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The oriented bicyclic graphs whose skew-spectral radii do not exceed 2

Guang-Hui Xu* and Shi-Cai Gong

*Correspondence:
ghxu@zafu.edu.cn
School of Science, Zhejiang A & F University, Hangzhou, 311300, China

Abstract

Let $S(G^\sigma)$ be the skew-adjacency matrix of the oriented graph G^σ on order n and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be all eigenvalues of $S(G^\sigma)$. The skew-spectral radius $\rho_s(G^\sigma)$ of G^σ is defined as $\max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}$. In this paper, we determine all the oriented bicyclic graphs whose skew-spectral radii do not exceed 2.

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Keywords: oriented graph; skew-adjacency matrix; skew-spectral radius

1 Introduction

Let G be a simple graph with n vertices. The *adjacency matrix* $A = A(G)$ is the symmetric matrix $[a_{ij}]_{n\times n}$, where $a_{ij} = a_{ji} = 1$ if v_iv_j is an edge of G, otherwise $a_{ij} = a_{ji} = 0$. We call $\det(\lambda I - A)$ the *characteristic polynomial* of G, denoted by $\phi(G;\lambda)$. Since A is symmetric, its eigenvalues $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ are real, and we assume that $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. We call $\rho(G) = \lambda_1(G)$ the *adjacency spectral radius* of G.

The class of all graphs G whose largest (adjacency) eigenvalue $\lambda_{\max}(G)$ is bounded by 2 has been completely determined by Smith; see, for example, [1, 2]. Later, Hoffman [3], Cvetković *et al.* [4] gave a nearly complete description of all graphs G with $2 < \lambda_{\max}(G) < \sqrt{2+\sqrt{5}}$ (≈ 2.0582). Their description was completed by Brouwer and Neumaier [5]. And Belardo *et al.* [6] ordered graphs with spectral radius in the interval $(2, \sqrt{2+\sqrt{5}})$. Then Woo and Neumaier [7] investigated the structure of graphs G with $\sqrt{2+\sqrt{5}} < \lambda_{\max}(G) < \frac{3}{2}\sqrt{2}$ (≈ 2.1312), Wang *et al.* [8] investigated the structure of graphs whose largest eigenvalue is close to $\frac{3}{2}\sqrt{2}$. In the paper [9], the first three bicyclic graphs on order n in terms of their larger spectral radii were determined.

The graph obtained from a simple undirected graph by assigning an orientation to each of its edges is referred to as the *oriented graph*. Let G^σ be an oriented graph with a vertex set $\{v_1, v_2, \ldots, v_n\}$ and an edge set $E(G^\sigma)$. The *skew-adjacency matrix* $S = S(G^\sigma) = [s_{ij}]_{n\times n}$ related to G^σ is defined as:

$$s_{ij} = \begin{cases} \mathbb{i} & \text{if there exists an edge with tail } v_i \text{ and head } v_j; \\ -\mathbb{i} & \text{if there exists an edge with head } v_i \text{ and tail } v_j; \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathbb{i} = \sqrt{-1}$ (note that the definition is slightly different from the one of the normal skew-adjacency matrix given by Adiga *et al.* [10]). Since $S(G^\sigma)$ is a Hermitian matrix, the eigenvalues $\lambda_1(G^\sigma), \lambda_2(G^\sigma), \ldots, \lambda_n(G^\sigma)$ of $S(G^\sigma)$ are all real numbers and thus can be arranged non-increase as

$$\lambda_1(G^\sigma) \geq \lambda_2(G^\sigma) \geq \cdots \geq \lambda_n(G^\sigma).$$

The *skew-spectral radius* and the *skew-characteristic polynomial* of G^σ are defined respectively as

$$\rho_s(G^\sigma) = \max\{|\lambda_1(G^\sigma)|, |\lambda_2(G^\sigma)|, \ldots, |\lambda_n(G^\sigma)|\}$$

and

$$\phi(G^\sigma; \lambda) = \det(\lambda I_n - S(G^\sigma)).$$

We denote by $\mathfrak{D}(G)$ the set of all the oriented graphs obtained from G by giving an arbitrary orientation to each edge. Recently, much attention has been devoted to the skew-adjacency matrix of an oriented graph. In 2009, Shader and So [11] investigated the spectra of the skew-adjacency matrix of an oriented graph. And in 2010, Adiga *et al.* [10] discussed the properties of the skew-energy of an oriented graph. In the papers [12, 13], all the coefficients of the skew-characteristic polynomial of G^σ in terms of G were interpreted. Cavers *et al.* [14] discussed the graphs whose skew-adjacency matrices are all cospectral and the relations between the matchings polynomial of a graph and the characteristic polynomials of its adjacency and skew-adjacency matrices. In [15], the author established a relation between $\rho_s(G^\sigma)$ and $\rho(G)$. Also, the author gave some results on the skew-spectral radii of G^σ and its oriented subgraphs. In the paper [16], the oriented graphs whose skew-spectral radii do not exceed 2 were investigated. In 2013, Chen *et al.* [17] ordered all the oriented unicyclic graphs with n vertices whose skew-spectral radii are bounded by 2.

A connected graph in which the number of edges equals the number of vertices plus one is called a bicyclic graph. In this paper, we will determine all the oriented bicyclic graphs whose skew-spectral radii do not exceed 2. The rest of the paper is organized as follows. In Section 2, we introduce some notations and preliminary results. In Section 3, we give some earlier results on the oriented graphs whose skew-spectral radii do not exceed 2. In Section 4, we determine all the oriented bicyclic graphs whose skew-spectral radii do not exceed 2.

