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Stability and Hopf bifurcation for a ratio-dependent predator-prey system with stage structure and time delay

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Abstract

A ratio-dependent predator-prey system with time delay due to the gestation of the predator and stage structure for both the predator and the prey is investigated. By analyzing the corresponding characteristic equations, the local stability of the predator-extinction equilibrium and the coexistence equilibrium of the system are discussed, respectively. Further, the existence of Hopf bifurcation at the coexistence equilibrium is also studied. By comparison arguments, sufficient conditions are obtained for the global stability of the predator-extinction equilibrium. By using an iteration technique, sufficient conditions are derived for the global stability of the coexistence equilibrium. Numerical simulations are carried out to illustrate the analytical results.

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Keywords: predator-prey system; stage structure; time delay; stability; Hopf bifurcation

1 Introduction

Since the pioneering works of Arditi *et al.* [1, 2], ratio-dependent predator-prey models have received much attention from scientists (see, for example, [3–7]). In [4], Kuang and Beretta studied the following ratio-dependent predator-prey model with the Michaelis-Menten type functional response

$$\begin{cases} \dot{x}(t) = rx(t)(1 - \frac{x(t)}{K}) - \frac{a_1 x(t)y(t)}{x(t)+my(t)}, \\ \dot{y}(t) = \frac{a_2 x(t)y(t)}{x(t)+my(t)} - dy(t), \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ are the densities of the prey and the predator population at time t, respectively. The parameters r, K, a_1, a_2, d and m are positive constants representing the prey intrinsic growth rate, carrying capacity, capturing rate, conversion rate, the predator death rate and half capturing saturation constant, respectively. System (1.1) was systematically studied by Kuang and Beretta [4], and the global stability of boundary equilibria, the positive equilibrium, and permanence of the system were discussed. In [3], Beretta and Kuang incorporated a time delay due to the gestation of the predator into system (1.1). Sufficient conditions were derived for the global stability of positive equilibrium of the delayed system. In [6], Xu and Ma incorporated stage structure for predator and time delay due to the gestation of the predator into system (1.1). Sufficient conditions were obtained for the global stability of the boundary equilibrium and the global attractivity of the positive equilibrium of the stage-structured system. Stage-structured models have received great attention in recent years (for example, [6–10]). Most of the researchers consider models with stage structure only for one species. However, it is of importance to discuss the effects of stage structure for both the predator and the prey species.

Based on the above discussions, in this paper, we incorporate stage structure for both the predator and the prey into system (1.1) and discuss the effects of time delay due to the gestation of the predator on the global dynamics of the model. To this end, we study the following delay differential system:

$$\begin{cases} \dot{x}_1(t) = rx_2(t) - (r_1 + d_1)x_1(t) - ax_1^2(t) - \frac{a_1 x_1(t) y_2(t)}{x_1(t) + m y_2(t)}, \\ \dot{x}_2(t) = r_1 x_1(t) - d_2 x_2(t), \\ \dot{y}_1(t) = \frac{a_2 x_1(t-\tau) y_2(t-\tau)}{x_1(t-\tau) + m y_2(t-\tau)} - (r_2 + d_3) y_1(t), \\ \dot{y}_2(t) = r_2 y_1(t) - d_4 y_2(t), \end{cases} \tag{1.2}$$

where $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$ represent the densities of the immature and the mature prey, the immature and the mature predator at time t, respectively. The parameters a, a_1, a_2, d_1, d_2, d_3, d_4, r, r_1, r_2 and m are positive constants, in which r is the birth rate of the prey; a is the intra-specific competition rate of the immature prey; d_1 and d_2 are the death rates of the immature prey and the mature prey, respectively; r_1 and r_2 are the transformation rates from the immature individuals to mature individuals for the prey and the predator, respectively; a_1 is the capturing rate of the predator, a_2/a_1 is the conversion rate of nutrients into the reproduction of the predator; d_3 and d_4 are the death rates of the immature predator and the mature predator, respectively; m is the half capturing saturation constant. $\tau \geq 0$ is a constant delay due to the gestation of the predator. We note that it is assumed in (1.2) that only mature predators capture immature prey and have the ability to reproduce.

