Hindawi
Mathematical Problems in Engineering
Volume 2018, Article ID 2850757, 9 pages
https://doi.org/10.1155/2018/2850757

Research Article

Consensus of Fractional-Order Multiagent Systems with Nonuniform Time Delays

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Received 9 October 2017; Revised 11 December 2017; Accepted 21 December 2017; Published 22 January 2018

Academic Editor: Luis Gracia

Due to the complex external environment, many multiagent systems cannot be precisely described or even cannot be described by an integer-order dynamical model and can only be described by a fractional-order dynamical model. In this paper, consensus problems are investigated for two types of fractional-order multiagent systems (FOMASs) with nonuniform time delays: FOMAS with symmetric time delays and undirected topology and FOMAS with asymmetric time delays and directed topology. Employing the Laplace transform and the frequency-domain theory, two delay margins are obtained to guarantee the consensus for the two types of FOMAS, respectively. These results are also suitable for the integer-order dynamical model. Finally, simulation results are provided to illustrate the effectiveness of our theoretical results.

1. Introduction

During recent years, consensus problems of multiagent systems have attracted a great deal of attention due to their enormous potential applications in many areas such as the formation control of multirobot systems [1], cooperative control of unmanned aerial vehicles [2], and distributed information filtering [3]. In the past decade, research results have been continuously springing up about consensus problems of various multiagent systems. Examples include consensus problems of multiagent systems with different dynamics such as first-order dynamics in [4], second-order dynamics in [5], and high-order dynamics in [6]; consensus problems of multiagent systems with different time delays such as fixed time delays in [7], time-varying delays in [8], and multiple time delays in [9]; and consensus problems of multiagent systems with different network topologies such as fixed topology in [10], switching topology in [11], and randomly switching topology in [12].

As is known to us, the essential characteristic or behavior of an object in the complex environment can be better revealed using the fractional-order dynamical model [13]. The reason why the system's actual order (fractional order) is neglected is the complexity and the lack of corresponding theories. But in recent years, this obstacle is being gradually solved in many fields, and relevant research results continue to emerge. The authors of [14] simulated the dynamical characteristics of self-similar protein in the fractional-order model due to the relaxation processes and the reaction kinematics of proteins deviated from the exponential behavior. In [15], Professor Liu described the relationship between weather and climate in fractional derivative. In [16, 17], Podlubny firstly proposed the fractional-order proportional-integral differential (PID) controllers, which had better dynamical performance and robustness than classical integer-order PID controllers. Fractional-order dynamical control has received some persuasive results in the field of linear/nonlinear dynamics [18, 19]. Furthermore, to the best of our knowledge, Cao et al. specifically studied distributed coordination of multiagent systems based on fractional order [13, 20], and they analyzed and summed up the relationship between the number of individuals and the fractional order in a stable multiagent system for the first time.

In practical applications, time delays often exist in fractional-order multiagent systems (FOMASs), which are caused by measuring, computing, or communication. Time delays are inevitable and they have a bad impact on consensus of FOMASs. Up to now, the consensus problems for FOMASs with time delays have been studied by some academic researchers. In [21], a new distributed control protocol based on the delayed state and delayed-state fractional-order derivative was introduced for improving the robustness against communication delays, and all the time delays were identical. Moreover, a sufficient condition for consensus in FOMAS with input delays has been presented in [22], and the time delays contained up to n different values when FOMAS was composed of n agents. A generalized form of FOMAS with self and communication uniform time delays was considered in [23], and all the time delays were identical. The fractional-order and compound-order multiagent cooperative control protocols with communication delays were studied in [24, 25], and the time delays contained up to n different values when FOMAS was composed of n agents. So far, there exists rare work considering the most common situation where all the time delays are of different values; that is, the asymmetric time delays contain up to $n(n-1)$ different values when FOMAS consists of n agents.

Motivated by the previous analysis, this paper investigates consensus problems for the two types of FOMAS with nonuniform time delays: the FOMAS with symmetric time delays and undirected topology and the FOMAS with asymmetric time delays and directed topology. Firstly, a distributed control protocol based on state feedback of neighbors is designed and the closed-loop dynamics are built. By applying graph theory tools, the model of FOMAS with nonuniform time delays is established and the consensus problems of FOMAS with nonuniform time delays are transformed into the problems of the transform function matrix eigenvalues of FOMAS. By employing Laplace transform of Caputo derivative, the transform function matrix of FOMAS is derived. Then, based on the frequency-domain analysis, two delay margins are obtained to guarantee consensus for the two types of FOMAS via the characteristic polynomial analysis of the transform function matrix and the matrix theory tools. The main innovation of this article lies in the research on the fractional-order dynamics which can better reveal the essential characteristic or behavior of an object in the complex environment and the consensus of FOMAS with asymmetric time delays which contain up to $n(n-1)$ different values when FOMAS consists of n agents.

The rest of the paper is organized as follows. In Section 2, we introduce some basic preliminaries about graph theory and fractional calculus knowledge. The fractional consensus algorithms for multiagent dynamical systems with different types of time delays are studied in Sections 3 and 4. In Section 5, some numerical examples are simulated to verify the theoretical results. Finally, conclusions are drawn in Section 6.