2 Preliminaries

Let $G = (V, E)$ be a simple graph with a vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and $e \in E(G)$. A graph H is called a *subgraph* of a graph G if $V(H) \in V(G)$ and $E(H) \in E(G)$. Further, H is called an *induced subgraph* of G if two vertices of $V(H)$ are adjacent in H if and only if they are adjacent in G. Denote by $G - U$, where $U \in V(G)$, the graph obtained from G by removing the vertices of U together with all edges incident to them. Denote by $G - e$ the subgraph obtained from G by deleting the edge e. We refer to [1, 18, 19] for more terminology and notation not defined here. Certainly, each subgraph of an oriented graph is referred to as an oriented graph and preserves the orientation of each edge.

Recall that the skew-adjacency matrix $S(G^\sigma)$ of any oriented graph G^σ is Hermitian, then the well-known interlacing theorem for Hermitian matrices applies equally well to oriented graphs; see, for example, Theorem 4.3.8 of [19].

Lemma 2.1 *Let G^σ be an arbitrary oriented graph on n vertices and $V' \subseteq V(G)$. Suppose that $|V'| = k$. Then*

$$\lambda_i(G^\sigma) \geq \lambda_i(G^\sigma - V') \geq \lambda_{i+k}(G^\sigma) \quad \text{for } i = 1, 2, \ldots, n-k.$$

Let $G^\sigma \in \mathfrak{D}(G)$ and $W = u_1u_2 \cdots u_ku_{k+1}$ be a walk of G. The sign of W^σ in G^σ, denoted by $\mathrm{sgn}(W^\sigma)$, is defined by

$$s_{1,2}s_{2,3} \cdots s_{k-1,k}s_{k,k+1}.$$

Let $\bar{W} = u_{k+1}u_k \cdots u_2u_1$ be the walk by inverting the order of the vertices along the walk W. Then one can find that

$$\mathrm{sgn}(\bar{W}^\sigma) = \begin{cases} -\mathrm{sgn}(W^\sigma) & \text{if } k \text{ is odd;} \\ \mathrm{sgn}(W^\sigma) & \text{if } k \text{ is even.} \end{cases}$$

Obviously, for an even closed walk (that is to say $u_{k+1} = u_1$), we can simply refer to it as a positive (or negative) even closed walk according to its sign, regardless of the order of its vertices. Similarly, we can define a positive (or negative) even cycle.

We now list some results related to this paper.

Lemma 2.2 ([15], Theorem 2.1) *Let G^σ be an arbitrary connected oriented graph. Denote by $\rho(G)$ the (adjacency) spectral radius of G. Then*

$$\rho_s(G^\sigma) \leq \rho(G)$$

with equality if and only if G is bipartite and each cycle of G^σ is positive.

Lemma 2.3 ([13], Theorem 2.4, [15], Theorem 3.1) *Let G^σ be an oriented graph and $\phi(G^\sigma, \lambda)$ be its skew-characteristic polynomial. Then*

(a) $$\phi(G^\sigma, \lambda) = \lambda\phi(G^\sigma - u, \lambda) - \sum_{v \in N(u)} \phi(G^\sigma - u - v, \lambda) - 2\sum_{u \in C} \mathrm{sgn}(C^\sigma)\phi(G^\sigma - C, \lambda),$$

where the first summation is over all the vertices in $N(u)$ and the second summation is over all even cycles of G containing the vertex u.

(b) $$\phi(G^\sigma, \lambda) = \phi(G^\sigma - e, \lambda) - \phi(G^\sigma - u - v, \lambda) - 2\sum_{(u,v) \in C} \mathrm{sgn}(C^\sigma)\phi(G^\sigma - C, \lambda),$$

where $e = uv$ and the summation is over all even cycles of G containing the edge e, and $\mathrm{sgn}(C^\sigma)$ *denotes the sign of the even cycle C^σ.*

Lemma 2.4 ([12], A part of Theorem 2.5) *Let G^σ be an oriented graph and $\phi(G^\sigma, \lambda)$ be its skew-characteristic polynomial. Then*

$$\frac{d}{d\lambda}\phi(G^\sigma, \lambda) = \sum_{v \in V(G)} \phi(G^\sigma - v, \lambda),$$

where $\frac{d}{d\lambda}\phi(G^\sigma, \lambda)$ denotes the derivative of $\phi(G^\sigma, \lambda)$.

3 Some earlier results on the oriented graphs whose skew-spectral radii do not exceed 2

Before proving the main theorems, we introduce some earlier results. By Lemmas 2.2 and 2.3 or the papers [10, 11], for a given graph G containing a cycle C_m, we know that the skew-spectral radius of G^σ is independent of its orientation if m is odd. Therefore we will briefly write $\vec{G}$ instead of the normal notation G^σ if each cycle of G is odd. If m is even, then essentially there exist two orientations σ_1 (the sign of the even cycle is positive) and σ_2 (the sign of the even cycle is negative) such that $\rho_s(G^{\sigma_1}) \neq \rho_s(G^{\sigma_2})$. Henceforth we will briefly write G^- (or G^+) instead of G^σ if the sign of each even cycle is negative (or positive). In particular, G will also denote the oriented graph if G is a tree since $\rho_s(G^\sigma) = \rho(G)$ in this case.

Firstly, we give a class of oriented graphs whose skew-spectral radii do not exceed 2.