The initial conditions for system (1.2) take the form

$$\begin{aligned} &x_1(\theta) = \phi_1(\theta) \geq 0, \qquad x_2(\theta) = \phi_2(\theta) \geq 0, \\ &y_1(\theta) = \varphi_1(\theta) \geq 0, \qquad y_2(\theta) = \varphi_2(\theta) \geq 0, \quad \theta \in [-\tau, 0), \\ &\phi_1(0) > 0, \qquad \phi_2(0) > 0, \qquad \varphi_1(0) > 0, \\ &\varphi_2(0) > 0, \quad \bigl(\phi_1(\theta), \phi_2(\theta), \varphi_1(\theta), \varphi_2(\theta)\bigr) \in C\bigl([-\tau, 0], R_{+0}^4\bigr), \end{aligned} \tag{1.3}$$

where $R_{+0}^4 = \{(x_1, x_2, x_3, x_4) : x_i \geq 0, i = 1, 2, 3, 4\}$.

It is well known by the fundamental theory of functional differential equations [11] that system (1.2) has a unique solution $(x_1(t), x_2(t), y_1(t), y_2(t))$ satisfying initial conditions (1.3).

The organization of this paper is as follows. In the next section, we investigate the local stability of the predator-extinction equilibrium and the coexistence equilibrium of system (1.2). Further, we study the existence of a Hopf bifurcation for system (1.2) at the coexistence equilibrium. In Section 3, by means of an iterative technique, sufficient conditions are derived for the global stability of the coexistence equilibrium of system (1.2). By comparison arguments, we discuss the global stability of the predator-extinction of system (1.2). Numerical simulations are carried out to illustrate the main results. A brief discussion is given in Section 4 to conclude this work.

2 Local stability and Hopf bifurcation

In this section, we discuss the local stability of equilibria and the existence of a Hopf bifurcation at the coexistence equilibrium of system (1.2).

It is easy to show that if $rr_1 > d_2(r_1 + d_1)$, system (1.2) admits a predator-extinction equilibrium $E_1(x_1^+, x_2^+, 0, 0)$, where

$$x_1^+ = \frac{rr_1 - d_2(r_1 + d_1)}{ad_2}, \qquad x_2^+ = \frac{r_1[rr_1 - d_2(r_1 + d_1)]}{ad_2^2}.$$

Further, if the following condition holds

(H1) $\frac{rr_1 - d_2(r_1+d_1)}{a_1 d_2} > \frac{a_2 r_2 - d_4(r_2+d_3)}{a_2 r_2 m} > 0$,

then system (1.2) has a unique coexistence equilibrium $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$, where

$$x_1^* = \frac{a_1}{a}\left[\frac{rr_1 - d_2(r_1 + d_1)}{a_1 d_2} - \frac{a_2 r_2 - d_4(r_2 + d_3)}{a_2 r_2 m}\right], \qquad x_2^* = \frac{r_1}{d_2} x_1^*,$$
$$y_1^* = \frac{a_2 r_2 - d_4(r_2 + d_3)}{m r_2 (r_2 + d_3)} x_1^*, \qquad y_2^* = \frac{a_2 r_2 - d_4(r_2 + d_3)}{m d_4 (r_2 + d_3)} x_1^*.$$

The characteristic equation of (1.2) at the equilibrium $E_1(x_1^+, x_2^+, 0, 0)$ is of the form

$$\begin{aligned} &\left[\lambda^2 + (r_2 + d_3 + d_4)\lambda + d_4(r_2 + d_3) - a_2 r_2 e^{-\lambda\tau}\right] \\ &\quad \times \left[\lambda^2 + \left(r_1 + d_1 + 2ax_1^+ + d_2\right)\lambda + rr_1 - d_2(r_1 + d_1)\right] = 0. \end{aligned} \tag{2.1}$$

Note that $rr_1 > d_2(r_1 + d_1)$, it is easy to show that the equation

$$\lambda^2 + \left(r_1 + d_1 + 2ax_1^+ + d_2\right)\lambda + rr_1 - d_2(r_1 + d_1) = 0$$

always has two negative real roots. All other roots are given by the roots of equation

$$\lambda^2 + (r_2 + d_3 + d_4)\lambda + d_4(r_2 + d_3) - a_2 r_2 e^{-\lambda\tau} = 0.$$

Let $f(\lambda) = \lambda^2 + (r_2 + d_3 + d_4)\lambda + d_4(r_2 + d_3) - a_2 r_2 e^{-\lambda\tau}$. If (H1) holds and λ is a real number, then we have

$$\begin{aligned} &f(0) = d_4(r_2 + d_3) - a_2 r_2 < 0, \\ &\lim_{x \to +\infty} f(x) = +\infty. \end{aligned}$$