2. Preliminaries

2.1. Graph Theory. In this section, some preliminary knowledge of graph theory is introduced for the following analysis. Let $\mathscr{G}(\mathscr{V},\mathscr{E},\mathscr{A})$ be an interaction graph of order n, where $\mathscr{V}=\{s_1,\dots,s_n\}$ is the set of nodes, $\mathscr{E}\subseteq\mathscr{V}\times\mathscr{V}$ is the set of edges, and $\mathscr{A}=[a_{ij}]\in\mathbb{R}^{n\times n}$ is a weighted adjacency matrix. The node indexes belong to a finite index set $\mathscr{I}=\{1,2,\dots,n\}$. If there is a directed edge $e_{ij}\in\mathscr{E}$ which is from node s_j to node s_i, then $a_{ij}>0$; otherwise, $a_{ij}=0$. Moreover, we assume $a_{ii}=0$ for $i=1,2,\dots,n$. If all $i,j\in\mathscr{I}$ have $a_{ij}=a_{ji}\geqslant 0$, then it is said that graph $\mathscr{G}$ is an undirected graph; otherwise, it is called a directed graph. The set of neighbors of node s_i is denoted by $N_i=\{s_j\in\mathscr{V}:(s_i,s_j)\in\mathscr{E}\}$. The Laplacian matrix of the interaction graph is defined as $L=\Delta-\mathscr{A}\in\mathbb{R}^{n\times n}$, where $\Delta\triangleq\operatorname{diag}\{\deg_{\text{out}}(h_1),\deg_{\text{out}}(h_2),\dots,\deg_{\text{out}}(h_i),\dots,\deg_{\text{out}}(h_n)\}$ is a diagonal matrix with $\deg_{\text{out}}(h_i)=\sum_{j=1}^{n}a_{ij}$. For two nodes i and k, if the subscript set $\{k_1,k_2,\dots,k_l\}$ satisfies $a_{ik_1}>0,a_{k_1k_2}>0,\dots,a_{k_lk}>0$, then there is a directed path from node i to node k, which is said to be strongly connected. If any two nodes in the graph are strongly connected, the graph is said to be strongly connected. If there exists a node such that there is a directed path from every other node to this node, this directed graph is said to have a spanning tree. If there are some graphs $\mathscr{G}_1,\mathscr{G}_2,\dots,\mathscr{G}_M$ and graph $\mathscr{G}$ with the same nodes, where the edge set of the graph $\mathscr{G}$ is the sum of the other graphs $\mathscr{G}_1,\mathscr{G}_2,\dots,\mathscr{G}_M$, then its Laplacian matrix of graph $\mathscr{G}$ is the sum of other graphs' Laplacian matrix; that is, $L=\sum_{m=1}^{M}L_m$.

Lemma 1 (see [26]). *If graph $\mathscr{G}$ is an undirected graph, then its Laplacian matrix L has a zero eigenvalue and the other eigenvalues are positive real numbers.*

Lemma 2 (see [26]). *If graph $\mathscr{G}$ is a directed graph and has a spanning tree, then its Laplacian matrix L has a zero eigenvalue and the other eigenvalues have a positive real part.*

2.2. Fractional Calculus. In modern science, fractional calculus has played a significant role. There are several different definitions of fractional calculus operators, such as Caputo fractional operator and Riemann-Liouville (R-L) fractional operator in [17]. In this paper, the Caputo fractional operator will be adopted to model the system dynamical characteristics. What needs to be supplemented is the Caputo fractional operator that contains the Caputo integral and Caputo derivative, and the Caputo integral is defined as

$$ {}_a^CD_t^{-\sigma}f(t)=\frac{1}{\Gamma(\sigma)}\int_a^t\frac{f(\eta)}{(t-\eta)^{1-\sigma}}d\eta, \tag{1}$$

where ${}_a^CD_t^{-\sigma}f(t)$ denotes the Caputo integral with order $\sigma\in(0,1]$, a is an arbitrary real number and denotes the initial value, and $\Gamma(\cdot)$ is the Gamma function

$$\Gamma(\sigma)=\int_0^{\infty}e^{-t}t^{\sigma-1}dt. \tag{2}$$

For a nonnegative real number α, the Caputo derivative takes on the same form with the traditional integer-order derivative in essence, but it is based on the Caputo integral:

$$ {}_a^CD_t^{\alpha}f(t)={}_a^CD_t^{-\sigma}\left[\frac{d^{[\alpha]+1}}{dt^{[\alpha]+1}}f(t)\right], \tag{3}$$

where $\sigma = [\alpha] + 1 - \alpha \in (0, 1]$ and $[\alpha]$ is the integral part of α. If α is an integer, then $\sigma = 1$ and the Caputo derivative is equivalent to the integer-order derivative. In this paper, let $f^{(\alpha)}(t)$ replace ${}_a^C D_t^\alpha f(t)$, and $F(s) = \mathfrak{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t) dt$ denotes the Laplace transform of the function $f(t)$; then, the Laplace transform of the Caputo derivative is obtained:

$$\mathfrak{L}\left\{f^{(\alpha)}(t)\right\} = \begin{cases} s^\alpha F(s) - s^{\alpha-1} f(0^-), & \alpha \in (0,1] \\ s^\alpha F(s) - s^{\alpha-1} f(0^-) - s^{\alpha-2} f'(0^-), & \alpha \in (1,2], \end{cases} \tag{4}$$

where $f(0^-) = \lim_{t\to 0^-} f(t)$ and $f'(0^-) = \lim_{t\to 0^-} f'(t)$.

3. Problem Statement

Consider a FOMAS consisting of n agents. Each agent is regarded as a node in a graph $\mathcal{G}$. Each edge $(s_i, s_j) \in \mathcal{E}$ corresponds to an available information channel between agents s_i and s_j. Suppose that the ith agent s_i ($i \in \mathcal{I}$, $\mathcal{I} \triangleq \{1, 2, \ldots, n\}$) has dynamics as follows:

$$x_i^{(\alpha)}(t) = u_i(t), \quad i \in \mathcal{I}, \tag{5}$$

where $x_i(t), u_i(t) \in \mathbb{R}$, respectively, denote the ith agent's state and control input and $x_i^{(\alpha)}(t)$ denotes the α order Caputo derivative of $x_i(t)$.

Definition 3. FOMAS (5) reaches consensus if and only if the states of agents satisfy

$$\lim_{t\to+\infty} \left(x_i(t) - x_j(t)\right) = 0 \tag{6}$$

$\forall i, j \in \mathcal{I}$.

To solve the consensus control problem for FOMAS (5), the distributed control protocol is given as

$$u_i(t) = \sum_{j\in N_i} a_{ij} \left[x_j\left(t - \tau_{ij}\right) - x_i\left(t - \tau_{ij}\right)\right], \quad i, j \in \mathcal{I}, \tag{7}$$

where a_{ij} denotes the adjacency elements of interaction graph $\mathcal{G}$, N_i represents the neighbors collection of the ith agent, and $\tau_{ij} > 0$ is the time delay for the ith agent to get the state information of the jth agent. If $\tau_{ij} = \tau_{ji}$ holds for all $i, j \in \mathcal{I}$, the time delays are said to be symmetric. Otherwise, the time delays are said to be asymmetric.

