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Strong convergence of a general iterative algorithm for a finite family of accretive operators in Banach spaces

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Abstract

The purpose of this paper is to present a new iterative scheme for finding a common solution to a variational inclusion problem with a finite family of accretive operators and a modified system of variational inequalities in infinite-dimensional Banach spaces. Under mild conditions, a strong convergence theorem for approximating this common solution is proved. The methods in the paper are novel and different from those in the early and recent literature.

MSC: 47H09; 47H10; 47H17

Keywords: iterative algorithm; strong convergence; fixed point; q-uniformly smooth Banach space; inverse-strongly accretive operator

1 Introduction

Variational inequalities theory, which was introduced by Stampacchia [1] in the early 1960s, has emerged as an interesting and fascinating branch of applicable mathematics with a wide range of applications in industry, finance, economics, social, pure and applied sciences. It has been shown that this theory provides the most natural, direct, simple, unified and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problems; see, for example, [2–5] and the references therein. Variational inequalities have been extended and generalized in several directions using novel and new techniques.

In 1968, Brézis [6] initiated the study of the existence theory of a class of variational inequalities, later known as variational inclusions, using proximal-point mappings due to Moreau [7]. Variational inclusions include variational, quasi-variational, variational-like inequalities as special cases. Variational inclusions can be viewed as an innovative and novel extension of the variational principles and thus have wide applications in the fields of optimization, control, economics and engineering sciences.

In recent years, much attention has been given to study the system of variational inclusions/inequalities, which occupies a central and significant role in the interdisciplinary research among analysis, geometry, biology, elasticity, optimization, imaging processing, biomedical sciences and mathematical physics. One can see an immense breadth of mathematics and its simplicity in the works of this research. A number of problems leading to the system of variational inclusions/inequalities arise in applications to variational problems and engineering. It is well known that the system of variational inclusions/inequalities can provide new insight regarding problems being studied and can stimulate new and innovative ideas for problem solving.

In 2000, Ansari and Yao [8] introduced the system of generalized implicit variational inequalities and proved the existence of its solution. They derived the existence results for a solution of system of generalized variational inequalities, from which they established the existence of a solution of system of optimization problems.

Ansari *et al.* [9] introduced the system of vector equilibrium problems and proved the existence of its solution. Moreover, they also applied their results to the system of vector variational inequalities. The results of [8] and [9] were used as tools to solve the Nash equilibrium problem for non-differentiable and (non)convex vector-valued functions.

Let $A, B : C \to E$ be two nonlinear mappings. In 2010, Yao *et al.* [10] introduced a system of general variational inequalities problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, \quad \forall x \in C, \\ \langle Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0, \quad \forall x \in C. \end{cases} \tag{1.1}$$

In 2-uniformly smooth Banach spaces, Kangtunyakarn [11], recently, introduced a new system of variational inequalities problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - (I - \lambda A)(ax^* + (1 - a)y^*), j(x - x^*)\rangle \geq 0, \quad \forall x \in C, \\ \langle y^* - (I - \mu B)x^*, j(x - y^*)\rangle \geq 0, \quad \forall x \in C. \end{cases} \tag{1.2}$$

If $a = 0$, then problem (1.2) reduces to the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j(x - x^*)\rangle \geq 0, \quad \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, j(x - y^*)\rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.3}$$

which is introduced by Cai and Bu [12]. In Hilbert spaces, problem (1.3) reduces to the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, \quad \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^*\rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.4}$$

which is introduced by Ceng *et al.* [13]. If $A = B$, then problem (1.4) collapses the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, \quad \forall x \in C, \\ \langle \mu Ax^* + y^* - x^*, x - y^*\rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.5}$$

which is introduced by Verma [14]. In particular, if we let $x^* = y^*$ in (1.5), then problem (1.4) is nothing but the classical variational inequality problem: find $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{1.6}$$

The set of solutions of problem (1.6) is denoted by $\mathrm{VI}(C, A)$.

Motivated by the works mentioned above, we shall consider the following problem in q-uniformly smooth Banach spaces: find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle x^* - (I - \lambda A)(ax^* + (1-a)y^*), j_q(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \mu B)x^*, j_q(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.7}$$

where $a \in [0,1]$, $\lambda > 0$ and $\mu > 0$ are three constants. This problem is called a modified system of variational inequalities, which clearly includes problems (1.1)-(1.6) as special cases.

In order to find a common element of the set of solutions of problem (1.2) and the set of fixed points of nonlinear operators, Kangtunyakarn [11] also studied the following algorithm in a 2-uniformly smooth Banach space:

$$x_{n+1} = G\bigl(\alpha_n u + \beta_n x_n + \gamma_n S^A x_n\bigr), \quad \forall n \geq 1, \tag{1.8}$$

where S^A is the S^A-mapping generated by $S_1, S_2, \ldots, S_N$, $T_1, T_2, \ldots, T_N$, $G : C \to C$ is the mapping defined by $Gx = Q_C(I - \lambda A)(aI + (1-a)Q_C(I - \mu B))x$, and Q_C is a sunny nonexpansive retraction of E onto C. Then, under mild conditions, they established a strong convergence theorem.

On the other hand, we know that the quasi-variational inclusion problem in the setting of Hilbert spaces has been extensively studied in the literature; see, for instance, [15–23]. There is, however, little work in the existing literature on this problem in the setting of Banach spaces. The main difficulties are due to the fact that the inner product structure of Hilbert spaces fails to be true in Banach spaces. To overcome these difficulties, López *et al.* [24] used a new technique to carry out certain initiative investigations on splitting methods for accretive operators in Banach spaces. They considered the following algorithms with errors in Banach spaces:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\bigl(J_{r_n}\bigl(x_n - r_n(Ax_n + a_n)\bigr) + b_n\bigr) \tag{1.9}$$

and

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\bigl(J_{r_n}\bigl(x_n - r_n(Ax_n + a_n)\bigr) + b_n\bigr), \tag{1.10}$$

where $u \in E$, $\{a_n\}, \{b_n\} \subset E$ and $J_{r_n} = (I + r_n B)^{-1}$ is the resolvent of B. Then they established the weak and strong convergence of algorithms (1.9) and (1.10), respectively.

Recently, Khuangsatung and Kangtunyakarn [25] introduced the following algorithm in Hilbert spaces for finding a common element of the set of fixed points of a k-strictly pseudononspreading mapping, the set of solutions of a finite family of variational inclusion problems and the set of solutions of a finite family of equilibrium problems:

$$\begin{cases} \sum_{i=1}^N \alpha_i \Psi_i(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - w_n\rangle \geq 0, \quad \forall y \in C, \\ w_{n+1} = \alpha_n \mu + \beta_n w_n + \gamma_n J_{M,\lambda}(I - \lambda \sum_{i=1}^N b_i A_i)w_n \\ \qquad + \eta_n(I - \rho_n(I - S))w_n + \delta_n z_n, \quad \forall n \geq 1. \end{cases} \tag{1.11}$$

And, under suitable conditions, they proved the strong convergence of the sequence $\{w_n\}$.

Motivated and inspired by Zhang *et al.* [19], Qin *et al.* [26], López *et al.* [24], Takahashi *et al.* [27] and Khuangsatung and Kangtunyakarn [25], we suggest and analyze a new iterative algorithm for finding a common solution to a variational inclusion problem with a finite family of accretive operators and a modified system of variational inequalities in infinite-dimensional Banach spaces. We also prove the convergence analysis of the proposed algorithm under some suitable conditions. The results obtained in this paper improve and extend the corresponding results announced by many others.

2 Preliminaries

Throughout this paper, we denote by E and E^* a real Banach space and the dual space of E, respectively. We use $\mathrm{Fix}(T)$ to denote the set of fixed points of T and $\mathscr{B}_r$ to denote the closed ball with center zero and radius r. Let C be a subset of E and $q > 1$ be a real number. The (generalized) duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}$$

for all $x \in E$, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*. It is well known that if E is smooth, then J_q is single-valued, which is denoted by j_q.

Let C be a nonempty closed convex subset of a real Banach space E. Let $A : E \to E$ be a single-valued nonlinear mapping, and let $M : E \to 2^E$ be a multivalued mapping. The so-called quasi-variational inclusion problem is to find a $z \in E$ such that

$$0 \in (A + M)z. \tag{2.1}$$

The set of solutions of (2.1) is denoted by $\mathrm{VI}(E, A, M)$.

Definition 2.1 Let E be a Banach space. Then a function $\delta_E : [0, 2] \to [0, 1]$ is said to be the modulus of convexity of E if

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x - y\| \ge \epsilon\right\}.$$

If $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$, then E is uniformly convex.

Definition 2.2 The function $\rho_E : [0, 1) \to [0, 1)$ is said to be the modulus of smoothness of E if

$$\rho_E(t) = \sup\left\{\frac{1}{2}\big(\|x + ty\| + \|x - ty\|\big) - 1 : \|x\| = \|y\| = 1\right\}.$$

A Banach space E is said to be:

(1) uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$;

(2) q-uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho_E(t) \le ct^q$, where $q \in (1, 2]$.