Denote by $P_{l_1,l_2,\ldots,l_k}$ a pathlike graph, which is defined as follows: we first draw k (≥ 2) paths $P_{l_1}, P_{l_2}, \ldots, P_{l_k}$ of orders $l_1, l_2, \ldots, l_k$ respectively along a line and put two isolated vertices between each pair of those paths, then add edges between the two isolated vertices and the nearest end vertices of such a pair of paths so that the four newly added edges form a cycle C_4, where $l_1, l_k \geq 0$ and $l_i \geq 1$ for $i = 2, 3, \ldots, k-1$. Then $P_{l_1,l_2,\ldots,l_k}$ contains $\sum_{i=1}^{k} l_i + 2k - 2$ vertices. Notice that if $l_i = 1$ ($i = 2, 3, \ldots, k-1$), the two end vertices of the path P_{l_i} are referred to as overlap; if $l_1 = 0$ ($l_k = 0$), the left (right) of the graph $P_{l_1,l_2,\ldots,l_k}$ has only two pendent vertices. Obviously, $P_{1,0} = K_{1,2}$, the star of order 3, and $P_{1,1} = C_4$. In general, P_{l_1,l_2}, P_{0,l_1,l_2}, $P_{0,l_1,l_2,0}$ are all unicyclic graphs containing C_4, where $l_1, l_2 \geq 1$. Meanwhile, P_{l_1,l_2,l_3}, P_{0,l_1,l_2,l_3}, $P_{0,l_1,l_2,l_3,0}$ are all bicyclic graphs containing C_4, where $l_1, l_2, l_3 \geq 1$.

Then we have the following.

Lemma 3.1 ([16]) *Let $P_{l_1,l_2,\ldots,l_k}$ ($k \geq 2$) be a pathlike graph described as above. Then*

$$\rho_s\bigl(P^-_{l_1,l_2,\ldots,l_k}\bigr) \leq 2.$$

Moreover, 2 is an eigenvalue of $P^-_{l_1,l_2,\ldots,l_k}$ with multiplicity $k-2$ and $\phi(P^-_{l_1,l_2};2) = 4$.

Now, we introduce more notations. Denote by T_{l_1,l_2,l_3} the starlike tree with exactly one vertex v of degree 3, and $T_{l_1,l_2,l_3} - v = P_{l_1} \cup P_{l_2} \cup P_{l_3}$, where P_{l_i} is the path of order l_i ($i = 1, 2, 3$).

Due to Smith, all undirected graphs whose (adjacency) spectral radii are bounded by 2 are completely determined as follows.

Lemma 3.2 ([2], or [1], Chapter 2.7.12) *All undirected graphs whose spectral radii do not exceed 2 are C_m, $P_{0,n-4,0}$, $T_{2,2,2}$, $T_{1,3,3}$, $T_{1,2,5}$ and their subgraphs, where $m \geq 3$ and $n \geq 5$.*

Consequently, combining with Lemma 2.2, the skew-spectral radius of each oriented graph whose underlying graph is described as Lemma 3.2, regardless of the orientation of the oriented cycle C_m^σ, does not exceed 2.

Let $C_m = v_1v_2\cdots v_mv_1$ be a cycle on m vertices and $P_{l_1}, P_{l_2}, \ldots, P_{l_m}$ be m paths with vertices $l_1, l_2, \ldots, l_m$, respectively (perhaps some of them are empty). Denote by $C_m^{l_1,l_2,\ldots,l_m}$ the unicyclic graph obtained from C_m by joining v_i to a pendent vertex of P_{l_i} for $i = 1, 2, \ldots, m$. For convenience, suppose without loss of generality that $l_1 = \max\{l_i : i = 1, 2, \ldots, m\}$, $l_2 \geq l_m$ and write $C_m^{l_1,l_2,\ldots,l_j}$ instead of the standard $C_m^{l_1,l_2,\ldots,l_j,0,\ldots,0}$ if $l_{j+1} = l_{j+2} = \cdots = l_m = 0$.

Denote by $\theta_{a,b,c}$ the undirected bicyclic graph obtained from paths P_a, P_b, P_c by identifying the three initial vertices and terminal vertices of them, where $\min\{a,b,c\} \geq 2$ and at most one of them is 2, and by $\infty_m^{p,q}$ ($m \geq 1, p, q \geq 3$) the undirected bicyclic graph obtained from cycles C_p and C_q joined by a path P_m. A bicyclic graph containing $\theta_{a,b,c}$ (or $\infty_m^{p,q}$) is called of θ-type (or ∞-type). The set of bicyclic graphs of θ-type (or ∞-type) is denoted by $\mathcal{G}_\theta$ (or $\mathcal{G}_\infty$). Furthermore, the subset of $\mathcal{G}_\theta$ (or $\mathcal{G}_\infty$) containing $\theta_{a,b,c}$ (or $\infty_m^{p,q}$) is denoted by $\mathcal{G}_\theta(a,b,c)$ (or $\mathcal{G}_\infty(p,q;m)$).

On C_4-free oriented graphs, we have the following.

Lemma 3.3 ([16]) *Let G^σ be an oriented graph and $\rho_s(G^\sigma) \leq 2$. Suppose that G is C_4-free, then G^σ is one of the graphs C_m^σ, $P_{0,n-4,0}$ ($n \geq 5$), $\vec{C}_3^2$, $(C_6^{2,0,0,2})^-$, $(C_6^{1,0,1,0,1})^-$, $(C_8^{1,0,0,0,1})^-$, $\theta_{3,5,5}^\sigma$ or their induced oriented subgraphs, where the orientation of C_m^σ is arbitrary. For induced even cycles C_6^σ, C_8^σ of $\theta_{3,5,5}^\sigma$, they satisfy* $\operatorname{sgn}(C_6^\sigma) = -1$ *and* $\operatorname{sgn}(C_8^\sigma) = 1$.

A connected graph in which the number of edges equals the number of vertices is called a unicyclic graph. For convenience, we write

$\mathcal{U}(m) = \{G | G$ is a unicyclic graph containing the cycle $C_m\}$.