Assume that there are M different time delays, which are denoted by $\tau_m \in \{\tau_{ij} : i, j \in \mathcal{I}\}$ $(m = 1, 2, \ldots, M)$. Then, the control protocol (7) can be rewritten to

$$u_i(t) = \sum_{j\in N_i} a_{ij} \left[x_j(t - \tau_m) - x_i(t - \tau_m)\right], \quad m = 1, 2, \ldots, M. \tag{8}$$

Define $\varphi(t) \triangleq [x_1(t), x_2(t), \ldots, x_n(t)]^T$. Using protocol (8), the closed-loop dynamics of FOMAS (5) can be written as

$$\varphi^{(\alpha)}(t) = -\sum_{m=1}^{M} L_m \varphi(t - \tau_m), \tag{9}$$

where $\varphi^{(\alpha)}(t)$ denotes the α order Caputo derivative of $\varphi(t)$ and L_m denotes the Laplacian matrix of a subgraph associated with the delay τ_m.

4. Main Results

4.1. Consensus of FOMAS with Symmetric Time Delays

Theorem 4. *Assume that the undirected interaction graph $\mathcal{G}$ is connected and the time delays are symmetric. By the distributed control protocol (8), FOMAS (9) can reach consensus if all the time delays τ_m $(m = 1, 2, \ldots, M)$ are less than $\overline{\tau}$, and FOMAS (9) cannot reach consensus if all the time delays τ_m $(m = 1, 2, \ldots, M)$ are greater than $\overline{\tau}$, where*

$$\overline{\tau} = \frac{\pi(2-\alpha)}{2\overline{\omega}}, \tag{10}$$

$\alpha \in (0, 2)$, $\overline{\omega} = \lambda_n^{1/\alpha}$, and λ_n is the maximum eigenvalue of the Laplacian matrix L of graph $\mathcal{G}$.

Proof. Applying the Laplace transform to FOMAS (9), the following equation can be obtained:

$$s^\alpha I_n \Psi(s) - \Omega + \sum_{m=1}^{M} L_m \Psi(s) e^{-s\tau_m} = 0, \tag{11}$$

where $\Psi(s)$ is the Laplace transform of $\varphi(t)$ and $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix.

$$\Psi(s)\left(s^\alpha I_n + \sum_{m=1}^{M} L_m e^{-s\tau_m}\right) = \Omega, \tag{12}$$

where

$$\Omega = \begin{cases} s^{\alpha-1}\varphi(0^-), & \alpha \in (0,1] \\ s^{\alpha-1}\varphi(0^-) + s^{\alpha-2}\varphi'(0^-), & \alpha \in (1,2]. \end{cases} \tag{13}$$

Equation (12) can be further simplified as

$$\Psi(s) = \frac{\Omega}{G_{\tau_m}(s)}, \tag{14}$$

where

$$G_{\tau_m}(s) = s^\alpha I_n + \sum_{m=1}^{M} L_m e^{-s\tau_m}. \tag{15}$$

For an integer-order system, the roots of the characteristic polynomial $\det[G_{\tau_m}(s)]$ when $\alpha = 1$ are called eigenvalues of the system; if all nonzero eigenvalues of the system lie in the left half plane (LHP) of the complex plane, the system can reach a stable state. However, for a fractional-order system, is this stability criterion equally applicable? Matignon firstly studied this problem and pointed out that the stability of the linear steady-state fractional-order system could be judged by the eigenvalues of the system in [27]. Two years later, he gave a conjecture for the stability condition of the general linear fractional-order system, which was that all the poles of the system had negative real parts in [28]. This conjecture was later rigorously proved in [29] and further extended to the stability analysis of the delays and neutral delays for the fractional-order system. Therefore, the stability criterion of an integer-order system is also applicable to the stability criterion of a fractional-order system. In addition, consensus conditions of FOMAS (9) without time delays ($\tau_m = 0,\ m \in \{1, 2, \ldots, M\}$) are obtained in [13]. Thus, as τ_m increases continuously from zero, the eigenvalues of FOMAS (9) will change continuously from the LHP to the right half plane (RHP). Once the trajectories of these eigenvalues reach the RHP through the imaginary axis, FOMAS (9) will no longer be stable, which results in the failure of the consensus condition. So what we need to consider is the time delay when the nonzero eigenvalues of FOMAS (9) appear on the imaginary axis for the first time, and the time delay will become the critical point of FOMAS (9) stability, which is known as the delay margin. Therefore, if we assume that $s = -j\omega$ is the imaginary eigenvalue of FOMAS (9), $u \in \mathbb{R}^n$ is the corresponding eigenvector, and $\|u\| = 1$, then there is the following equation:

$$\left[(-j\omega)^{\alpha} I_n + \sum_{m=1}^{M} L_m e^{j\omega\tau_m}\right] u = 0. \quad (16)$$

Note that all the complex roots of each $G_{\tau_m}(s)$ appeared in conjugated pairs and we only need to study the situation where $\omega > 0$. On the left side of (16), multiply u^H (the conjugate transpose of u); the following equation can be obtained:

$$\begin{aligned}
&u^H \left[(-j\omega)^{\alpha} I_n + \sum_{m=1}^{M} L_m e^{j\omega\tau_m}\right] u = 0,\\
&u^H u (-j\omega)^{\alpha} + u^H \sum_{m=1}^{M} L_m e^{j\omega\tau_m} u = 0,\\
&\sum_{m=1}^{M} u^H L_m u e^{j\omega\tau_m} = -u^H u (-j\omega)^{\alpha},\\
&\sum_{m=1}^{M} \frac{u^H L_m u}{u^H u} e^{j\omega\tau_m} = -(-j\omega)^{\alpha} = -\omega^{\alpha}(-j)^{\alpha}\\
&\quad = -\omega^{\alpha}\left\{\cos\left(-\frac{\pi}{2}\right) + j\sin\left(-\frac{\pi}{2}\right)\right\}^{\alpha} = -\omega^{\alpha} e^{j(-\pi\alpha/2)}\\
&\quad = \omega^{\alpha} e^{j(\pi(2-\alpha)/2)}.
\end{aligned} \quad (17)$$