It is known that a uniformly convex Banach space is reflexive and strictly convex.

Definition 2.3 A mapping $T : C \to E$ is said to be:

(1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C;$$

(2) r-contractive if there exists $r \in [0,1)$ such that

$$\|Tx - Ty\| \leq r\|x - y\| \quad \text{for all } x, y \in C;$$

(3) accretive if for all $x, y \in C$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \geq 0;$$

(4) η-strongly accretive if for all $x, y \in C$, there exist $\eta > 0$ and $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \geq \eta\|x - y\|^q;$$

(5) μ-inverse-strongly accretive if for all $x, y \in C$, there exist $\mu > 0$ and $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \geq \mu\|Tx - Ty\|^q.$$

Definition 2.4 A set-valued mapping $T : \mathrm{Dom}(T) \to 2^E$ is said to be:

(1) accretive if for any $x, y \in \mathrm{Dom}(T)$, there exists $j_q(x-y) \in J_q(x-y)$ such that for all $u \in T(x)$ and $v \in T(y)$,

$$\langle u - v, j_q(x-y)\rangle \geq 0;$$

(2) m-accretive if T is accretive and $(I + \rho T)(\mathrm{Dom}(T)) = E$ for every (equivalently, for some) $\rho > 0$, where I is the identity mapping.

Let $M : \mathrm{Dom}(M) \to 2^E$ be m-accretive. The mapping $J_{M,\rho} : E \to \mathrm{Dom}(M)$ defined by

$$J_{M,\rho}(u) = (I + \rho M)^{-1}(u), \quad \forall u \in E,$$

is called the resolvent operator associated with M, where ρ is any positive number and I is the identity mapping. It is well known that $J_{M,\rho}$ is single-valued and nonexpansive.

We need some facts and tools which are listed as lemmas below.

Lemma 2.5 ([28]) *Let E be a Banach space and J_q be a generalized duality mapping. Then, for any given $x, y \in E$, the following inequality holds*:

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x+y)\rangle, \quad j_q(x+y) \in J_q(x+y).$$

Lemma 2.6 ([29]) *Let $\{\alpha_n\}$ be a sequence of nonnegative numbers satisfying the property*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + b_n + \gamma_n c_n, \quad n \in \mathbb{N},$$

where $\{\gamma_n\}$, $\{b_n\}$, $\{c_n\}$ satisfy the restrictions:

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$, $\lim_{n\to\infty} \gamma_n = 0$;
(ii) $b_n \geq 0$, $\sum_{n=1}^{\infty} b_n < \infty$;
(iii) $\limsup_{n\to\infty} c_n \leq 0$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.7 ([28]) *Let* $1 < p < \infty$, $q \in (1,2]$, $r > 0$ *be given. If* E *is a real* q*-uniformly smooth Banach space, then there exists a constant* $C_q > 0$ *such that*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + C_q\|y\|^q, \quad \forall x, y \in E.$$

Lemma 2.8 ([30]) *Let* C *be a nonempty closed convex subset of a real* q*-uniformly smooth Banach space* E. *Let the mapping* $A : C \to E$ *be an* α*-inverse-strongly accretive operator. Then the following inequality holds*:

$$\|(I - \lambda A)x - (I - \lambda A)y\|^q \leq \|x - y\|^q - \lambda(q\alpha - C_q\lambda^{q-1})\|Ax - Ay\|^q.$$

In particular, if $0 < \lambda \leq (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$, *then* $I - \lambda A$ *is nonexpansive.*

Recall that if C and D are nonempty subsets of a Banach space E such that C is closed convex and $D \subset C$, then a mapping $Q : C \to D$ is sunny [31] provided

$$Q(x + t(x - Q(x))) = Q(x)$$

for all $x \in C$ and $t \geq 0$, whenever $Qx + t(x - Q(x)) \in C$. A mapping $Q : C \to D$ is called a retraction if $Qx = x$ for all $x \in D$. Furthermore, Q is a sunny nonexpansive retraction from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive. A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D. The following lemma collects some properties of the sunny nonexpansive retraction.

Lemma 2.9 ([30, 31]) *Let* C *be a closed convex subset of a smooth Banach space* E. *Let* D *be a nonempty subset of* C. *Let* $Q : C \to D$ *be a retraction and let* j, j_q *be the normalized duality mapping and generalized duality mapping on* E, *respectively. Then the following are equivalent*:

(i) Q *is sunny and nonexpansive*;
(ii) $\|Qx - Qy\|^2 \leq \langle x - y, j(Qx - Qy)\rangle$, $\forall x, y \in C$;
(iii) $\langle x - Qx, j(y - Qx)\rangle \leq 0$, $\forall x \in C$, $y \in D$;
(iv) $\langle x - Qx, j_q(y - Qx)\rangle \leq 0$, $\forall x \in C$, $y \in D$.

Lemma 2.10 *Let* $A : C \to E$ *and* $M : C \supseteq \mathrm{Dom}(M) \to 2^E$ *be two nonlinear operators. Denote* J_r *by*

$$J_r := J_{M,r} = (I + rM)^{-1}$$

and T_r *by*

$$T_r := J_r(I - rA) = (I + rM)^{-1}(I - rA).$$

Then it holds for all $r > 0$ *that* $\mathrm{Fix}(T_r) = \mathrm{VI}(E, A, M)$.

Proof From the definition of T_r, it follows that

$$\begin{aligned} x = T_r x \quad &\Longleftrightarrow \quad x = (I + rM)^{-1}(I - rA)x \\ &\Longleftrightarrow \quad (I - rA)x \in (I + rM)x \\ &\Longleftrightarrow \quad 0 \in (A + M)x \\ &\Longleftrightarrow \quad x \in \mathrm{VI}(E, A, M). \end{aligned}$$

This completes the proof. □

Lemma 2.11 ([24]) *Assume that C is a nonempty closed subset of a real uniformly convex and q-uniformly smooth Banach space E. Suppose that* $A : C \to E$ *is* α*-inverse-strongly accretive and M is an m-accretive operator in E, with* $\mathrm{Dom}(M) \subseteq C$. *Then it holds that*:

(i) *Given* $0 < s \le r$ *and* $x \in E$,

$$\|T_s x - T_r x\| \le \left|1 - \frac{s}{r}\right| \|x - T_r x\| \quad \text{and} \quad \|x - T_s x\| \le 2\|x - T_r x\|.$$

(ii) *Given* $k > 0$, *there exists a continuous, strictly increasing and convex function* $\phi_q : [0, \infty) \to [0, \infty)$ *with* $\phi_q(0) = 0$ *such that for all* $x, y \in \mathscr{B}_k$,

$$\begin{aligned} \|T_r x - T_r y\|^q \le{} & \|x - y\|^q - r\bigl(\alpha q - r^{q-1}C_q\bigr)\|Ax - Ay\|^q \\ & - \phi_q\bigl(\bigl\|(I - J_r)(I - rA)x - (I - J_r)(I - rA)y\bigr\|\bigr). \end{aligned}$$

3 Main results

For every $i = 1, 2, \ldots, N$, let $A_i : C \to E$ and $M : C \supseteq \mathrm{Dom}(M) \to 2^E$ be nonlinear mappings. From (2.1), we introduce the combination of variational inclusion problems in Banach spaces as follows: find a point $x^* \in C$ such that

$$0 \in \left(\sum_{i=1}^{N} \lambda_i A_i + M\right) x^*, \tag{3.1}$$

where λ_i is a real positive number for all $i = 1, 2, \ldots, N$ with $\sum_{i=1}^{N} \lambda_i = 1$. The set of solutions of (3.1) in Banach spaces is denoted by $\mathrm{VI}(E, \sum_{i=1}^{N} \lambda_i A_i, M)$.

To prove the strong convergence results, we also need the following four lemmas.