Hence, $f(\lambda) = 0$ has at least one positive real root. Therefore, the equilibrium E_1 is unstable. If $a_2 r_2 < d_4(r_2 + d_3)$, we have $f(0) > 0$ and $f'(\lambda) = 2\lambda + r_2 + d_3 + d_4 + a_2 r_2 \tau e^{-\lambda\tau}$, then it is easily seen that all roots of $f(\lambda) = 0$ have only negative real parts, that is, the equilibrium E_1 is stable. Therefore, if $a_2 r_2 < d_4(r_2 + d_3)$, by Kuang and So ([12], Lemma B), we see that the equilibrium E_1 is locally stable for all $\tau > 0$.

The characteristic equation of system (1.2) at the equilibrium E^* takes the form

$$\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 + \left(q_2\lambda^2 + q_1\lambda + q_0\right)e^{-\lambda\tau} = 0, \tag{2.2}$$

where

$$\begin{aligned} &p_3 = \gamma + d_2 + r_2 + d_3 + d_4, \\ &p_2 = d_4(\gamma + d_2 + r_2 + d_3) + (\gamma + d_2)(r_2 + d_3) + \gamma d_2 - rr_1, \\ &p_1 = d_4(\gamma + d_2)(r_2 + d_3) + (r_2 + d_3 + d_4)(\gamma d_2 - rr_1), \\ &p_0 = d_4(r_2 + d_3)(\gamma d_2 - rr_1), \\ &q_2 = -a_2 r_2 \beta^2, \qquad q_1 = -a_2 r_2 \beta^2 \bigl(2ax_1^* + r_1 + d_1 + d_2\bigr), \\ &q_0 = a_2 r_2 \beta^2 \bigl(rr_1 + d_2 a_1 m \alpha^2 - \gamma d_2\bigr), \\ &\alpha = \frac{a_2 r_2 - d_4(r_2 + d_3)}{a_2 r_2 m}, \qquad \beta = \frac{d_4(r_2 + d_3)}{a_2 r_2}, \qquad \gamma = r_1 + d_1 + 2ax_1^* + a_1 m \alpha^2. \end{aligned}$$

When $\tau = 0$, equation (2.2) becomes

$$\lambda^4 + p_3\lambda^3 + (p_2 + q_2)\lambda + p_0 + q_0 = 0. \tag{2.3}$$

Clearly $p_3 > 0$. If the following condition holds

(H2) $\frac{rr_1 - d_2(r_1+d_1)}{a_1 d_2} > \frac{a_2 r_2 - d_4(r_2+d_3)}{a_2 r_2 m}(1 + \frac{d_4(r_2+d_3)}{a_2 r_2}) > 0$,

then we have

$$\begin{aligned} &p_3(p_2 + q_2) - (p_1 + q_1) > 0, \\ &(p_1 + q_1)\bigl[p_3(p_2 + q_2) - (p_1 + q_1)\bigr] > p_3^2(p_0 + q_0). \end{aligned}$$

Hence, the coexistence equilibrium E^* of system (1.2) is locally asymptotically stable.

If $i\omega$ ($\omega > 0$) is a solution of (2.2), separating real and imaginary parts, we have the following:

$$\begin{aligned} &\bigl(q_2\omega^2 - q_0\bigr)\sin\omega\tau + q_1\omega\cos\omega\tau = p_3\omega^3 - p_1\omega, \\ &\bigl(q_2\omega^2 - q_0\bigr)\cos\omega\tau - q_1\omega\sin\omega\tau = \omega^4 - p_2\omega^2 + p_0. \end{aligned} \tag{2.4}$$

Squaring and adding the two equations of (2.4), it follows that

$$\omega^8 + h_3\omega^6 + h_2\omega^4 + h_1\omega^2 + h_0 = 0, \tag{2.5}$$

where

$$\begin{aligned} &h_0 = p_0^2 - q_0^2, \qquad h_1 = p_1^2 - 2p_0p_2 + 2q_0q_2 - q_1^2, \\ &h_2 = p_2^2 + 2p_0 - 2p_1p_3 - q_2^2, \qquad h_3 = p_3^2 - 2p_2. \end{aligned}$$