Then, we get

$$\sum_{m=1}^{M} a_m e^{j\omega\tau_m} = \omega^{\alpha} e^{j(\pi(2-\alpha)/2)} \triangleq F(\omega), \quad (18)$$

where

$$a_m = \frac{u^H L_m u}{u^H u}. \quad (19)$$

Taking the modulus of both sides of (18), we can get the following inequality:

$$M(\omega) \triangleq |F(\omega)| = \left|\sum_{m=1}^{M} a_m e^{j\omega\tau_m}\right| \le \sum_{m=1}^{M} a_m = \frac{u^H L u}{u^H u}. \quad (20)$$

According to Lemma 1, we can get

$$\frac{u^H L u}{u^H u} \le \lambda_n. \quad (21)$$

In addition, because $|F(\omega)| = \omega^{\alpha}$, therefore $\omega^{\alpha} \le \lambda_n$; that is, $\omega \le \lambda_n^{1/\alpha}$.

Next, we analyze the principal value of the argument of $F(\omega)$. According to (18), there is

$$\theta(\omega) \triangleq \arg[F(\omega)] = \frac{\pi(2-\alpha)}{2}, \quad (22)$$

where $\theta(\omega) \in [0, \pi]$. Let

$$\tau(\omega) \triangleq \frac{\theta(\omega)}{\omega} = \frac{\pi(2-\alpha)}{2\omega}, \quad (23)$$

and calculate the derivative of $\tau(\omega)$ about ω:

$$D_1(\omega) \triangleq \frac{d\tau(\omega)}{d\omega} = -\frac{\pi(2-\alpha)}{2\omega^2} < 0. \quad (24)$$

It shows that $\tau(\omega)$ is the decreasing function of ω. So, when $\omega \le \overline{\omega}$, there is

$$\overline{\tau} = \tau(\overline{\omega}) \le \tau(\omega). \quad (25)$$

What we need to be aware of is that the above conclusion is based on the assumption that the system eigenvalues exist on the imaginary axis. If we make all $\tau_m < \overline{\tau}$, there is

$$\begin{aligned}
\tau(\omega) &= \frac{\theta(\omega)}{\omega} = \frac{\arg\left(\sum_{m=1}^{M} a_m e^{j\omega\tau_m}\right)}{\omega} \le \frac{\max\{\omega\tau_m\}}{\omega}\\
&< \frac{\omega\overline{\tau}}{\omega} = \overline{\tau}.
\end{aligned} \quad (26)$$

Inequality (26) contradicts inequality (25). That is, as long as all τ_m are less than $\overline{\tau}$, we can avoid the eigenvalues of the FOMAS crossing the imaginary axis to reach the unstable RHP and the FOMAS can still reach consensus. On the other hand, when all $\tau_m = \overline{\tau}$, $-j\overline{\omega}$ is an imaginary eigenvalue of the FOMAS, whose corresponding eigenvector $u(\overline{\omega})$ makes $|\sum_{m=1}^{M} a_m| = \lambda_n$ hold, so $\overline{\tau}$ is the critical delay which is called the delay margin. When all $\tau_m > \overline{\tau}$, there must exist at least one eigenvalue of the FOMAS in the RHP. Then, according to the principle of stability, the states of the FOMAS are no longer convergent and the FOMAS with symmetric time delays cannot reach consensus. Proof is completed. □

Remark 5. For FOMAS (9) without time delays ($\tau_m = 0,\ m \in \{1, 2, \ldots, M\}$), consensus is achieved if an undirected interaction graph is connected and $\alpha \in (0, 2)$ in [13]. In addition, because $\overline{\tau} > 0$ in (10), we can derive that $\alpha < 2$. So, in order to achieve the consensus of FOMAS (9) with symmetric time delays, the fractional order should also satisfy $\alpha \in (0, 2)$.

Corollary 6. *Assume that the undirected interaction graph $\mathcal{G}$ is connected and the time delays are symmetric. When $\alpha = 1$, by the distributed control protocol (8), FOMAS (9) can reach consensus if all the time delays τ_m ($m = 1, 2, \ldots, M$) are less than $\overline{\tau}$, and FOMAS (9) cannot reach consensus if all the time delays τ_m ($m = 1, 2, \ldots, M$) are greater than $\overline{\tau}$, where*

$$\overline{\tau} = \frac{1}{2}\pi\lambda_n^{-1/\alpha}. \tag{27}$$

4.2. Consensus of FOMAS with Asymmetric Time Delays

Theorem 7. *Assume that the directed interaction graph $\mathcal{G}$ has a spanning tree and the time delays are asymmetric. By the distributed control protocol (8), FOMAS (9) can reach consensus if all the time delays τ_m ($m = 1, 2, \ldots, M$) are less than $\overline{\tau}$, and FOMAS (9) cannot reach consensus if all the time delays τ_m ($m = 1, 2, \ldots, M$) are greater than $\overline{\tau}$, where*

$$\overline{\tau} = \min_{\|\lambda_i\|\neq 0}\left\{\frac{\pi(2-\alpha)/2 - \arg(\lambda_i)}{\overline{\omega}_i}\right\}, \tag{28}$$

$\alpha \in (0, 2\theta/\pi)$, $\theta = \min\{\pi - \arg(\lambda_i)\}$ ($\lambda_i \neq 0$, $i = 1, 2, \ldots, n$), $\overline{\omega}_i = \|\lambda_i\|^{1/\alpha}$, and λ_i is the ith eigenvalue of the Laplacian matrix L of the graph $\mathcal{G}$.