Lemma 3.1 *Let C be a nonempty closed convex subset of a real smooth Banach space E. Let* $N \ge 1$ *be some positive integer,* $A_i : C \to E$ *be* η_i*-inverse-strongly accretive with* $\eta = \min\{\eta_1, \eta_2, \ldots, \eta_N\}$, *and M be m-accretive in E with* $\mathrm{Dom}(M) \subseteq C$. *Let* $\{\lambda_i\}$ *be a real number sequence in* $(0, 1)$ *with* $\sum_{i=1}^{N} \lambda_i = 1$ *and* $\mathrm{VI}(E, \sum_{i=1}^{N} \lambda_i A_i, M) \neq \emptyset$. *Then*

$$\mathrm{VI}\left(E, \sum_{i=1}^{N} \lambda_i A_i, M\right) = \bigcap_{i=1}^{N} \mathrm{VI}(E, A_i, M).$$

Proof It is obvious that $\bigcap_{i=1}^N \mathrm{VI}(E, A_i, M) \subseteq \mathrm{VI}(E, \sum_{i=1}^N \lambda_i A_i, M)$. Next we prove that

$$\mathrm{VI}\left(E, \sum_{i=1}^N \lambda_i A_i, M\right) \subseteq \bigcap_{i=1}^N \mathrm{VI}(E, A_i, M).$$

Suppose that $x_1 \in \mathrm{VI}(E, \sum_{i=1}^N \lambda_i A_i, M)$ and $x_2 \in \bigcap_{i=1}^N \mathrm{VI}(E, A_i, M)$. We have from Lemma 2.10 that

$$x_1 \in \mathrm{Fix}\left(J_r\left(I - r\sum_{i=1}^N \lambda_i A_i\right)\right).$$

Since $\bigcap_{i=1}^N \mathrm{VI}(E, A_i, M) \subseteq \mathrm{VI}(E, \sum_{i=1}^N \lambda_i A_i, M)$, we have $x_2 \in \mathrm{VI}(E, \sum_{i=1}^N \lambda_i A_i, M)$. Again, from Lemma 2.10, we have

$$x_2 \in \mathrm{Fix}\left(J_r\left(I - r\sum_{i=1}^N \lambda_i A_i\right)\right).$$

In light of the nonexpansiveness of J_r, we deduce that

$$\begin{aligned}
\|x_1 - x_2\|^q &= \left\| J_r\left(I - r\sum_{i=1}^N \lambda_i A_i\right)x_1 - J_r\left(I - r\sum_{i=1}^N \lambda_i A_i\right)x_2 \right\|^q \\
&\le \left\| \left(I - r\sum_{i=1}^N \lambda_i A_i\right)x_1 - \left(I - r\sum_{i=1}^N \lambda_i A_i\right)x_2 \right\|^q \\
&= \left\| (x_1 - x_2) - r\left(\sum_{i=1}^N \lambda_i A_i x_1 - \sum_{i=1}^N \lambda_i A_i x_2\right) \right\|^q \\
&\le \|x_1 - x_2\|^q - qr\sum_{i=1}^N \lambda_i \langle A_i x_1 - A_i x_2, j_q(x_1 - x_2)\rangle \\
&\quad + C_q r^q \sum_{i=1}^N \lambda_i \|A_i x_1 - A_i x_2\|^q \\
&\le \|x_1 - x_2\|^q - qr\sum_{i=1}^N \lambda_i \eta_i \|A_i x_1 - A_i x_2\|^q + C_q r^q \sum_{i=1}^N \lambda_i \|A_i x_1 - A_i x_2\|^q \\
&\le \|x_1 - x_2\|^q - qr\eta\sum_{i=1}^N \lambda_i \|A_i x_1 - A_i x_2\|^q + C_q r^q \sum_{i=1}^N \lambda_i \|A_i x_1 - A_i x_2\|^q \\
&\le \|x_1 - x_2\|^q - r\sum_{i=1}^N \lambda_i \left(q\eta - C_q r^{q-1}\right) \|A_i x_1 - A_i x_2\|^q,
\end{aligned}$$

which means that

$$r\sum_{i=1}^N \lambda_i \left(q\eta - C_q r^{q-1}\right) \|A_i x_1 - A_i x_2\|^q \le 0.$$

By Lemma 2.10, without loss of generality, we may assume $r \in (0, (\frac{q\eta}{C_q})^{\frac{1}{q-1}})$. We then deduce that

$$A_i x_1 = A_i x_2, \quad \forall i = 1, 2, \ldots, N. \tag{3.2}$$

Again since $x_1 \in \mathrm{VI}(E, \sum_{i=1}^N \lambda_i A_i, M)$ and $x_2 \in \bigcap_{i=1}^N \mathrm{VI}(E, A_i, M)$, we find that

$$0 \in \sum_{i=1}^N \lambda_i A_i x_1 + M x_1 \tag{3.3}$$

and

$$0 \in \sum_{i=1}^N \lambda_i A_i x_2 + M x_2. \tag{3.4}$$

We derive from (3.3) and (3.4) that

$$0 \in \sum_{i=1}^N \lambda_i A_i x_1 + M x_1 - \sum_{i=1}^N \lambda_i A_i x_2 - M x_2. \tag{3.5}$$

It then follows from (3.2) and (3.5) that

$$0 \in M x_1 - M x_2.$$

By virtue of $x_2 \in \bigcap_{i=1}^N \mathrm{VI}(E, A_i, M)$ and (3.2), we see

$$0 \in M x_1 - M x_2 + M x_2 + A_i x_2 = M x_1 + A_i x_2 = M x_1 + A_i x_1 \tag{3.6}$$

for all $i = 1, 2, \ldots, N$, which yields that

$$x_1 \in \bigcap_{i=1}^N \mathrm{VI}(E, A_i, M).$$

Hence, we obtain the desired result. □

Lemma 3.2 *Let* E, C, M, η, λ_i *and* A_i *be the same as those in Lemma* 3.1. *Then the mapping* $\sum_{i=1}^N \lambda_i A_i$ *is* η*-inverse-strongly accretive.*

Proof Let $x, y \in C$. It follows that

$$\begin{aligned} &\left\langle \sum_{i=1}^N \lambda_i A_i x - \sum_{i=1}^N \lambda_i A_i y, j_q(x - y) \right\rangle \\ &\quad = \sum_{i=1}^N \lambda_i \langle A_i x - A_i y, j_q(x - y) \rangle \\ &\quad \geq \sum_{i=1}^N \lambda_i \eta_i \|A_i x - A_i y\|^q \end{aligned}$$

$$\geq \sum_{i=1}^{N} \lambda_i \eta \|A_i x - A_i y\|^q$$
$$\geq \eta \left\| \sum_{i=1}^{N} \lambda_i A_i x - \sum_{i=1}^{N} \lambda_i A_i y \right\|^q .$$

Consequently, the mapping $\sum_{i=1}^{N} \lambda_i A_i$ is η-inverse-strongly accretive. □

Lemma 3.3 *Assume that C is a nonempty closed subset of a real uniformly convex and q-uniformly smooth Banach space E. Let $S : C \to C$ be nonexpansive, $A : C \to E$ be η-inverse-strongly accretive, and $M : \mathrm{Dom}(M) \to 2^E$ be m-accretive with $\mathrm{Dom}(M) \subseteq C$. Assume $r \in (0, (\frac{q\eta}{C_q})^{\frac{1}{q-1}})$ and $\mathrm{Fix}(S) \cap \mathrm{Fix}(T_r) \neq \emptyset$. Then $\mathrm{Fix}(ST_r) = F(T_r S) = \mathrm{Fix}(S) \cap \mathrm{Fix}(T_r)$.*

Proof It is easy to check that $\mathrm{Fix}(S) \cap \mathrm{Fix}(T_r) \subseteq \mathrm{Fix}(ST_r)$ and $\mathrm{Fix}(S) \cap \mathrm{Fix}(T_r) \subseteq \mathrm{Fix}(T_r S)$. We are left to show that $\mathrm{Fix}(ST_r) \subseteq \mathrm{Fix}(S) \cap \mathrm{Fix}(T_r)$ and $\mathrm{Fix}(T_r S) \subseteq \mathrm{Fix}(S) \cap \mathrm{Fix}(T_r)$.

We first prove $\mathrm{Fix}(ST_r) \subseteq \mathrm{Fix}(S) \cap \mathrm{Fix}(T_r)$. Suppose that $\hat{x} \in \mathrm{Fix}(ST_r)$ and $\tilde{x} \in \mathrm{Fix}(S) \cap \mathrm{Fix}(T_r)$. We have by Lemma 2.11 that

$$\begin{aligned} \|\hat{x} - \tilde{x}\|^q &= \|ST_r\hat{x} - ST_r\tilde{x}\|^q \\ &\leq \|T_r\hat{x} - T_r\tilde{x}\|^q \\ &\leq \|\hat{x} - \tilde{x}\|^q - r\bigl(\eta q - r^{q-1}C_q\bigr)\|A\hat{x} - A\tilde{x}\|^q \\ &\quad - \phi_q \bigl\|(I - J_r)(I - rA)\hat{x} - (I - J_r)(I - rA)\tilde{x}\bigr\|. \end{aligned}$$

Hence, we have from $r \in (0, (\frac{q\eta}{C_q})^{\frac{1}{q-1}})$ and the property of ϕ_q that

$$\|A\hat{x} - A\tilde{x}\| = \bigl\|(I - J_r)(I - rA)\hat{x} - (I - J_r)(I - rA)\tilde{x}\bigr\| = 0.$$

It follows that

$$\|\hat{x} - T_r\hat{x} - \tilde{x} + T_r\tilde{x}\| = 0.$$

Hence, we have

$$T_r\hat{x} = \hat{x}.$$

By the assumption of $\hat{x} \in \mathrm{Fix}(ST_r)$, we have $\hat{x} = S\hat{x}$. This means that $\hat{x} \in \mathrm{Fix}(S) \cap \mathrm{Fix}(T_r)$.