Let (H1) hold, it is easy to show that

$$\begin{aligned} &h_3 = \gamma^2 + d_2^2 + (r_2 + d_3)^2 + d_4^2 + 2rr_1 > 0, \\ &h_2 = \bigl(d_2^2 + \gamma^2 + 2rr_1\bigr)\bigl[(r_2 + d_3)^2 + d_4^2\bigr] + (\gamma d_2 - rr_1)^2 + d_4^2(r_2 + d_3)^2 - \bigl(a_2 r_2\beta^2\bigr)^2 > 0, \end{aligned}$$

$$
\begin{aligned}
h_1 &= (\gamma d_2 - rr_1)^2\big[d_4^2 + (r_2 + d_3)^2\big] \\
&\quad + \big[d_4^2(r_2 + d_3)^2 - \big(a_2 r_2 \beta^2\big)^2\big]\big(d_2^2 + \gamma^2 + 2rr_1\big) \\
&\quad + 2\big(a_2 r_2 \beta^2\big)^2 a_1 m \alpha^2 \big(r_1 + d_1 + 2a x_1^*\big) > 0.
\end{aligned}
$$

If the following condition holds

(H3) $\frac{rr_1 - d_2(r_1+d_1)}{a_1 d_2} > \frac{a_2 r_2 - d_4(r_2+d_3)}{a_2 r_2 m}(1 + 2\frac{d_4(r_2+d_3)}{a_2 r_2 + d_4(r_2+d_3)}) > 0$,

then $h_0 > 0$. Hence, the coexistence equilibrium E^* of system (1.2) is locally asymptotically stable for all $\tau > 0$. If the following condition holds

(H4) $\frac{rr_1 - d_2(r_1+d_1)}{a_1 d_2} < \frac{a_2 r_2 - d_4(r_2+d_3)}{a_2 r_2 m}(1 + 2\frac{d_4(r_2+d_3)}{a_2 r_2 + d_4(r_2+d_3)})$,

then $h_0 < 0$. Hence, equation (2.5) has a unique positive root ω_0, that is, the characteristic equation (2.2) admits a pair of purely imaginary roots of the form $\pm i\omega_0$. From (2.4), we see that τ_k corresponding to ω_0 is

$$
\begin{aligned}
\tau_k &= \frac{1}{\omega_0} \arccos \frac{q_1 \omega_0^2 (p_3 \omega_0^2 - p_1) + (q_2 \omega_0^2 - q_0)(\omega_0^4 - p_2 \omega_0^2 + p_0)}{(q_1 \omega_0)^2 + (q_2 \omega_0^2 - q_0)^2} \\
&\quad + \frac{2k\pi}{\omega_0}, \quad k = 0, 1, 2, \ldots.
\end{aligned}
\tag{2.6}
$$

Hence, by the general theory on characteristic equation of delay differential equation from [13], if (H2) and (H4) hold, E^* remains stable for $\tau < \tau_0$.

We now claim that

$$
\left.\frac{d(\mathrm{Re}(\lambda))}{d\tau}\right|_{\tau=\tau_0} > 0.
$$

This will signify that there exists at least one eigenvalue with positive real part for $\tau > \tau_0$. Moreover, the conditions for the existence of a Hopf bifurcation [14] are then satisfied. To this end, differentiating equation (2.2) with respect to τ, it follows that

$$
\begin{aligned}
&\big[\big(4\lambda^3 + 3p_3\lambda^2 + 2p_2\lambda + p_1\big) + (2q_2\lambda + q_1)e^{-\lambda\tau} - \tau\big(q_2\lambda^2 + q_1\lambda + q_0\big)e^{-\lambda\tau}\big]\frac{d\lambda}{d\tau} \\
&\quad = \lambda\big(q_2\lambda^2 + q_1\lambda + q_0\big)e^{-\lambda\tau}.
\end{aligned}
$$

From this equation, we can obtain

$$
\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{4\lambda^3 + 3p_3\lambda^2 + 2p_2\lambda + p_1}{-\lambda(\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)} + \frac{2q_2\lambda + q_1}{\lambda(q_2\lambda^2 + q_1\lambda + q_0)} - \frac{\tau}{\lambda}.
$$