Proof. We use the frequency-domain proof method, which has been used in proving Theorem 4. Suppose that $s = -j\omega \neq 0$ is the eigenvalue of FOMAS (9) on the imaginary axis, $u \in \mathbb{R}^n$ is the corresponding eigenvector, and $\|u\| = 1$. By similar methods and Lemma 2, we can get the following equation:

$$\begin{aligned} B_a &\triangleq \sum_{m=1}^{M} a_m e^{j\omega\tau_m} = -(-j\omega)^{\alpha} = (-1)(-j)^{\alpha}(\omega)^{\alpha} \\ &= e^{j\pi}e^{j(-\pi\alpha/2)}(\omega)^{\alpha} = \omega^{\alpha}e^{j((2\pi-\pi\alpha)/2)} \\ &= \omega^{\alpha}e^{j(\pi(2-\alpha)/2)}. \end{aligned} \tag{29}$$

By taking the modulus of both sides of (29) and regarding ω as the function of $\|B_a\|$, we can get

$$\omega(\|B_a\|) = \|B_a\|^{1/\alpha}, \tag{30}$$

where $\omega(\|B_a\|)$ is an increasing function of $\|B_a\|$.

Calculating the principal value of the argument of (29) on both sides separately, we can get

$$\arg(B_a) = \frac{\pi(2-\alpha)}{2}. \tag{31}$$

According to the definition of B_a in (29), we can get

$$\arg(B_a) \leq \arg\left(\sum_{m=1}^{M} a_m\right) + \max(\omega\tau_m), \tag{32}$$

so, there is

$$\max(\omega\tau_m) \geq \frac{\pi(2-\alpha)}{2} - \arg\left(\sum_{m=1}^{M} a_m\right). \tag{33}$$

Due to $\sum_{m=1}^{M} a_m = u^H Lu/u^H u$, the possible values of $\sum_{m=1}^{M} a_m$ must be nonzero eigenvalues of the Laplacian matrix L of the graph $\mathcal{G}$; that is, $\sum_{m=1}^{M} a_m = \lambda_i$, ($\lambda_i \neq 0$). So, when $\|B_a\| \leq \|\lambda_i\|$, $\omega(\|B_a\|) \leq \omega(\|\lambda_i\|) = \overline{\omega}_i = \|\lambda_i\|^{1/\alpha}$. If we let all $\tau_m < \overline{\tau}$, there is

$$\begin{aligned} \max(\omega\tau_m) &< \overline{\omega}_i\overline{\tau} \\ &= \min_{\|\lambda_i\|\neq 0}\left\{\frac{[\pi(2-\alpha)/2 - \arg(\lambda_i)]}{\overline{\omega}_i}\right\}\overline{\omega}_i \\ &= \min\left\{\frac{[\pi(2-\alpha)/2 - \arg(\sum_{m=1}^{M} a_m)]}{\overline{\omega}_i}\right\}\overline{\omega}_i \\ &\leq \frac{\pi(2-\alpha)}{2} - \arg\left(\sum_{m=1}^{M} a_m\right). \end{aligned} \tag{34}$$

Inequality (34) contradicts inequality (33). That is, when all $\tau_m < \overline{\tau}$, the eigenvalues of the FOMAS with asymmetric time delays cannot reach or cross the imaginary axis, and the FOMAS will remain stable and the consensus of the FOMAS can be achieved. On the other hand, when all $\tau_m > \overline{\tau}$, there must exist at least one eigenvalue of the FOMAS in the RHP, and then the states of the FOMAS are no longer convergent and the FOMAS with asymmetric time delays cannot reach consensus. Proof is completed. □

Remark 8. For FOMAS (9) without time delays ($\tau_m = 0$, $m \in \{1, 2, \ldots, M\}$), consensus is achieved if the directed interaction graph $\mathcal{G}$ has a spanning tree and $\alpha \in (0, 2\theta/\pi)$, $\theta = \min\{\pi - \arg(\lambda_i)\}$ ($\lambda_i \neq 0$, $i = 1, 2, \ldots, n$) in [13]. In addition, because $\overline{\tau} > 0$ in (28), we can derive that $\alpha < 2\theta/\pi$, $\theta = \min\{\pi - \arg(\lambda_i)\}$ ($\lambda_i \neq 0$, $i = 1, 2, \ldots, n$). So, in order to achieve the consensus of FOMAS (9) with asymmetric time delays, the fractional order should also satisfy $\alpha \in (0, 2\theta/\pi)$, $\theta = \min\{\pi - \arg(\lambda_i)\}$ ($\lambda_i \neq 0$, $i = 1, 2, \ldots, n$).

Corollary 9. *Assume that the directed interaction graph $\mathcal{G}$ has a spanning tree and the time delays are asymmetric. When $\alpha = 1$, by the distributed protocol (8), FOMAS (9) can reach consensus if all the time delays τ_m ($m = 1, 2, \ldots, M$) are less than $\overline{\tau}$, and FOMAS (9) cannot reach consensus if all the time delays τ_m ($m = 1, 2, \ldots, M$) are greater than $\overline{\tau}$, where*

$$\overline{\tau} = \min_{\|\lambda_i\|\neq 0}\left\{\frac{\pi/2 - \arg(\lambda_i)}{\|\lambda_i\|^{1/\alpha}}\right\}. \tag{35}$$

5. Simulation Results

To illustrate the correctness of the theoretical results, numerical simulations will be given in this section.

5.1. Example 1: Simulations for Theorem 4. Consider a FOMAS with four agents, whose dynamics is described by (9) with $\alpha = 0.7$. The connected interaction graph $\mathcal{G}$ of FOMAS

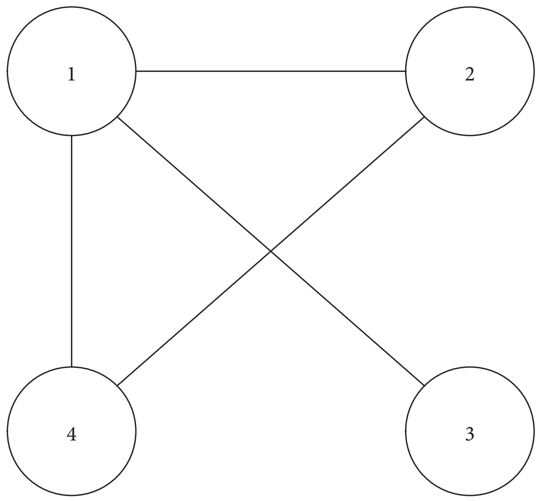

Figure 1: The connected interaction topology in Example 1.