We now prove $\mathrm{Fix}(T_r S) \subseteq \mathrm{Fix}(S) \cap \mathrm{Fix}(T_r)$. Suppose that $\tilde{u} \in \mathrm{Fix}(T_r S)$ and $\hat{u} \in \mathrm{Fix}(S) \cap \mathrm{Fix}(T_r)$. Repeating the above proof again, we get that

$$\|AS\tilde{u} - AS\hat{u}\| = \bigl\|(I - J_r)(I - rA)S\tilde{u} - (I - J_r)(I - rA)S\hat{u}\bigr\| = 0.$$

It follows that

$$\|S\tilde{u} - T_r S\tilde{u} - \hat{u} + T_r S\hat{u}\| = 0.$$

Hence, we have

$$S\tilde{u} = T_r S\tilde{u}.$$

By the assumption of $\tilde{u} \in \mathrm{Fix}(T_r S)$, we have $\tilde{u} = S\tilde{u}$ and $\tilde{u} = T_r\tilde{u}$. This means that $\tilde{u} \in \mathrm{Fix}(S) \cap \mathrm{Fix}(T_r)$, which implies $\mathrm{Fix}(T_r S) \subseteq \mathrm{Fix}(S) \cap \mathrm{Fix}(T_r)$. □

Lemma 3.4 *Let C be a nonempty closed convex subset of a q-uniformly smooth Banach space E, and let $A, B : C \to E$ be two nonlinear mappings. Let Q_C be a sunny nonexpansive retraction from E onto C. For $\forall\lambda, \mu > 0$ and $a \in [0,1]$, define a mapping*

$$Gx := Q_C(I - \lambda A)\big(aI + (1-a)Q_C(I - \mu B)\big)x, \quad \forall x \in C.$$

Then (x^*, y^*) is a solution of problem (1.7) *if and only if $x^* = Gx^*$, where $y^* = Q_C(I - \mu B)x^*$.*

Proof First, we prove '$\Longrightarrow$'.

Let (x^*, y^*) be a solution of (1.7), and we have

$$\begin{cases} \langle x^* - (I - \lambda A)(ax^* + (1-a)y^*), j_q(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \mu B)x^*, j_q(x - y^*)\rangle \geq 0, & \forall x \in C. \end{cases}$$

From Lemma 2.9, we have

$$x^* = Q_C(I - \lambda A)\big(ax^* + (1-a)y^*\big)$$

and $y^* = Q_C(I - \mu B)x^*$.

It follows that

$$x^* = Q_C(I - \lambda A)\big(aI + (1-a)Q_C(I - \mu B)\big)x^* = Gx^*,$$

which implies that $x^* \in \mathrm{Fix}(G)$, where $y^* = Q_C(I - \mu B)x^*$.

Next we prove '$\Longleftarrow$'.

Let $x^* \in \mathrm{Fix}(G)$ and $y^* = Q_C(I - \mu B)x^*$. Then

$$x^* = Gx^* = Q_C(I - \lambda A)\big(aI + (1-a)Q_C(I - \mu B)\big)x^* = Q_C(I - \lambda A)\big(ax^* + (1-a)y^*\big).$$

It follows from Lemma 2.9 that

$$\begin{cases} \langle x^* - (I - \lambda A)(ax^* + (1-a)y^*), j_q(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \mu B)x^*, j_q(x - y^*)\rangle \geq 0, & \forall x \in C. \end{cases}$$

Then we find that (x^*, y^*) is a solution of problem (1.7). □

Example 3.5 ([11]) Let $\mathbb{R}$ be a real line with the Euclidean norm and let $A, B : \mathbb{R} \to \mathbb{R}$ be defined by $Ax = \frac{x-1}{4}$ and $Bx = \frac{x-1}{2}$ for all $x \in \mathbb{R}$. The mapping $G : \mathbb{R} \to \mathbb{R}$ is defined by

$$Gx := (I - 2A)\left(\frac{1}{2}I + \frac{1}{2}(I - 3B)\right)x$$

for all $x \in \mathbb{R}$. Then $1 \in \mathrm{Fix}(G)$ and $(1,1)$ is a solution of problem (1.7).

Theorem 3.6 *Let E be a uniformly convex and q-uniformly smooth Banach space. Let $N \geq 1$ be some positive integer and let $A_i : C \to E$ be η_i-inverse-strongly accretive with $\eta = \min\{\eta_1, \eta_2, \dots, \eta_N\}$. Let M be m-accretive on E with* $\mathrm{Dom}(M) \subseteq C$, $f : C \to C$ *be r-contractive. Let $A, B : C \to E$ be α- and β-inverse-strongly accretive, respectively. Define a mapping $Gx := Q_C(I - \lambda A)(aI + (1-a)Q_C(I - \mu B))x$ for all $x \in C$ and $a \in [0,1]$. Assume $\lambda \in (0, (\frac{q\alpha}{C_q})^{\frac{1}{q-1}})$, $\mu \in (0, (\frac{q\beta}{C_q})^{\frac{1}{q-1}})$ and $\mathrm{Fix}(G) \cap \bigcap_{i=1}^N \mathrm{VI}(E, A_i, M)$. For arbitrarily given $x_1 \in C$, let $\{x_n\}$ be the sequence generated iteratively by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) J_{r_n}\left(\left(I - r_n \sum_{i=1}^N \lambda_i A_i\right) G x_n + e_n\right), \tag{3.7}$$

where $\{e_n\}_1^\infty \subset E$, $\{\alpha_n\}_1^\infty \subset [0,1]$, $\{\lambda_n\}_1^N \subset (0,1)$ and $\{r_n\}_1^\infty \subset (0, +\infty)$ satisfy the following conditions:

(i) $\sum_{n=1}^\infty \|e_n\| < \infty$;
(ii) $\sum_{n=1}^\infty \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < (\frac{q\eta}{C_q})^{\frac{1}{q-1}}$ *and* $\sum_{n=1}^\infty |r_{n+1} - r_n| < \infty$;
(iv) $\sum_{n=1}^N \lambda_i = 1$.

Then $\{x_n\}$ converges strongly to some point $x \in \mathrm{Fix}(G) \cap \bigcap_{i=1}^N \mathrm{VI}(E, A_i, M)$, which solves the variational inequality

$$\langle f(x) - x, j_q(p - x)\rangle \leq 0, \quad p \in \mathrm{Fix}(G) \cap \bigcap_{i=1}^N \mathrm{VI}(E, A_i, M).$$

And (x, y) solves problem (1.7), *where $y = Q_C(I - \mu B)x$.*

Proof Let $\{y_n\}$ be a sequence generated by

$$y_{n+1} = \alpha_n f(y_n) + (1-\alpha_n) T_n G y_n, \tag{3.8}$$

where $T_n := J_{r_n}(I - r_n \sum_{i=1}^N \lambda_i A_i)$. Hence to show the desired result, it suffices to prove that $y_n \to x$. Indeed, by virtue of Lemma 2.8, Lemma 3.2, (iii), $\lambda \in (0, (\frac{q\alpha}{C_q})^{\frac{1}{q-1}})$ and $\mu \in (0, (\frac{q\beta}{C_q})^{\frac{1}{q-1}})$, we find that $T_n : C \to C$ and $G : C \to C$ are nonexpansive. And hence,

$$\begin{aligned}
&\|y_{n+1} - x_{n+1}\| \\
&\quad\leq \alpha_n r\|y_n - x_n\| + (1-\alpha_n)\left\| J_{r_n}\left(I - r_n \sum_{i=1}^N \lambda_i A_i\right) G y_n \right. \\
&\qquad \left. - J_{r_n}\left(\left(I - r_n \sum_{i=1}^N \lambda_i A_i\right) G x_n + e_n\right)\right\| \\
&\quad\leq \alpha_n r\|y_n - x_n\| + (1-\alpha_n)\left\|\left(I - r_n \sum_{i=1}^N \lambda_i A_i\right) G y_n - \left(I - r_n \sum_{i=1}^N \lambda_i A_i\right) G x_n\right\| + \|e_n\| \\
&\quad\leq \bigl[1 - \alpha_n(1-r)\bigr]\|y_n - x_n\| + \|e_n\|.
\end{aligned} \tag{3.9}$$

By virtue of Lemma 2.6 and (3.9), we see $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

First, we prove that the sequence $\{y_n\}$ is bounded.

Taking $x \in \mathrm{Fix}(G) \cap \bigcap_{i=1}^N (A_i + M)^{-1}(0)$, we find $x \in \mathrm{Fix}(G) \cap \mathrm{Fix}(T_n)$ by Lemma 2.10 and Lemma 3.1. It follows from (3.8) and Lemma 3.3 that

$$\begin{aligned} \|y_{n+1} - x\| &= \|\alpha_n f(y_n) + (1-\alpha_n)T_n G y_n - x\| \\ &\le \alpha_n \|f(y_n) - x\| + (1-\alpha_n)\|T_n G y_n - x\| \\ &\le \alpha_n \|f(y_n) - f(x)\| + \alpha_n \|f(x) - x\| + (1-\alpha_n)\|T_n G y_n - x\| \\ &\le \alpha_n r\|y_n - x\| + \alpha_n \|f(x) - x\| + (1-\alpha_n)\|y_n - x\| \\ &= [1 - \alpha_n(1-r)]\|y_n - x\| + \alpha_n \|f(x) - x\| \\ &\le \max\left\{ \frac{\|f(x) - x\|}{1-r}, \|y_n - x\| \right\}. \end{aligned}$$

By induction, we have

$$\|y_n - x\| \le \max\left\{ \frac{\|f(x) - x\|}{1-r}, \|y_1 - x\| \right\}, \quad \forall n \ge 1.$$

Hence, $\{y_n\}$ is bounded, so are $\{f(y_n)\}$, $\{T_n(y_n)\}$ and $\{T_n G(y_n)\}$.