On oriented unicyclic graphs, we have the following.

Lemma 3.4 ([17]) *Let G^σ be an oriented unicyclic graph and $\rho_s(G^\sigma) \leq 2$. Then G^σ is one of the graphs C_m^σ, $\vec{C}_3^2$, $(C_4^{4,1})^-$, $(C_4^{3,1,1})^-$, $(C_4^{2,1,2,1})^-$, $(C_4^{2,2})^-$, $(C_6^{2,0,0,2})^-$, $(C_6^{1,0,1,0,1})^-$, $(C_8^{1,0,0,0,1})^-$ and $P_{0,l_1,l_2,0}^-$ or their induced oriented unicyclic subgraphs, where the orientation of C_m^σ is arbitrary.*

4 The oriented bicyclic graphs whose skew-spectral radii do not exceed 2

In this section, we determine all the oriented bicyclic graphs whose skew-spectral radii do not exceed 2. Let

$\mathcal{B}_2 = \{G^\sigma | G$ is a bicyclic graph and $\rho_s(G^\sigma) \leq 2\}$,

$\mathcal{B}_{21} = \{G^\sigma | G$ is a C_4-free bicyclic graph and $\rho_s(G^\sigma) \leq 2\}$,

$\mathcal{B}_{22} = \{G^\sigma | G$ is a bicyclic graph containing C_4 and $\rho_s(G^\sigma) \leq 2\}$.

The main purpose of this paper is to determine the set $\mathcal{B}_2$. Obviously, $\mathcal{B}_2 = \mathcal{B}_{21} \cup \mathcal{B}_{22}$. By Lemma 3.3, we immediately have the following result on the set $\mathcal{B}_{21}$.

Theorem 4.1 $\mathcal{B}_{21} = \{\theta_{3,5,5}^\sigma\}$, *where* $\operatorname{sgn}(C_6^\sigma) = -1$ *and* $\operatorname{sgn}(C_8^\sigma) = 1$ *for induced even cycles C_6^σ, C_8^σ of $\theta_{3,5,5}^\sigma$.*

Now, we will determine the set $\mathcal{B}_{22}$. Let

$\mathcal{B}_{22}^\infty = \{G^\sigma | G^\sigma \in \mathcal{B}_{22}$ and contain $\infty_m^{4,q}$ ($m \geq 1, q \geq 3$)$\}$,

$\mathcal{B}_{22}^\theta = \{G^\sigma | G^\sigma \in \mathcal{B}_{22}$ and contain $\theta_{a,b,c}$ ($c \geq b \geq a \geq 2$)$\}$.

Obviously, $\mathcal{B}_{22} = \mathcal{B}_{22}^\infty \cup \mathcal{B}_{22}^\theta$. On the set $\mathcal{B}_{22}^\infty$, we have the following.

Theorem 4.2 $\mathcal{B}_{22}^\infty = \{G^\sigma | G^\sigma$ *is* $P_{0,l_1,l_2,l_3,0}^-$ ($l_i \geq 1, i = 1,2,3$) *or its induced oriented bicyclic subgraphs*$\}$.

Proof We will complete the proof by considering following cases.

Case 1. $q = 3$.

We consider the graph $\infty_m^{4,3}$ ($m \geq 1$). Obviously, $\infty_m^{4,3}$ ($m \geq 1$) contains a unicyclic graph $U_1 \in \mathcal{U}(3)$, but U_1 is not C_3^2 or its induced unicyclic graphs. By Lemma 3.4, we know $\rho_s((\infty_m^{4,3})^\sigma) > 2$. Thus $\rho_s(G^\sigma) > 2$ for $G \in \mathcal{G}_\infty(4,3;m)$.

Case 2. $q \geq 5$.

If $q \neq 6, 8$, then there exists a unicyclic graph $U_2 \in \mathcal{U}(q) - C_q$ in $\infty_m^{4,q}$ ($m \geq 1$). If $q = 6$, then there exists a unicyclic graph $U_3 \in \mathcal{U}(6)$ in $\infty_m^{4,6}$ ($m \geq 1$), but U_3 is not any of the graphs $C_6^{2,0,0,2}$, $C_6^{1,0,1,0,1}$ or their induced unicyclic graphs. If $q = 8$, then there exists a unicyclic graph $U_4 \in \mathcal{U}(8)$ in $\infty_m^{4,8}$ ($m \geq 1$) but U_4 is not $C_8^{1,0,0,0,1}$ or its induced unicyclic graphs. So, by Lemma 3.4, we also have $\rho_s((\infty_m^{4,q})^\sigma) > 2$ ($q \geq 5$). Therefore $\rho_s(G^\sigma) > 2$ for $G \in \mathcal{G}_\infty(4,q;m)$ ($q \geq 5$).

Case 3. $q = 4$.

The sign of each induced cycle C_4^σ of G^σ must be negative. Furthermore, G^σ must be $P_{0,l_1,l_2,l_3,0}^-$ or its induced oriented bicyclic subgraphs. Otherwise, G has an induced tree T such that T contains a proper subgraph $P_{0,l,0}$. Hence $\rho_s(G^\sigma) > \rho_s(P_{0,l,0}) = 2$. □

The rest of this manuscript is to determine the set $\mathcal{B}_{22}^\theta$. Firstly, we have the following.

Lemma 4.1 *If all the three cycles of $\theta_{a,b,c}^\sigma$ are even, then there exists a cycle such that its sign is positive.*

Proof Suppose without loss of generality that $\mathrm{sgn}(\theta_{a,b}^\sigma) = \mathrm{sgn}(\theta_{a,c}^\sigma) = -1$, we will show $\mathrm{sgn}(\theta_{b,c}^\sigma) = 1$.