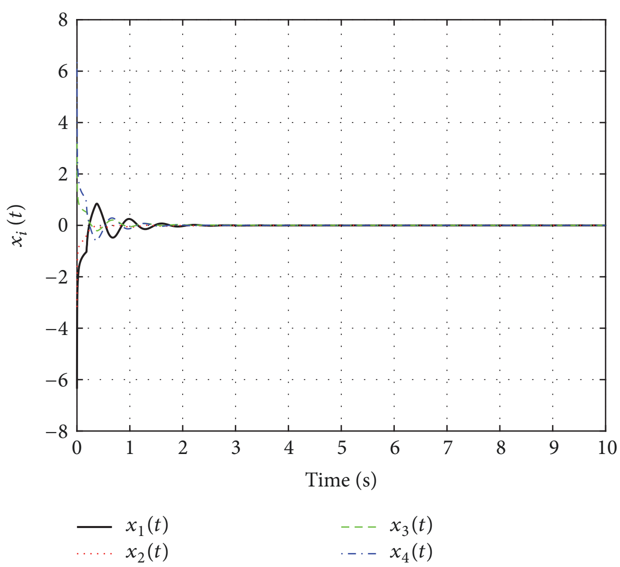

Figure 2: The trajectories of $x_i(t)$ when (1) all $\tau_m < \overline{\tau}$ in Example 1.

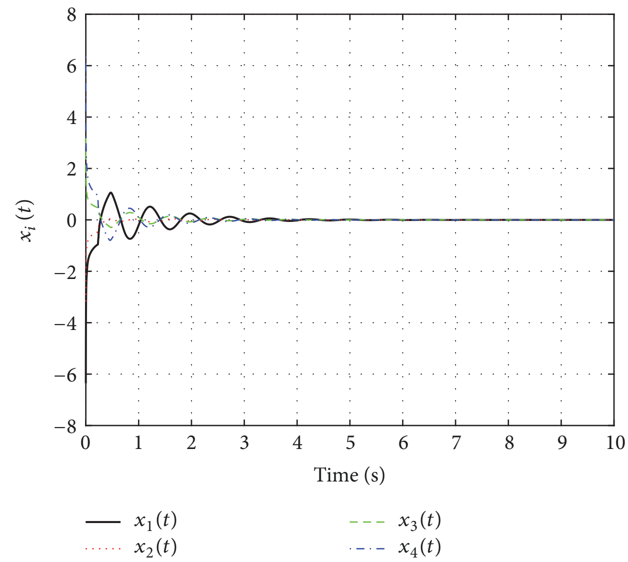

Figure 3: The trajectories of $x_i(t)$ when (2) all $\tau_m < \overline{\tau}$ in Example 1.

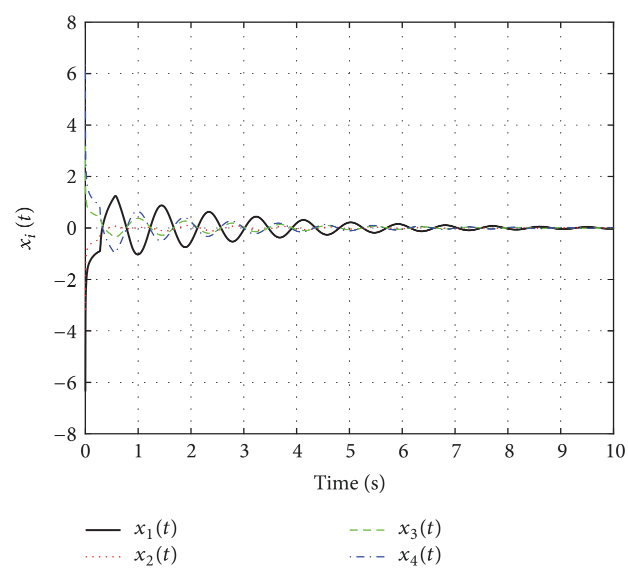

Figure 4: The trajectories of $x_i(t)$ when (3) all $\tau_m < \overline{\tau}$ in Example 1.

(9) is undirected and shown in Figure 1, whose Laplacian matrix is as follows:

$$L = \begin{bmatrix} 2.6 & -0.7 & -0.9 & -1 \\ -0.7 & 1.5 & 0 & -0.8 \\ -0.9 & 0 & 0.9 & 0 \\ -1 & -0.8 & 0 & 1.8 \end{bmatrix}. \quad (36)$$

According to Theorem 4, the delay margin $\overline{\tau}$ of the FOMAS is 0.3379 s. Assume that the initial states of the FOMAS are taken as $x_1(t = 0) = -6.4$, $x_2(t = 0) = -3.2$, $x_3(t = 0) = 3.2$, and $x_4(t = 0) = 6.4$. Then, three sets of different time delays are used in the simulation:

(1) $\tau_{12} = \tau_{21} = 0.2$ s, $\tau_{13} = \tau_{31} = 0.19$ s, $\tau_{14} = \tau_{41} = 0.18$ s, $\tau_{24} = \tau_{42} = 0.17$ s.

(2) $\tau_{12} = \tau_{21} = 0.25$ s, $\tau_{13} = \tau_{31} = 0.24$ s, $\tau_{14} = \tau_{41} = 0.23$ s, $\tau_{24} = \tau_{42} = 0.22$ s.

(3) $\tau_{12} = \tau_{21} = 0.3$ s, $\tau_{13} = \tau_{31} = 0.29$ s, $\tau_{14} = \tau_{41} = 0.28$ s, $\tau_{24} = \tau_{42} = 0.27$ s.