Next, we prove that

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| \to 0. \tag{3.10}$$

Write $V = \sum_{i=1}^N \lambda_i A_i$. Noticing Lemma 3.2, we get that the mapping V is η-inverse-strongly accretive. Putting $z_n = T_n G y_n$, we derive from Lemma 2.11 that

$$\begin{aligned} &\|z_{n+1} - z_n\| \\ &\quad = \|T_{n+1} G y_{n+1} - T_n G y_n\| \\ &\quad \le \|T_{n+1} G y_{n+1} - T_n G y_{n+1}\| + \|T_n G y_{n+1} - T_n G y_n\| \\ &\quad \le \left|1 - \frac{r_{\alpha_n}}{r_{\beta_n}}\right| \|G y_{n+1} - J_{r_{\beta_n}}(1 - r_{\beta_n} V) G y_{n+1}\| + \|G y_{n+1} - G y_n\| \\ &\quad \le |r_{\beta_n} - r_{\alpha_n}| \frac{\|G y_{n+1} - J_{r_{\beta_n}}(1 - r_{\beta_n} V) G y_{n+1}\|}{r_{\beta_n}} + \|y_{n+1} - y_n\| \\ &\quad \le |r_{n+1} - r_n| M_1 + \|y_{n+1} - y_n\|, \end{aligned} \tag{3.11}$$

where $M_1 > \sup_{n\ge1}\{\frac{\|G y_{n+1} - J_{r_{\beta_n}}(1 - r_{\beta_n} V) G y_{n+1}\|}{r_{\beta_n}}\}$, $r_{\alpha_n} = \min\{r_{n+1}, r_n\}$ and $r_{\beta_n} = \max\{r_{n+1}, r_n\}$.

Combining (3.8) and (3.11), we find that

$$\begin{aligned} &\|y_{n+1} - y_n\| \\ &\quad = \|\alpha_n f(y_n) + (1-\alpha_n) z_n - \alpha_{n-1} f(y_{n-1}) - (1-\alpha_{n-1}) z_{n-1}\| \\ &\quad = \|(\alpha_n - \alpha_{n-1})(f(y_{n-1}) - z_{n-1}) + (1-\alpha_n)(z_n - z_{n-1})\| \\ &\qquad + \alpha_n \|f(y_n) - f(y_{n-1})\| \end{aligned}$$

$$\begin{aligned}
&\leq |\alpha_n - \alpha_{n-1}| \big\| f(y_{n-1}) - z_{n-1} \big\| + (1-\alpha_n)\|z_n - z_{n-1}\| + \alpha_n r \|y_n - y_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}| M_2 + (1-\alpha_n)\|z_n - z_{n-1}\| + \alpha_n r \|y_n - y_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}| M_2 + |r_n - r_{n-1}| M_1 + \big[1 - \alpha_n(1-r)\big]\|y_n - y_{n-1}\|,
\end{aligned}$$

where $M_2 > \sup_{n\geq 1}\{\|f(y_n) - z_n\|\}$. It follows from Lemma 2.6, (ii) and (iii) that $\lim_{n\to\infty}\|y_{n+1} - y_n\| = 0$.

Again, using Lemma 2.5, Lemma 2.11 and Lemma 3.3, we obtain

$$\begin{aligned}
&\|y_{n+1} - x\|^q \\
&\quad = \big\|\alpha_n\big(f(y_n) - x\big) + (1-\alpha_n)(T_n G y_n - x)\big\|^q \\
&\quad \leq (1-\alpha_n)\|T_n G y_n - x\|^q + q\alpha_n \big\langle f(y_n) - x, j_q(y_{n+1} - x)\big\rangle \\
&\quad \leq \|T_n G y_n - x\|^q + q\alpha_n M_3 \\
&\quad \leq \|y_n - x\|^q - r_n\big(\alpha q - r_n^{q-1} C_q\big)\|VGy_n - VGx\|^q \\
&\qquad - \phi_q\big(\|Gy_n - r_n VGy_n - T_n G y_n + r_n VGx\|\big) + q\alpha_n M_3,
\end{aligned}$$

where $M_3 > \sup_{n\geq 1}\{\langle f(y_n) - x, j_q(y_{n+1} - x)\rangle\}$. Meanwhile, by the fact that $a^r - b^r \leq r a^{r-1}(a-b)$ for all $r \geq 1$, we find that

$$\begin{aligned}
&r_n\big(\alpha q - r_n^{q-1} C_q\big)\|VGy_n - VGx\|^q \\
&\qquad + \phi_q\big(\|Gy_n - r_n VGy_n - T_n G y_n + r_n VGx\|\big) \\
&\quad \leq \|y_n - x\|^q - \|y_{n+1} - x\|^q + q\alpha_n M_3 \\
&\quad \leq q\|y_n - x\|^{q-1}\big(\|y_n - x\| - \|y_{n+1} - x\|\big) + q\alpha_n M_3.
\end{aligned} \tag{3.12}$$

It follows immediately from (ii), (iii), (3.12) and the property of ϕ_q that

$$\lim_{n\to\infty}\|VGy_n - VGx\| = \lim_{n\to\infty}\|Gy_n - r_n VGy_n - T_n G y_n + r_n VGx\| = 0,$$

which implies that

$$\lim_{n\to\infty}\|T_n G y_n - G y_n\| = 0. \tag{3.13}$$

In view of condition (iii), there exists $\varepsilon > 0$ such that $r_n \geq \varepsilon$ for all $n \geq 1$. Then we get, by Lemma 2.11, that

$$\lim_{n\to\infty}\|T_\varepsilon G y_n - G y_n\| \leq \lim_{n\to\infty} 2\|T_n G y_n - G y_n\| = 0. \tag{3.14}$$

We show $\lim_{n\to\infty}\|T_\varepsilon G y_n - y_n\| = 0$.

Thanks to (3.10), (3.13), (3.14) and (ii), we see

$$\begin{aligned}
&\|T_\varepsilon G y_n - y_n\| \\
&\quad \leq \|T_\varepsilon G y_n - T_n G y_n\| + \|T_n G y_n - y_n\|
\end{aligned}$$

$$\begin{aligned}
&\leq \|T_\varepsilon Gy_n - Gy_n\| + \|Gy_n - T_nGy_n\| + \|T_nGy_n - y_{n+1}\| + \|y_{n+1} - y_n\| \\
&\leq \|T_\varepsilon Gy_n - Gy_n\| + \|Gy_n - T_nGy_n\| + \alpha_n \big\|f(y_n) - T_nGy_n\big\| + \|y_{n+1} - y_n\| \\
&\to 0.
\end{aligned} \tag{3.15}$$

Next we prove that

$$\limsup_{n\to\infty} \big\langle f(x) - x, j_q(y_n - x)\big\rangle \leq 0.$$

Equivalently (should $\|y_n - x\| \neq 0$), we need to prove that

$$\limsup_{n\to\infty} \big\langle f(x) - x, j(y_n - x)\big\rangle \leq 0.$$

To this end, let x_t satisfy $x_t = tf(x_t) + (1-t)T_\varepsilon Gx_t$. By Xu's Theorem 4.1 in [32], we get $x_t \to x \in \mathrm{Fix}(T_\varepsilon G) = \mathrm{Fix}(G) \cap \bigcap_{i=1}^N \mathrm{VI}(E, A_i, M)$ (by Lemma 2.10, Lemma 3.1 and Lemma 3.3) as $t \to 0$, which x solves the variational inequality

$$\big\langle f(x) - x, j(p - x)\big\rangle \leq 0, \quad \forall p \in \mathrm{Fix}(T_\varepsilon G).$$