Since

$$\mathrm{sgn}(\theta_{a,b}^\sigma) = \mathrm{sgn}(P_a)\,\mathrm{sgn}(\bar{P}_b),$$
$$\mathrm{sgn}(\theta_{a,c}^\sigma) = \mathrm{sgn}(P_a)\,\mathrm{sgn}(\bar{P}_c).$$

Thus

$$\begin{aligned}\mathrm{sgn}(\theta_{a,b}^\sigma)\,\mathrm{sgn}(\theta_{a,c}^\sigma) &= (\mathrm{sgn}(P_a))^2\,\mathrm{sgn}(\bar{P}_b)\,\mathrm{sgn}(\bar{P}_c)\\ &= (-1)^{a-1}\,\mathrm{sgn}(\bar{P}_b)\,\mathrm{sgn}(\bar{P}_c) = 1.\end{aligned}$$

So

$$\mathrm{sgn}(\bar{P}_b)\,\mathrm{sgn}(\bar{P}_c) = (-1)^{a-1}.$$

On the other hand,

$$\begin{aligned}\mathrm{sgn}(\theta_{b,c}^\sigma) &= \mathrm{sgn}(P_b)\,\mathrm{sgn}(\bar{P}_c)\\ &= (-1)^{b-1}\,\mathrm{sgn}(\bar{P}_b)\,\mathrm{sgn}(\bar{P}_c)\\ &= (-1)^{a+b-2}.\end{aligned}$$

But we know $a + b$ is even. Then

$$\mathrm{sgn}(\theta_{b,c}^\sigma) = 1.$$ □

Let G^σ be an oriented graph with the property

$$\rho_s\left(G^\sigma\right) \le 2. \tag{4.1}$$

The property (4.1) is hereditary because, as a direct consequence of Lemma 2.1, for any induced subgraph $H \subset G$, H^σ also satisfies (4.1). The inheritance (hereditary) of property (4.1) implies that there are minimal connected oriented graphs that do not obey (4.1); such graphs are called *forbidden oriented subgraphs*.

Let $v_1v_2\cdots v_bv_{b+1}\cdots v_{b+c-2}v_1$ ($c \ge b \ge 3$) be the longest cycle in $\theta_{2,b,c}$ and $P_b = v_1v_2\cdots v_b$, $P_c = v_bv_{b+1}\cdots v_{b+c-2}v_1$. Denoted by $\theta_{2,b,c}^{l_1,l_2,\ldots,l_bl_{b+1}\cdots l_{b+c-2}}$ is the bicyclic graph obtained from $\theta_{2,b,c}$ by joining its vertex v_i to a pendent vertex of P_{l_i} with vertices l_i ($i = 1,2,\ldots,b+c-2$), where $l_i \ge 0$, $l_1 \ge l_b$. Moreover, if $b = c$, we will suppose $l_2 \ge l_{b+c-2}$.

Lemma 4.2 *Let* $F_1 = \theta_{2,3,3}^{1,0,0,0}$, $F_2 = \theta_{2,3,3}^{0,1,0,0}$, $F_3 = \theta_{2,3,4}^{1,0,0,0,0}$, $F_4 = \theta_{2,3,4}^{0,1,0,0,0}$, $F_5 = \theta_{2,3,4}^{0,0,0,1,0}$, $F_6 = \theta_{2,4,4}^{2,0,0,0,0,0}$, $F_7 = \theta_{2,4,4}^{0,2,0,0,2,0}$ *and* $F_8 = \theta_{2,4,6}^{0,2,0,0,0,0,0,0}$. *Then* F_i^σ ($i = 1,2,\ldots,8$) *are forbidden oriented subgraphs on the property* (4.1), *where each subgraph* C_4^σ *of* F_i^σ ($i = 1,2,\ldots,8$) *is negative. Also the subgraph* C_6^σ *of* F_8^σ *is negative.*

Proof By Lemma 2.3, it is not difficult to know that

$$\phi\left(\theta_{2,3,3}^-;\lambda\right) = \phi\left(C_4^-;\lambda\right) - \lambda^2.$$

Thus, by Lemma 3.1, $\phi(\theta_{2,3,3}^-;2) = 0$. And then $\rho_s(\theta_{2,3,3}^-) = 2$. Moreover, we have

$$\phi\left(F_1^\sigma;\lambda\right) = \phi\left(F_1^-;\lambda\right) = \lambda\phi\left(\theta_{2,3,3}^-;\lambda\right) - \phi(K_{1,2};\lambda).$$

Then $\phi(F_1^\sigma;2) = -4$, and thus $\rho_s(F_1^\sigma) > 2$. Also, we have

$$\phi\left(F_2^\sigma;\lambda\right) = \phi\left(F_2^-;\lambda\right) = \lambda\phi\left(\theta_{2,3,3}^-;\lambda\right) - \phi(\vec{C}_3;\lambda).$$

Then $\phi(F_2^\sigma;2) = -2$, and therefore $\rho_s(F_2^\sigma) > 2$.

Similarly, we have $\rho_s(\theta_{2,3,4}^-) = \rho_s((\theta_{2,4,4}^{1,0,0,0,0,0})^\sigma) = \rho_s((\theta_{2,4,4}^{0,2,0,0,1,0})^\sigma) = \rho_s((\theta_{2,4,6}^{0,1,0,0,0,0,0,0})^\sigma) = 2$ and $\rho_s(F_i^\sigma) > 2$ ($i = 3,4,\ldots,8$). □

In order to determine the set $\mathcal{B}_{22}^\theta$, we will first consider the oriented graphs containing $\theta_{2,4,4}^\sigma$ or $\theta_{3,3,c}^\sigma$ ($c \ge 3$). Combining with Lemma 4.2 and the fact that $\rho_s(G^\sigma) > 2$ if the oriented tree G^σ contains an arbitrary tree described in Lemma 3.2 as a proper subgraph, we have the following result.