Figures 2, 3, and 4 show the trajectories of all the agents' states $x_i(t)$ in Example 1. It is obvious that the FOMAS with symmetric time delays can reach consensus. Furthermore, by comparing the simulation results of Figures 2, 3, and 4, we can find that when all the time delays stay away from the delay margin $\overline{\tau}$ when the communication topology and the fractional order α of the FOMAS stay the same, the convergence speed will become faster, whereas the convergence speed will become slower when all the time delays get close to the delay margin $\overline{\tau}$. Similarly, we change the communication topology, which changes the eigenvalues of the Laplace matrix of the communication topology, and the delay margin $\overline{\tau}$ also increases (decreases) with the decrease

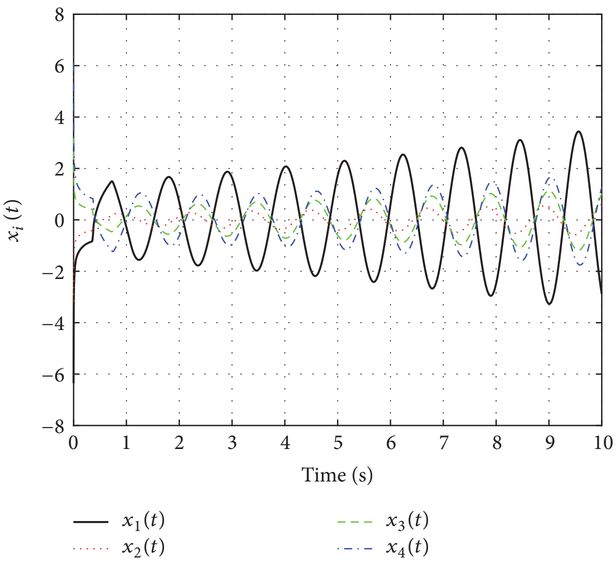

Figure 5: The trajectories of $x_i(t)$ when all $\tau_m > \overline{\tau}$ in Example 1.

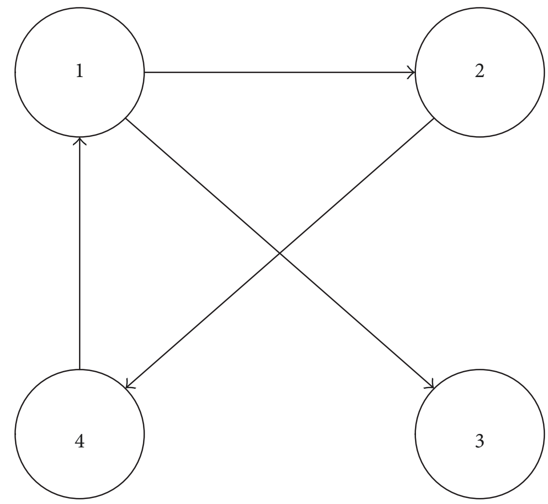

Figure 6: The connected interaction topology in Example 2.

(increase) of the maximum eigenvalue of all the eigenvalues of the Laplace matrix of the communication topology when a set of valid time delays and the fractional order α of the FOMAS stay the same. This will result in the change of distance between the taken valid time delays and the delay margin $\overline{\tau}$, which will eventually lead to the similar rules above.

On the other hand, in order to make a comparison, under the same conditions, we suppose that $\tau_{12} = \tau_{21} = 0.35$ s, $\tau_{13} = \tau_{31} = 0.36$ s, $\tau_{14} = \tau_{41} = 0.37$ s, and $\tau_{24} = \tau_{42} = 0.38$ s.

Figure 5 shows the trajectories of all the agents' states $x_i(t)$. It is clear that the FOMAS cannot reach consensus.

All of the above simulation results are consistent with Theorem 4. So, the correctness of Theorem 4 is validated.

5.2. Example 2: Simulations for Theorem 7. Consider a FOMAS with four agents, whose dynamics is described by (9) with $\alpha = 0.7$. The connected interaction graph $\mathscr{G}$ of FOMAS (9) is directed and shown in Figure 6, whose Laplacian matrix is as follows:

$$L = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -0.7 & 0.7 & 0 & 0 \\ -0.9 & 0 & 0.9 & 0 \\ 0 & -0.8 & 0 & 0.8 \end{bmatrix}. \quad (37)$$

According to Theorem 7, the delay margin $\overline{\tau}$ of the FOMAS is 0.9120 s. Assume that the initial states of the FOMAS are taken as $x_1(t = 0) = -6.4$, $x_2(t = 0) = -3.2$, $x_3(t = 0) = 3.2$, and $x_4(t = 0) = 6.4$. Similarly, three sets of different time delays are used in the simulation:

(1) $\tau_{14} = 0.61$ s, $\tau_{21} = 0.64$ s, $\tau_{31} = 0.67$ s, $\tau_{42} = 0.7$ s.

(2) $\tau_{14} = 0.71$ s, $\tau_{21} = 0.74$ s, $\tau_{31} = 0.77$ s, $\tau_{42} = 0.8$ s.

(3) $\tau_{14} = 0.81$ s, $\tau_{21} = 0.84$ s, $\tau_{31} = 0.87$ s, $\tau_{42} = 0.9$ s.

Figures 7, 8, and 9 show the trajectories of all the agents' states $x_i(t)$ in Example 2. It is obvious that the FOMAS with asymmetric time delays can reach consensus. Moreover, by comparing the simulation results of Figures 7, 8, and 9, we also find that when all the time delays stay away from the delay margin $\overline{\tau}$ when the communication topology and the fractional order α of the FOMAS stay the same, the convergence speed will become faster, whereas the convergence speed will become slower when all the time delays get close to the delay margin $\overline{\tau}$. Similarly, we change the communication topology, which changes the eigenvalues of the Laplace matrix of the communication topology, and the delay margin $\overline{\tau}$ is also changed with the change of eigenvalues when a set of valid time delays and the fractional order α of the FOMAS stay the same. This will result in the change of distance between the taken valid time delays and the delay margin $\overline{\tau}$, which will eventually lead to the similar rules above, too.

Now, in order to make a comparison, under the same conditions, we suppose that $\tau_{14} = 1$ s, $\tau_{21} = 1.1$ s, $\tau_{31} = 1.2$ s, and $\tau_{42} = 1.3$ s.

Figure 10 shows the trajectories of all the agents' states $x_i(t)$. It is clear that the FOMAS cannot reach consensus.

All of the above simulation results are consistent with Theorem 7. So, the correctness of Theorem 7 is validated.