Using subdifferential inequality, we deduce that

$$\begin{aligned}
&\|x_t - y_n\|^2 \\
&\quad = t\big\langle f(x_t) - y_n, j(x_t - y_n)\big\rangle + (1-t)\big\langle T_\varepsilon Gx_t - y_n, j(x_t - y_n)\big\rangle \\
&\quad = t\big\langle f(x_t) - z_t, j(x_t - y_n)\big\rangle + t\big\langle x_t - y_n, j(x_t - y_n)\big\rangle \\
&\qquad + (1-t)\big\langle T_\varepsilon Gx_t - T_\varepsilon Gy_n, j(x_t - y_n)\big\rangle + (1-t)\big\langle T_\varepsilon Gy_n - y_n, j(x_t - y_n)\big\rangle \\
&\quad \leq t\big\langle f(x_t) - x_t, j(x_t - y_n)\big\rangle + t\|x_t - y_n\|^2 + (1-t)\|x_t - y_n\|^2 \\
&\qquad + (1-t)\|T_\varepsilon Gy_n - y_n\|\|x_t - y_n\| \\
&\quad \leq t\big\langle f(x_t) - x_t, j(x_t - y_n)\big\rangle + \|x_t - y_n\|^2 + \|T_\varepsilon Gy_n - y_n\|\|x_t - y_n\|,
\end{aligned}$$

which implies that

$$\big\langle f(x_t) - x_t, j(y_n - x_t)\big\rangle \leq \frac{\|T_\varepsilon Gy_n - y_n\|}{t}\|x_t - y_n\|. \tag{3.16}$$

Using (3.15), taking the upper limit as $n \to \infty$ firstly, and then as $t \to 0$ in (3.16), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty} \big\langle f(x_t) - x_t, j(y_n - x_t)\big\rangle \leq 0.$$

Since E is a uniformly smooth Banach space, we have that the duality mapping j is norm-to-norm uniform on any bounded subset of E, which ensures that the limits $\limsup_{t\to 0}$ and $\limsup_{n\to\infty}$ are interchangeable. Then we have

$$\limsup_{n\to\infty} \big\langle f(x) - x, j(y_n - x)\big\rangle \leq 0.$$

Finally, we show $\|y_n - x\| \to 0$.
By Lemma 3.3 and the fact that $ab \leq \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}}$, we get

$$\begin{aligned}
&\|y_{n+1} - x\|^q \\
&\quad = \bigl\|\alpha_n f(y_n) + (1-\alpha_n)T_n G y_n - x\bigr\|^q \\
&\quad = \bigl\langle \alpha_n f(y_n) + (1-\alpha_n)T_n G y_n - x, j_q(y_{n+1} - x)\bigr\rangle \\
&\quad = \alpha_n \bigl\langle f(y_n) - f(x), j_q(y_{n+1} - x)\bigr\rangle + \alpha_n \bigl\langle f(x) - x, j_q(y_{n+1} - x)\bigr\rangle \\
&\qquad + (1-\alpha_n)\bigl\langle T_n G y_n - x, j_q(y_{n+1} - x)\bigr\rangle \\
&\quad \leq \alpha_n \bigl\|f(y_n) - f(x)\bigr\| \|y_{n+1} - x\|^{q-1} + \alpha_n \bigl\langle f(x) - x, j_q(y_{n+1} - x)\bigr\rangle \\
&\qquad + (1-\alpha_n)\|y_n - x\| \|y_{n+1} - x\|^{q-1} \\
&\quad \leq \alpha_n r \|y_n - x\| \|y_{n+1} - x\|^{q-1} + \alpha_n \bigl\langle f(x) - x, j_q(y_{n+1} - x)\bigr\rangle \\
&\qquad + (1-\alpha_n)\|y_n - x\| \|y_{n+1} - x\|^{q-1} \\
&\quad \leq \bigl[1 - \alpha_n(1-r)\bigr]\|y_n - x\| \|y_{n+1} - x\|^{q-1} + \alpha_n \bigl\langle f(x) - x, j_q(y_{n+1} - x)\bigr\rangle \\
&\quad \leq \bigl[1 - \alpha_n(1-r)\bigr]\frac{1}{q}\|y_n - x\|^q + \frac{q-1}{q}\|y_{n+1} - x\|^q \\
&\qquad + \alpha_n \bigl\langle f(x) - x, j_q(y_{n+1} - x)\bigr\rangle,
\end{aligned}$$

which implies that

$$\|y_{n+1} - x\|^q \leq \bigl[1 - \alpha_n(1-r)\bigr]\|y_n - x\|^q + q\alpha_n \bigl\langle f(x) - x, j_q(y_{n+1} - x)\bigr\rangle. \tag{3.17}$$

Apply Lemma 2.6 to (3.17) to conclude $y_n \to x \in \operatorname{Fix}(G) \cap \bigcap_{i=1}^N \mathrm{VI}(E, A_i, M)$ as $n \to \infty$, which solves the variational inequality

$$\bigl\langle f(x) - x, j_q(p - x)\bigr\rangle \leq 0, \quad p \in \operatorname{Fix}(G) \cap \bigcap_{i=1}^N \mathrm{VI}(E, A_i, M).$$

And (x, y) is a solution of the modified system of variational inequalities problem (1.7) due to Lemma 3.4, where $y = Q_C(I - \mu B)x$. This completes the proof. □

Remark 3.7 Theorem 3.6 improves and extends Theorem 3.7 of López *et al.* [24] in the sense:

- From the problem of finding a solution for a variational inclusion problem with two accretive operators to problem of finding a common solution for a variational inclusion problem with a finite family of accretive operators and a modified system of variational inequalities.

Remark 3.8 Theorem 3.6 improves and extends Theorem 2.1 of Zhang *et al.* [19], Theorem 3.1 of Qin *et al.* [26], Theorem 3.1 of Takahashi *et al.* [27] and Theorem 3.1 of Khuangsatung and Kangtunyakarn [25] in the following senses:

- From Hilbert spaces to uniformly convex and q-uniformly smooth Banach spaces.

- From finding a common element of the set of solutions for the variational inclusion problem with two accretive operators and the set of fixed points of nonexpansive mappings to finding a common solution to a variational inclusion problem with a finite family of accretive operators and a modified system of variational inequalities.

As a direct consequence of Theorem 3.6, we obtain the following corollary.

Corollary 3.9 *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $N \geq 1$ be some positive integer and let $A_i : C \to H$ be η_i-inverse-strongly monotone with $\eta = \min\{\eta_1, \eta_2, \dots, \eta_N\}$. Let M be maximal monotone in H with $\operatorname{Dom}(M) \subseteq C$, $f : C \to C$ be r-contractive. Let $A, B : C \to H$ be α- and β-inverse-strongly monotone, respectively. Define a mapping $Gx := \operatorname{Proj}_C(I - \lambda A)(aI + (1 - a)\operatorname{Proj}_C(I - \mu B))x$ for all $x \in C$ and $a \in [0,1]$, where Proj_C is the metric projection from H onto C. Assume that $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ and $\operatorname{Fix}(G) \cap \bigcap_{i=1}^{N} \operatorname{VI}(H, A_i, M) \neq \emptyset$. For arbitrarily given $x_1 \in C$, let $\{x_n\}$ be the sequence generated iteratively by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) J_{r_n}\left(\left(I - r_n \sum_{i=1}^{N} \lambda_i A_i\right) G x_n + e_n\right),$$

where $\{e_n\}_1^\infty \subset E$, $\{\alpha_n\}_1^\infty \subset [0,1]$, $\{\lambda_n\}_1^N \subset [0,1]$ and $\{r_n\}_1^\infty \subset (0, +\infty)$ satisfy the following conditions:

(i) $\sum_{n=1}^{\infty} \|e_n\| < \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\eta$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(iv) $\sum_{n=1}^{N} \lambda_i = 1$.

Then $\{x_n\}$ converges strongly to some point $x \in \operatorname{Fix}(G) \cap \bigcap_{i=1}^{N} \operatorname{VI}(H, A_i, M)$, which solves the variational inequality

$$\langle f(x) - x, p - x \rangle \leq 0, \quad p \in \operatorname{Fix}(G) \cap \bigcap_{i=1}^{N} \operatorname{VI}(H, A_i, M).$$

4 Applications

In this section, we give some applications of our main results in the framework of Hilbert spaces. Let C be a nonempty, closed and convex subset of a Hilbert space, and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, *i.e.*, $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,

$$\limsup_{t\downarrow 0} f\bigl(tz + (1 - t)x, y\bigr) \leq f(x,y);$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semi-continuous.

Then the mathematical model related to equilibrium problems (with respect to C) is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0$$

for all $y \in C$. The set of such solutions $\hat{x}$ is denoted by EP(f). The following lemma appears implicitly in Blum and Oettli [33].

Lemma 4.1 *Let C be a nonempty, closed and convex subset of H and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

The following lemma is given in Combettes and Hirstoaga [34].

Lemma 4.2 *Assume that $f : C \times C \to \mathbb{R}$ satisfies* (A1)-(A4). *For $r > 0$ and $x \in H$, define a mapping $S_r : H \to C$ as follows*:

$$S_r x = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}$$

for all $x \in H$. Then the following hold:

(i) *S_r is single-valued*;

(ii) *S_r is a firmly nonexpansive mapping, i.e., for all $x, y \in H$,*
$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y\rangle$;

(iii) $\mathrm{Fix}(S_r) = \mathrm{EP}(f)$;

(iv) EP(f) *is closed and convex.*

We call such S_r the resolvent of f for $r > 0$. Using Lemma 4.1 and Lemma 4.2, Takahashi *et al.* [27] proved the following result.