Lemma 4.3 *Let* $G \in \mathcal{G}_\theta(2,4,4)$ *and* $G_1 = \theta_{2,4,4}^{1,0,3,0,0,0}$, $G_2 = \theta_{2,4,4}^{1,0,1,0,1,0}$, $G_3 = \theta_{2,4,4}^{0,2,0,0,1,0}$ *and* $G_4 = \theta_{2,4,4}^{0,1,1,0,1,0}$. *If* $\rho_s(G^\sigma) \le 2$, *then* G^σ *is one of the graphs* G_i^σ ($i = 1,2,3,4$) *or their induced oriented bicyclic subgraphs, where each subgraph* C_4^σ *of* G_i^σ ($i = 1,2,3,4$) *is negative.*

Proof Obviously, the sign of C_6^σ in $\theta_{2,4,4}^\sigma$ must be positive. If $G \ne \theta_{2,4,4}^{l_1,l_2,\ldots,l_6}$, we know $\rho_s(G^\sigma) > 2$ by Lemma 3.4.

Now, let $G = \theta_{2,4,4}^{l_1,l_2,\ldots,l_6}$. By Lemma 4.2, we know $l_1 \le 1$.

Claim 1. If $l_1 = 1$, then G^σ is one of the graphs G_1^σ, G_2^σ or their induced oriented bicyclic subgraphs, where each subgraph C_4^σ of G_1^σ and G_2^σ is negative.

Obviously, we have $l_2 = l_4 = l_6 = 0$. Otherwise, $\rho_s(G^\sigma) > 2$ by Lemma 3.4. Suppose $l_3 \geq l_5$. Then $l_3 \leq 3$. Otherwise $\rho_s(G^\sigma) \geq \rho(T_{1,2,6}) > 2$. Now, we consider the following two cases.

Case 1. $l_3 = 2$ or 3.

Then $l_5 = 0$. Otherwise $\rho_s(G^\sigma) \geq \rho(T_{1,3,4}) > 2$.

It is not difficult to know that $\phi(G_1^\sigma; 2) = 0$. Thus $\rho_s(G_1^\sigma) = 2$. Therefore G^σ is G_1^σ or its induced oriented bicyclic subgraphs in this case.

Case 2. $l_3 = 1$.

We can obtain that $\rho_s(G_2^\sigma) = 2$. Therefore G^σ is G_2^σ or its induced oriented bicyclic subgraphs in this case.

Claim 2. If $l_1 = 0$. Then G^σ is one of the graphs $(\theta_{2,4,4}^{0,3,0,0,0,0})^\sigma$, $(\theta_{2,4,4}^{0,1,0,0,0,1})^\sigma$, G_3^σ, G_4^σ or their induced oriented bicyclic subgraphs, where each subgraph C_4^σ of the mentioned oriented graphs is negative.

Firstly, we can suppose $l_2 = \max\{l_2, l_3, l_5, l_6\} \geq 1$. It is easy to see that $l_2 \leq 3$, otherwise $\rho_s(G^\sigma) \geq \rho(T_{1,3,4}) > 2$ by Lemma 3.2. Now, we consider the following three cases.

Case 1. $l_2 = 3$.

Then $l_3 = l_5 = l_6 = 0$. Otherwise $\rho_s(G^\sigma) > 2$ by Lemma 3.2. On the other hand, $\rho_s((\theta_{2,4,4}^{0,3,0,0,0,0})^\sigma) = 2$. Therefore G^σ is $(\theta_{2,4,4}^{0,3,0,0,0,0})^\sigma$ or its induced oriented bicyclic subgraphs in this case. Obviously, $\theta_{2,4,4}^{0,3,0,0,0,0}$ is an induced subgraph of G_1.

Case 2. $l_2 = 2$.

Then $l_3 = l_6 = 0$ and $l_5 \leq 2$ by Lemma 3.2. Moreover, by Lemma 4.2, we know $l_5 \leq 1$. Since $\rho_s(G_3^\sigma) = 2$. Therefore G^σ is G_3^σ or its induced oriented bicyclic subgraphs in this case.

Case 3. $l_2 = 1$.

Subcase 3.1. $l_6 = 1$.

Then $l_3 = l_5 = 0$. Since $\rho_s((\theta_{2,4,4}^{0,1,0,0,0,1})^\sigma) = 2$. Therefore G^σ is $(\theta_{2,4,4}^{0,1,0,0,0,1})^\sigma$ or its induced oriented bicyclic subgraphs in this case. Obviously, $\theta_{2,4,4}^{0,1,0,0,0,1}$ is an induced subgraph of G_2.

Subcase 3.2. $l_6 = 0$.

Then $l_3 \leq 1$ and $l_5 \leq 1$. But we have $\rho_s(G_4^\sigma) = 2$. Therefore G^σ is G_4^σ or its induced oriented bicyclic subgraphs in this case.

