. Math. Anal. **31**(3), 693-699 (2000)
29. Alzer, H, Qiu, S-L: Monotonicity theorems and inequalities for the complete elliptic integrals. J. Comput. Appl. Math. **172**(2), 289-312 (2004)
30. Jagers, AA: Solution of problem 887. Nieuw Arch. Wiskd. (4) **12**(2), 230-231 (1994)
31. Hästö, PA: A monotonicity property of ratios of symmetric homogeneous means. JIPAM. J. Inequal. Pure Appl. Math. **3**(5), Article 71 (2002)
32. Hästö, PA: Optimal inequalities between Seiffert's mean and power means. Math. Inequal. Appl. **7**(1), 47-53 (2004)
33. Costin, I, Toader, G: Optimal evaluations of some Seiffert-type means by power means. Appl. Math. Comput. **219**(9), 4745-4754 (2013)
34. Li, Y-M, Wang, M-K, Chu, Y-M: Sharp power mean bounds for Seiffert mean. Appl. Math. J. Chin. Univ. Ser. B **29**(1), 101-107 (2014)
35. Yang, Z-H: Estimates for Neuman-Sándor mean by power means and their relative errors. J. Math. Inequal. **7**(4), 711-726 (2013)
36. Chu, Y-M, Long, B-Y: Bounds of the Neuman-Sándor mean using power and identric means. Abstr. Appl. Anal. **2013**, Article ID 832591 (2013)
37. Yang, Z-H, Wu, L-M, Chu, Y-M: Optimal power mean bounds for Yang mean. J. Inequal. Appl. **2014**, Article ID 401 (2014)
38. Chu, Y-M, Yang, Z-H, Wu, L-M: Sharp power mean bounds for Sándor mean. Abstr. Appl. Anal. **2015**, Article ID 172867 (2015)