6. Conclusion

In this paper, the consensus problems of the FOMAS with nonuniform time delays are studied. First of all, the consensus problem is investigated for the FOMAS with symmetric time delays and undirected topology, and the time-delay margin is obtained to guarantee the consensus for this FOMAS. Then, the consensus problem is investigated for the FOMAS with asymmetric time delays and directed topology, and the time-delay margin is obtained to guarantee the consensus for this FOMAS. Furthermore, the relationship between the speed of convergence and communication topology and the

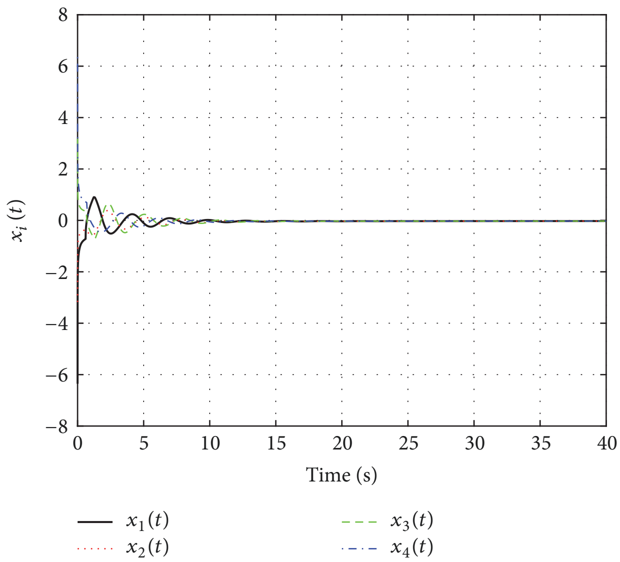

FIGURE 7: The trajectories of $x_i(t)$ when (1) all $\tau_m < \overline{\tau}$ in Example 2.

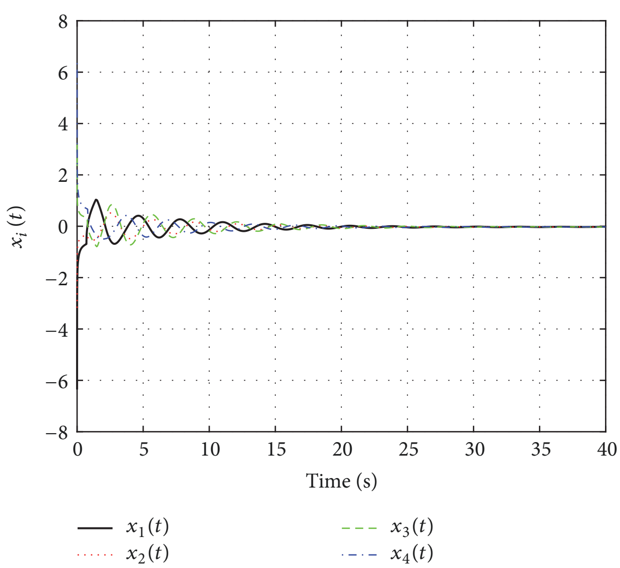

FIGURE 8: The trajectories of $x_i(t)$ when (2) all $\tau_m < \overline{\tau}$ in Example 2.

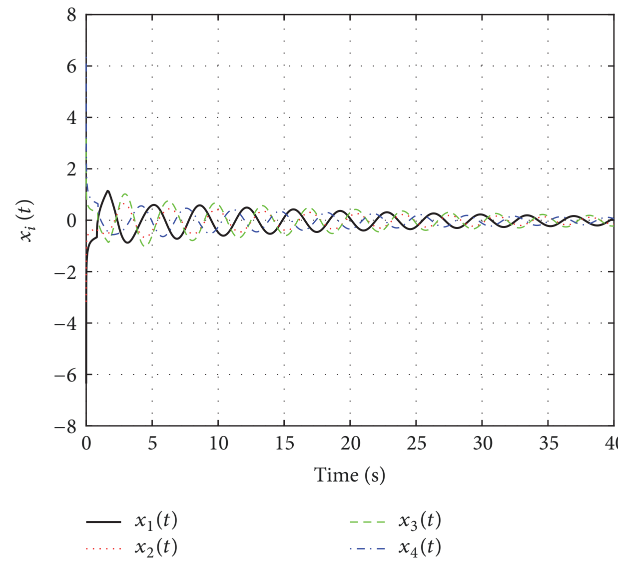

FIGURE 9: The trajectories of $x_i(t)$ when (3) all $\tau_m < \overline{\tau}$ in Example 2.

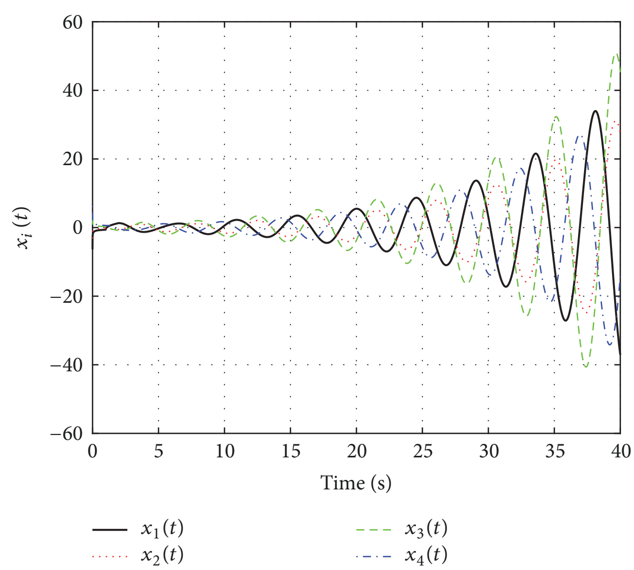

FIGURE 10: The trajectories of $x_i(t)$ when all $\tau_m > \overline{\tau}$ in Example 2.

time delays is revealed. And extending the above results, two corollaries are obtained for the corresponding integer-order multiagent systems, which are the same as traditional integer-order systems. Finally, the correctness of our theoretical results is validated by the simulations.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the Science and Technology Department Project of Sichuan Province of China (Grants nos. 2014FZ0041, 2017GZ0305, and 2016GZ0220), the Education Department Key Project of Sichuan Province of China (Grant no. 17ZA0077), and the National Natural Science Foundation of China (Grant no. U1733103).

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