Hence, we have completed the proof of this theorem. □

Now, we consider the oriented bicyclic graphs containing $\theta_{3,3,c}^\sigma$ ($c \geq 3$). For the graph $\theta_{a,b,c} = \theta_{3,3,c}$, we suppose $P_a = u_1u_2u_3$, $P_b = v_1v_2v_3$, $P_c = w_1w_2 \cdots w_c$ and $u_1 = v_1 = w_1$, $u_3 = v_3 = w_c$. Also, in $\theta_{3,3,c}^\sigma$, we suppose that the cycle C_4^σ is negative if c is even and only the cycle $v_1v_2v_3w_{c-1} \cdots w_2v_1$ is positive if c is odd. Furthermore, let $T_{1,1,l} - v = P_1 \cup P_1 \cup P_l$ and $P_l = x_1x_2 \cdots x_l$ ($l \geq 0$). Denoted by G_5 is the graph obtained from $\theta_{3,3,3}$ by identifying u_2 with x_l (or v if $l = 0$) of $T_{1,1,l}$. Suppose that G_6 is the graph obtained from $\theta_{3,3,5}$ by joining u_2 to a pendent vertex of P_2, and G_7 is the graph obtained from $\theta_{3,3,7}$ by joining u_2 to an isolated vertex.

Lemma 4.4 *Let* $G \in \mathcal{G}_\theta(3,3,c)$ ($c \geq 3$) *and* $\rho_s(G^\sigma) \leq 2$. *Then* G^σ *is one of the graphs* $\theta_{3,3,c}^\sigma$ ($c \geq 4$, $c \neq 5, 7$), G_i^σ ($i = 5, 6, 7$) *or the induced oriented bicyclic subgraphs of* G_i^σ ($i = 5, 6, 7$), *where the cycle* C_4^σ *is negative if c is even and only one cycle* C_{c+1} *in each* $\theta_{3,3,c}^\sigma$ *is positive if c is odd.*

Proof We will complete the proof by proving the following five claims.

Claim 1. $\rho_s(\theta^\sigma_{3,3,c}) = 2$ $(c \geq 3)$.

By Lemma 2.3, we have

$$\phi(\theta^\sigma_{3,3,c};\lambda) = \lambda\phi(P^-_{c-2,1};\lambda) - \phi(P^-_{c-3,1};\lambda) - \phi(P^-_{c-2,0};\lambda).$$

Thus $\phi(\theta^\sigma_{3,3,c};2) = 0$ because of $\phi(P^-_{l_1,l_2};2) = 4$ $(l_1, l_2 \geq 0)$. And then $\rho_s(\theta^\sigma_{3,3,c}) = 2$ $(c \geq 3)$.

Claim 2. If $c \neq 3,5,7$, then G^σ is one of the graphs $\theta^\sigma_{3,3,c}$ $(c \geq 3, c \neq 3,5,7)$.

Suppose that H is the graph obtained from $\theta_{3,3,c}$ by joining its one vertex to a pendent vertex. Then we know H^σ has an induced subgraph $(C^1_k)^\sigma$, where $k = 5,7$ or $k \geq 9$. Thus $\rho_s(H^\sigma) > 2$ and the result holds.

Claim 3. If $c = 3$, then G^σ is G^σ_5 or its induced oriented bicyclic subgraphs.

If $c = 3$, then G^σ must be some oriented graph by joining a tree T to the vertex u_2 of $\theta_{3,3,3}$. By Lemma 3.4, we know G^σ can only be G^σ_5 or its induced oriented bicyclic subgraphs. Now we will show $\rho_s(G^\sigma_5) = 2$.

Let $\theta^l_{3,3,3}$ be the graph obtained from G_5 by deleting its two pendent vertices. We show by induction on l that $\rho_s((\theta^l_{3,3,3})^\sigma) = 2$. If $l = 0$, then $\rho_s((\theta^0_{3,3,3})^\sigma) = \rho_s(\theta^\sigma_{3,3,3}) = 2$. Suppose now that $l \geq 0$ and the result is true for the order no more than l. Then we have

$$\phi((\theta^{l+1}_{3,3,3})^\sigma;\lambda) = \lambda\phi((\theta^l_{3,3,3})^\sigma;\lambda) - \phi((\theta^{l-1}_{3,3,3})^\sigma;\lambda).$$

So $\phi((\theta^{l+1}_{3,3,3})^\sigma;2) = 0$ if $l = 0$ because of $(\theta^{-1}_{3,3,3})^\sigma = C^+_4$. Therefore $\rho_s((\theta^l_{3,3,3})^\sigma) = 2$.

Now, we will prove $\rho_s(G^\sigma_5) = 2$. Firstly, we have

$$\phi(G^\sigma_5;\lambda) = \lambda\phi((\theta^{l+1}_{3,3,3})^\sigma;\lambda) - \lambda\phi((\theta^{l-1}_{3,3,3})^\sigma;\lambda).$$

Then 2 is an eigenvalue of G_5.

Also, by Lemma 2.4, we have

$$\frac{d}{d\lambda}\phi(G^\sigma_5;\lambda) = \sum_{v\in V(G_5)} \phi(G^\sigma_5 - v;\lambda).$$

Since $\rho_s(G^\sigma_5 - v) = 2$ for each $v \in V(G_5)$, we know 2 is an eigenvalue of G_5 with multiplicity 2. But $\lambda_3(G_5) < 2$, so we can confirm the result holds.

Claim 4. If $C = 5$, then G^σ is G^σ_6 or its induced oriented bicyclic subgraphs.

In this case, G^σ must be some oriented graph by joining a path to the vertex u_2 of $\theta^\sigma_{3,3,5}$. By Lemma 3.4, we know G^σ can only be G^σ_6 or its induced oriented bicyclic subgraphs. To confirm the result holds, we only need to show $\rho_s(G^\sigma_6) = 2$. In fact, similar to the proof of Claim 3, we can show 2 is an eigenvalue of G_6 with multiplicity 2. And thus $\rho_s(G^\sigma_6) = 2$.

Claim 5. If $c = 7$, then G^σ is G^σ_7 or its induced oriented bicyclic subgraphs.