Hence, we derive that

$$
\begin{aligned}
&\mathrm{sign}\left\{\frac{d(\mathrm{Re}\,\lambda)}{d\tau}\right\}_{\lambda=i\omega_0} \\
&\quad = \mathrm{sign}\left\{\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega_0} \\
&\quad = \mathrm{sign}\left\{\mathrm{Re}\left[\frac{4\lambda^3 + 3p_3\lambda^2 + 2p_2\lambda + p_1}{-\lambda(\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)} + \frac{2q_2\lambda + q_1}{\lambda(q_2\lambda^2 + q_1\lambda + q_0)}\right]_{\lambda=i\omega_0}\right\}
\end{aligned}
$$

$$= \operatorname{sign}\left\{ \frac{2(2\omega_0^2 - p_2)(\omega_0^4 - p_2\omega_0^4 + p_0) + (3p_3\omega_0^2 - p_1)(p_3\omega_0^2 - p_1)}{(\omega_0^4 - p_2\omega_0^2 + p_0)^2 + \omega_0^2(p_3\omega_0^2 - p_1)^2} + \frac{-q_1^2 - 2q_2(q_2\omega_0^2 - q_0)}{(q_2\omega_0^2 - q_0)^2 + (q_1\omega_0)^2} \right\}.$$

Note that $(\omega_0^4 - p_2\omega_0^2 + p_0)^2 + \omega_0^2(p_3\omega_0^2 - p_1)^2 = (q_2\omega_0^2 - q_0)^2 + (q_1\omega_0)^2$, then

$$\operatorname{sign}\left\{ \frac{d(\operatorname{Re}\lambda)}{d\tau} \right\}_{\lambda = i\omega_0} = \operatorname{sign}\left\{ \frac{4\omega_0^6 + 3h_3\omega_0^4 + 2h_2\omega_0^2 + h_1}{(q_2\omega_0^2 - q_0)^2 + (q_1\omega_0)^2} \right\}.$$

Accordingly, if (H1) holds, then we have that

$$\operatorname{sign}\left\{ \frac{d(\operatorname{Re}\lambda)}{d\tau} \right\}_{\lambda = i\omega_0} > 0.$$

Therefore, the transversal condition holds and a Hopf bifurcation occurs at $\tau = \tau_0$, $\omega = \omega_0$.

We therefore obtain the following results.

Theorem 2.1 *For system* (1.2), *we have the following*:

(i) *Let* $rr_1 > d_2(r_1 + d_1)$, *if* $a_2r_2 < d_4(r_2 + d_3)$, *then the predator-extinction equilibrium* $E_1(x_1^+, x_2^+, 0, 0)$ *is locally asymptotically stable*; *if* $a_2r_2 > d_4(r_2 + d_3)$, *then the equilibrium* E_1 *is unstable*.

(ii) *If* (H3) *holds, then the positive equilibrium* E^* *is locally asymptotically stable for all* $\tau \geq 0$.

(iii) *If* (H2) *and* (H4) *hold, then there exists a positive number* τ_0 *such that the coexistence equilibrium* E^* *is locally asymptotically stable if* $0 < \tau < \tau_0$ *and unstable if* $\tau > \tau_0$. *Further, system* (1.2) *undergoes a Hopf bifurcation at* E^* *when* $\tau = \tau_0$.

3 Global stability

In this section, we are concerned with the global stability of the coexistence equilibrium E^* and the predator-extinction equilibrium E_1 of system (1.2), respectively.

Theorem 3.1 *Let* (H3) *hold, then the coexistence equilibrium* $E^*(x_1^*, x_2^*, y_1^*, y_2^*)$ *of system* (1.2) *is globally stable provided that*

(H5) $a_2r_2 < 2d_4(r_2 + d_3)$, $\frac{rr_1 - d_2(r_1+d_1)}{a_1d_2} > \frac{1}{m}$.

Proof Let $(x_1(t), x_2(t), y_1(t), y_2(t))$ be any positive solution of system (1.2) with initial conditions (1.3). Let

$$U_{x_i} = \limsup_{t\to+\infty} x_i(t), \qquad L_{x_i} = \liminf_{t\to+\infty} x_i(t),$$

$$U_{y_i} = \limsup_{t\to+\infty} y_i(t), \qquad L_{y_i} = \liminf_{t\to+\infty} y_i(t) \quad (i = 1, 2).$$

We now claim that $U_{x_i} = L_{x_i} = x_i^*$, $U_{y_i} = L_{y_i} = y_i^*$. The strategy of the proof is to use an iteration technique.