Lemma 4.3 *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $f : C \times C \to \mathbb{R}$ satisfy* (A1)-(A4). *Let A_f be a multivalued mapping of H into itself defined by*

$$A_f x = \begin{cases} \{z \in H : f(x,y) \geq \langle y - x, z\rangle, \forall y \in C\}, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Then $\mathrm{EP}(f) = A_f^{-1}0$ *and A_f is a maximal monotone operator. Further, for any $x \in H$ and $r > 0$, the resolvent S_r of f coincides with the resolvent of A_f; i.e., $S_r x = (I + rA_f)^{-1}x$.*

Theorem 4.4 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4) *and let S_δ be the resolvent of f for $\delta > 0$. Let $\psi : C \to C$ be r-contractive, $A, B : C \to H$ be α- and β-inverse-strongly monotone, respectively. Define a mapping* $Gx := \mathrm{Proj}_C(I - \lambda A)(aI + (1 - a)\,\mathrm{Proj}_C(I - \mu B))x$ *for all $x \in C$ and $a \in [0,1]$. Assume that $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ and* $\mathrm{Fix}(G) \cap \mathrm{EP}(f) \neq \emptyset$. *For arbitrarily given $x_1 \in C$, let $\{x_n\}$ be the sequence generated iteratively by*

$$x_{n+1} = \alpha_n \psi(x_n) + (1 - \alpha_n) S_{r_n}(Gx_n + e_n),$$

where $\{e_n\}_1^\infty \subset E$, $\{\alpha_n\}_1^\infty \subset [0,1]$ and $\{r_n\}_1^\infty \subset (0, +\infty)$ for all $n \in \mathbb{N}$ satisfy the following conditions:

(i) $\sum_{i=1}^{\infty} \|e_n\| < \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iii) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ *converges strongly to some point* $x \in \mathrm{Fix}(G) \cap \mathrm{EP}(f)$, *which solves the variational inequality*

$$\langle f(x) - x, p - x\rangle \leq 0, \quad p \in \mathrm{Fix}(G) \cap \mathrm{EP}(f).$$

Proof Put $A_i = 0$ for $i = 1, 2, \dots, N$ in Corollary 3.9. From Lemma 4.3, we know that $J_{r_n} = S_{r_n}$ for all $n \in \mathbb{N}$. So, we obtain the desired result by Corollary 3.9. □

Let $g : H \to (-\infty, +\infty]$ be a proper convex lower semi-continuous function. Then, the subdifferential ∂g of g is defined as follows:

$$\partial g = \bigl\{y \in H : g(z) \geq g(x) + \langle z - x, y\rangle, z \in H\bigr\}, \quad \forall x \in H.$$

From Rockafellar [35], we know that ∂g is maximal monotone. It is easy to verify that $0 \in \partial g(x)$ if and only if $g(x) = \min_{y\in H} g(y)$. Let I_C be the indicator function of C, *i.e.*,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases}$$

Then I_C is a proper lower semi-continuous convex function on H, and the subdifferential ∂I_C of I_C is a maximal monotone operator. Furthermore, suppose that C is a nonempty closed convex subset. Then

$$(I + \lambda \partial I_C)^{-1} x = \mathrm{Proj}_C x, \quad \forall x \in H, \lambda > 0.$$

For more details, one can refer to [27].

Theorem 4.5 *Let* C *be a nonempty, closed and convex subset of a Hilbert space* H. *Let* $N \geq 1$ *be some positive integer and let* $A_i : C \to H$ *be* η_i*-inverse-strongly monotone with* $\eta = \min\{\eta_1, \eta_2, \dots, \eta_N\}$ *for each* $i \in \{1, 2, \dots, N\}$. *Let* $f : C \to C$ *be* r*-contractive. Let* $A, B : C \to H$ *be* α*- and* β*-inverse-strongly monotone, respectively. Define a mapping* $Gx := \mathrm{Proj}_C(I - \lambda A)(aI + (1 - a)\mathrm{Proj}_C(I - \mu B))x$ *for all* $x \in C$ *and* $a \in [0, 1]$. *Assume that* $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ *and* $\mathrm{Fix}(G) \cap \bigcap_{i=1}^{N} \mathrm{VI}(C, A_i) \neq \emptyset$. *For arbitrarily given* $x_1 \in C$, *let* $\{x_n\}$ *be the sequence generated iteratively by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\mathrm{Proj}_C\left(\left(I - r_n \sum_{i=1}^{N} \lambda_i A_i\right) G x_n + e_n\right),$$

where $\{e_n\}_1^{\infty} \subset E$, $\{\alpha_n\}_1^{\infty} \subset [0, 1]$, $\{\lambda_n\}_1^{N} \subset [0, 1]$ *and* $\{r_n\}_1^{\infty} \subset (0, +\infty)$ *satisfy the following conditions*:

(i) $\sum_{n=1}^{\infty} \|e_n\| < \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\eta$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
(iv) $\sum_{n=1}^{N} \lambda_i = 1$.

Then $\{x_n\}$ *converges strongly to some point* $x \in \mathrm{Fix}(G) \cap \bigcap_{i=1}^N \mathrm{VI}(C, A_i)$, *which solves the variational inequality*

$$\langle f(x) - x, p - x \rangle \le 0, \quad p \in \mathrm{Fix}(G) \cap \bigcap_{i=1}^N \mathrm{VI}(C, A_i).$$

Proof Put $B = \partial I_C$. Next, we show that $\mathrm{VI}(C, A_i) = \mathrm{VI}(H, A_i, \partial I_C)$. Notice that

$$\begin{aligned} x \in \mathrm{VI}(H, A_i, \partial I_C) \quad &\Longleftrightarrow \quad 0 \in A_i x + \partial I_C x \\ &\Longleftrightarrow \quad -A_i x \in \partial I_C x \\ &\Longleftrightarrow \quad \langle A_i x, y - x \rangle \ge 0 \\ &\Longleftrightarrow \quad x \in \mathrm{VI}(C, A_i). \end{aligned}$$

In view of Theorem 3.6, we find the desired result immediately. □

Let $W : H \to \mathbb{R}$ be a convex and differentiable function and $M : H \to \mathbb{R}$ be a convex function. Consider the convex minimization problem $\min_{x \in H}(Wx + Mx)$. From [35], we know if ∇W is $\frac{1}{L}$-Lipschitz continuous, then it is L-inverse-strongly monotone. Hence, we have the following theorem.

Theorem 4.6 *Let* C *be a nonempty, closed and convex subset of a Hilbert space* H. *Let* $N \ge 1$ *be some positive integer. Let* $W_i : H \to \mathbb{R}$ *be a convex and differentiable function and* ∇W_i *be* $\frac{1}{L_i}$-*Lipschitz continuous with* $L = \min\{L_1, L_2, \dots, L_N\}$ *for each* $i \in \{1, 2, \dots, N\}$. *Let* M *be a convex and lower semi-continuous function,* $f : C \to C$ *be r-contractive. Let* $A, B : C \to H$ *be a convex and differentiable function and let* ∇A, ∇B *be* α- *and* β-*Lipschitz continuous, respectively. Define a mapping*

$$G' x := \mathrm{Proj}_C(I - \lambda \nabla A)\bigl(aI + (1 - a)\,\mathrm{Proj}_C(I - \mu \nabla B)\bigr)x, \quad \forall x \in C, a \in [0, 1].$$

Assume that $\lambda \in (0, 2\alpha)$, $\mu \in (0, 2\beta)$ *and* $\mathrm{Fix}(G') \cap \bigcap_{i=1}^N \mathrm{VI}(H, \nabla W_i, \partial M) \ne \emptyset$. *For arbitrarily given* $x_1 \in C$, *let* $\{x_n\}$ *be the sequence generated iteratively by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)(I + r_n \partial M)^{-1}\left(\left(I - r_n \sum_{i=1}^N \lambda_i \nabla W_i\right) G x_n + e_n\right),$$

where $\{e_n\}_1^\infty \subset E$, $\{\alpha_n\}_1^\infty \subset [0, 1]$, $\{\lambda_n\}_1^N \subset [0, 1]$ *and* $\{r_n\}_1^\infty \subset (0, +\infty)$ *satisfy the following conditions*:

(i) $\sum_{n=1}^\infty \|e_n\| < \infty$;
(ii) $\sum_{n=1}^\infty \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
(iii) $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2L$ *and* $\sum_{n=1}^\infty |r_{n+1} - r_n| < \infty$;
(iv) $\sum_{n=1}^N \lambda_i = 1$.

Then $\{x_n\}$ *converges strongly to some point* $x \in \mathrm{Fix}(G') \cap \bigcap_{i=1}^N \mathrm{VI}(H, \nabla W_i, \partial M)$, *which solves the variational inequality*

$$\langle f(x) - x, p - x \rangle \le 0, \quad p \in \mathrm{Fix}\bigl(G'\bigr) \cap \bigcap_{i=1}^N \mathrm{VI}(H, \nabla W_i, \partial M).$$

Proof Put $M = \partial M$, $A = \nabla A$, $B = \nabla B$, $A_i = \nabla W_i$ for each $i \in \{1, 2, \dots, N\}$ in Theorem 3.6. Then we get the desired conclusions immediately. □

5 Numerical examples

The purpose of this section is to give two numerical examples supporting Theorem 3.6.