Similarly, we know the result holds. □

Theorem 4.3 $\mathcal{B}^\theta_{22} = \{\theta^\sigma_{2,3,3}, \theta^\sigma_{2,3,4}, (\theta^{0,1,0,0,0,0,0,0}_{2,4,6})^\sigma, \theta^\sigma_{3,3,c}$ $(c \geq 4, c \neq 5,7), G^\sigma_i$ $(i = 1,2,\ldots,7)$ *or their induced oriented bicyclic subgraphs*$\}$, *where the cycle* C^σ_4 *is negative if* c *is even and only one cycle* C_{c+1} *in each* $\theta^\sigma_{3,3,c}$ *is positive if* c *is odd.*

Proof Firstly, since $2 \leq a \leq b \leq c$ and G^σ containing C_4, we know $2 \leq a \leq 3$, $3 \leq b \leq 4$. Furthermore, the radius of each graph mentioned in this theorem is 2. The discussion is divided into three parts according to different cases of a and b.

Case 1. $a = 2$, $b = 3$.

Subcase 1.1. $c = 3$ or $c = 4$.

By Lemma 4.2, we know G^σ must be $\theta^\sigma_{2,3,3}$ or $\theta^\sigma_{2,3,4}$.

Subcase 1.2. $c \geq 5$.

Then $\rho_s(G^\sigma) \geq \rho_s((C_3^{1,1})^\sigma) > 2$.

Case 2. $a = 2$, $b = 4$.

By Lemma 3.4 and Lemma 4.2, we know $c = 4$ or $c = 6$.

Subcase 2.1. $c = 4$.

By Lemma 4.3, G^σ must be one of the graphs G_i^σ $(i = 1, 2, 3, 4)$ or their induced oriented bicyclic subgraphs.

Subcase 2.2. $c = 6$.

By Lemma 4.2, G^σ must be $(\theta^{0,1,0,0,0,0,0,0}_{2,4,6})^\sigma$ or its induced oriented bicyclic subgraph.

Case 3. $a = 3$.

Then $b = 3$, otherwise G^σ does not contain C_4. By Lemma 4.4, G^σ must be $\theta^\sigma_{3,3,c}$ $(c \geq 3$, $c \neq 3, 5, 7)$, G_i^σ $(i = 5, 6, 7)$ or their induced oriented bicyclic subgraphs. □

Combining with Theorem 4.1, Theorem 4.2 and Theorem 4.3, we have the following main result.

Theorem 4.4 $\mathcal{B}_2 = \{\theta^\sigma_{3,5,5}, P^-_{0,l_1,l_2,l_3,0}, \theta^\sigma_{2,3,3}, \theta^\sigma_{2,3,4}, (\theta^{0,1,0,0,0,0,0,0}_{2,4,6})^\sigma, \theta^\sigma_{3,3,c}\ (c \geq 4, c \neq 5, 7), G_i^\sigma\ (i = 1, 2, \ldots, 7)$ *or their induced oriented bicyclic subgraphs*$\}$, *where the cycle* C_4^σ *is negative if* c *is even and only one cycle* C_{c+1} *in each* $\theta^\sigma_{3,3,c}$ *is positive if* c *is odd*.

By this theorem, we have the following.

Theorem 4.5 *Let* G^σ *be an oriented bicyclic graph and* $\rho_s(G^\sigma) < 2$. *Then* G^σ *is one of the graphs* $(\theta^{0,1,0,0,1,0}_{2,4,4})^\sigma$, $(\theta^{0,2,0,0,0,0}_{2,4,4})^\sigma$, $\theta^\sigma_{2,4,6}$ *or their induced oriented bicyclic subgraphs*.

Proof Of course, the radius of each graph mentioned in this theorem is less than 2. Obviously, by Theorem 4.4, G must contain $\theta_{2,4,4}$ or $\theta_{2,4,6}$ if $\rho_s(G^\sigma) < 2$. Firstly, by taking some direct calculations, we know $\phi(\theta^\sigma_{2,4,6}; 2) = 1$, and then $\rho_s(\theta^\sigma_{2,4,6}) < 2$.

Now suppose that G contains $\theta_{2,4,4}$. By taking some direct calculations, we know $\rho_s((\theta^{1,0,0,0,0,0}_{2,4,4})^\sigma) = 2$. Thus $l_1 = l_4 = 0$. Similarly, we have $\rho_s((\theta^{0,3,0,0,0,0}_{2,4,4})^\sigma) = 2$. Thus $l_2 = \max\{l_2, l_3, l_5, l_6\} \leq 2$.

Case 1. $l_2 = 2$.

Since $\rho_s((\theta^{0,2,1,0,0,0}_{2,4,4})^\sigma) > 2$, $\rho_s((\theta^{0,2,0,0,0,1}_{2,4,4})^\sigma) > 2$ and $\rho_s((\theta^{0,2,0,0,1,0}_{2,4,4})^\sigma) = 2$. Thus G^σ must be $(\theta^{0,2,0,0,0,0}_{2,4,4})^\sigma$ or its induced oriented bicyclic subgraphs in this case.

Case 2. $l_2 = 1$.

Since $\rho_s((\theta^{0,1,0,0,0,1}_{2,4,4})^\sigma) = 2$ and $\rho_s((\theta^{0,1,1,0,0,0}_{2,4,4})^\sigma) = 2$. Thus G^σ must be $(\theta^{0,1,0,0,1,0}_{2,4,4})^\sigma$ or its induced oriented bicyclic subgraphs in this case. □

By this theorem, we immediately obtain the following sharp lower bound on the skew-spectral radii of oriented bicyclic graphs.

Corollary 4.1 *Let* G^σ *be an oriented bicyclic graph on order* n *and* $n \geq 9$. *Then* $\rho_s(G^\sigma) \geq 2$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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