Example 5.1 Let $\mathbb{R}$ be a real line with the Euclidean norm. For all $x \in \mathbb{R}$, let $A, B, M, f : \mathbb{R} \to \mathbb{R}$ be defined by $Ax = \frac{1}{3}x$, $Bx = \frac{1}{6}x$, $Mx = x$ and $f(x) = \frac{1}{2}x$, respectively. For each $i \in \{1, 2, \dots, N\}$, let $A_i : \mathbb{R} \to \mathbb{R}$ be defined by $A_i x = \frac{i}{6}x$ for all $x \in \mathbb{R}$. Let $a = \frac{1}{2}$, $\lambda = 2$, $\mu = 3$, $\lambda_i = \frac{2}{3^i} + \frac{1}{N3^N}$ for each $i \in \{1, 2, \dots, N\}$, and $e_n = \frac{e_1}{n^2}$ $(i = 1, 2, \dots)$, where $|e_1| < \infty$. Let the sequence $\{x_n\}$ be generated iteratively by (3.7), where $\alpha_n = \frac{1}{n}$ and $r_n = \frac{1}{n+2} + \frac{1}{N}$. Then the sequence $\{x_n\}$ converges strongly to 0.

Solution: It can be observed that all the assumptions of Theorem 3.6 are satisfied. It is also easy to check that

$$\operatorname{Fix}(G) \cap \bigcap_{i=1}^{N} \operatorname{VI}(E, A_i, M) = \{0\}.$$

We rewrite (3.7) as follows:

$$\begin{aligned} x_{n+1} = {} & \frac{1}{2n}x_n + \left(1 - \frac{1}{n}\right)\frac{(n+2)N}{n(N+1) + 3N + 2} \\ & \times \left[\frac{1}{4}x_n - \left(\frac{1}{n+2} + \frac{1}{N}\right)\sum_{i=1}^{N}\left(\frac{2}{3^i} + \frac{1}{N3^N}\right)\frac{i}{24}x_n + \frac{e_1}{n^2}\right]. \end{aligned} \tag{5.1}$$

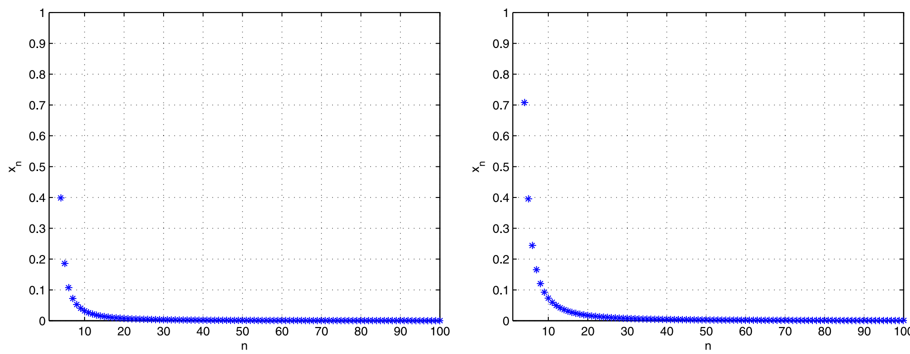
Using algorithm (5.1) and choosing $e_1 = x_1 = 5$ with $N = 1$ and $N = 100$ (see Table 1), we see that Figure 1 and numerical results demonstrate Theorem 3.6.

Next, we present a numerical example in $\mathbb{R}^3$ that also supports our result.

Example 5.2 Let the inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be defined by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$ and the usual norm $\| \cdot \| : \mathbb{R}^3 \to \mathbb{R}$ be defined by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$

Table 1 The values of the sequence $\{x_n\}$

n	$N = 1$	$N = 100$
	x_n	x_n
1	5.000000000000000	5.000000000000000
2	2.500000000000000	2.500000000000000
3	1.012731481481481	1.352926587301587
4	0.398516414141414	0.708148944121956
5	0.185768821022727	0.395562727289485
⋮	⋮	⋮
50	0.001141959256001	0.002664953434053
⋮	⋮	⋮
96	0.000306349806735	0.000718002934223
97	0.000300029191229	0.000703224190461
98	0.000293902199966	0.000688897141470
99	0.000287961003866	0.000675003563404
100	0.000282198165612	0.000661526142407



(a) N=1 (b) N=100

Figure 1 The convergence of $\{x_n\}$ with initial values $e_1 = x_1 = 5$.

Table 2 The values of the sequence $\{\mathbf{x}_n\}$

n	x_n^1	x_n^2	x_n^3
1	1.000000000000000	6.000000000000000	12.000000000000000
2	0.166666666666667	1.000000000000000	2.000000000000000
3	0.123544973544974	0.741269841269841	1.482539682539682
4	0.078950180010786	0.473701080064716	0.947402160129433
5	0.051217417354718	0.307304504128309	0.614609008256618
⋮	⋮	⋮	⋮
50	0.000476097239794	0.002856583438761	0.005713166877522
⋮	⋮	⋮	⋮
96	0.000128869584585	0.000773217507511	0.001546435015021
97	0.000126223335925	0.000757340015550	0.001514680031100
98	0.000123657771457	0.000741946628743	0.001483893257487
99	0.000121169643936	0.000727017863616	0.001454035727232
100	0.000118755867858	0.000712535207149	0.001425070414298

for all $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$. Let $A, B, M, f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $A\mathbf{x} = \frac{1}{4}\mathbf{x}$, $B\mathbf{x} = f\mathbf{x} = \frac{1}{6}\mathbf{x}$ and $M\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$, respectively. For each $i \in \{1, 2, \ldots, N\}$, let $A_i : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $A_i\mathbf{x} = \frac{i}{6}\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$. Let $a = \frac{1}{2}$, $\lambda = 3$, $\mu = 6$, $\lambda_i = \frac{5}{6^i} + \frac{1}{N6^N}$ for each $i \in \{1, 2, \ldots, N\}$ and $e_n = \frac{e_1}{n^2}$ $(n = 1, 2, \ldots)$, where $e_1 \in \mathbb{R}^3$ and $\|e_1\| < \infty$. Let the sequence $\{\mathbf{x}_n\}$ be generated iteratively by (3.7), where $\alpha_n = \frac{1}{n}$ and $r_n = \frac{1}{n+2} + \frac{1}{N}$. Then the sequence $\{\mathbf{x}_n\}$ converges strongly to 0.

Solution: It can be observed that all the assumptions of Theorem 3.6 are satisfied. It is also easy to check $\operatorname{Fix}(G) \cap \bigcap_{i=1}^{N} \operatorname{VI}(E, A_i, M) = \{0\}$.

We rewrite (3.7) as follows:

$$\begin{aligned} \mathbf{x}_{n+1} = {} & \frac{1}{6n}\mathbf{x}_n + \left(1 - \frac{1}{n}\right)\frac{(n+2)N}{n(N+1) + 3N + 2} \\ & \times \left[\frac{1}{6}\mathbf{x}_n - \left(\frac{1}{n+2} + \frac{1}{N}\right)\sum_{i=1}^{N}\left(\frac{5}{6^i} + \frac{1}{N6^N}\right)\frac{i}{36}\mathbf{x}_n + \frac{e_1}{n^2}\right]. \end{aligned} \tag{5.2}$$

Utilizing algorithm (5.2) and choosing $\mathbf{x}_1 = e_1 = (1, 6, 12)$ with $N = 100$, we report the numerical results in Table 2. In addition, Figure 2 also demonstrates Theorem 3.6.

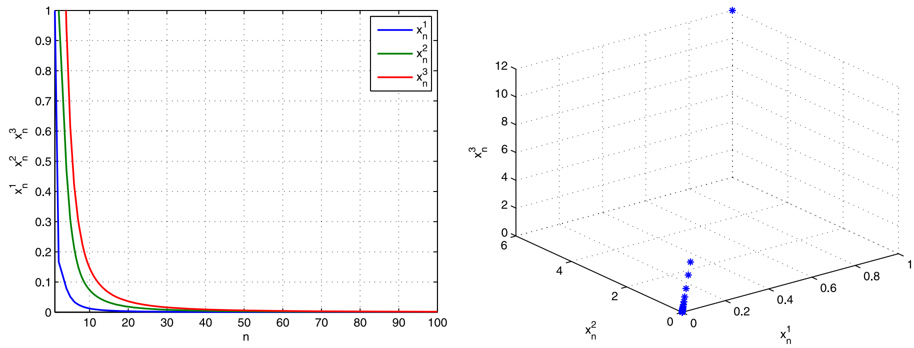


Figure 2 The convergence of $\{x_n\}$ with initial values $x_1 = e_1 = (1, 6, 12)$ and $N = 100$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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