We derive from the first and the second equations of system (1.2) that

$$\dot{x}_1(t) \leq rx_2(t) - (r_1 + d_1)x_1(t) - ax_1^2(t),$$

$$\dot{x}_2(t) = r_1x_1(t) - d_2x_2(t).$$

Consider the following auxiliary equations:

$$\begin{aligned} \dot{z}_1(t) &= rz_2(t) - (r_1 + d_1)z_1(t) - az_1^2(t), \\ \dot{z}_2(t) &= r_1 z_1(t) - d_2 z_2(t). \end{aligned} \tag{3.1}$$

Since $rr_1 > d_2(r_1 + d_1)$ holds, by Lemma 2.2 of [10], it follows from (3.1) that

$$\lim_{t\to+\infty} z_1(t) = \frac{rr_1 - d_2(r_1 + d_1)}{ad_2}, \qquad \lim_{t\to+\infty} z_2(t) = \frac{r_1[rr_1 - d_2(r_1 + d_1)]}{ad_2^2}.$$

A comparison argument shows that

$$\begin{aligned} U_{x_1} &= \limsup_{t\to+\infty} x_1(t) \le \frac{rr_1 - d_2(r_1 + d_1)}{ad_2} := M_1^{x_1}, \\ U_{x_2} &= \limsup_{t\to+\infty} x_2(t) \le \frac{r_1[rr_1 - d_2(r_1 + d_1)]}{ad_2^2} := M_1^{x_2}. \end{aligned} \tag{3.2}$$

Hence, for $\varepsilon > 0$ sufficiently small, there exists $T_1 > 0$ such that if $t > T_1$, $x_1(t) \le M_1^{x_1} + \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the third and the fourth equations of system (1.2) that for $t > T_1 + \tau$,

$$\begin{aligned} \dot{y}_1(t) &\le \frac{a_2(M_1^{x_1} + \varepsilon)y_2(t-\tau)}{M_1^{x_1} + \varepsilon + my_2(t-\tau)} - (d_3 + r_2)y_1(t), \\ \dot{y}_2(t) &= r_2 y_1(t) - d_4 y_2(t). \end{aligned}$$

Consider the following auxiliary equations:

$$\begin{aligned} \dot{u}_1(t) &= \frac{a_2(M_1^{x_1} + \varepsilon)u_2(t-\tau)}{M_1^{x_1} + \varepsilon + mu_2(t-\tau)} - (d_3 + r_2)u_1(t), \\ \dot{u}_2(t) &= r_2 u_1(t) - d_4 u_2(t). \end{aligned} \tag{3.3}$$

Since (H3) holds, by Lemma 2.4 of [6], it follows from (3.3) that

$$\begin{aligned} \lim_{t\to+\infty} u_1(t) &= \frac{(M_1^{x_1} + \varepsilon)[a_2 r_2 - d_4(r_2 + d_3)]}{mr_2(r_2 + d_3)}, \\ \lim_{t\to+\infty} u_2(t) &= \frac{(M_1^{x_1} + \varepsilon)[a_2 r_2 - d_4(r_2 + d_3)]}{md_4(r_2 + d_3)}. \end{aligned}$$

By comparison, we derive that

$$\begin{aligned} U_{y_1} &= \limsup_{t\to+\infty} y_1(t) \le \frac{(M_1^{x_1} + \varepsilon)[a_2 r_2 - d_4(r_2 + d_3)]}{mr_2(r_2 + d_3)}, \\ U_{y_2} &= \limsup_{t\to+\infty} y_2(t) \le \frac{(M_1^{x_1} + \varepsilon)[a_2 r_2 - d_4(r_2 + d_3)]}{md_4(r_2 + d_3)}. \end{aligned}$$

Since these inequalities are true for arbitrary $\varepsilon > 0$, it follows that $U_{y_1} \le M_1^{y_1}$, $U_{y_2} \le M_1^{y_2}$, where

$$M_1^{y_1} = \frac{a_2 r_2 - d_4(r_2 + d_3)}{mr_2(r_2 + d_3)} M_1^{x_1}, \qquad M_1^{y_2} = \frac{a_2 r_2 - d_4(r_2 + d_3)}{md_4(r_2 + d_3)} M_1^{x_1}. \tag{3.4}$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is $T_2 \geq T_1 + \tau$ such that if $t > T_2$, $y_2(t) \leq M_1^{y_2} + \varepsilon$.

For $\varepsilon > 0$ sufficiently small, it follows from the first and the second equations of system (1.2) that for $t > T_2$,

$$\begin{aligned} \dot{x}_1(t) &\geq rx_2(t) - (r_1 + d_1)x_1(t) - ax_1^2(t) - \frac{a_1}{m}x_1(t), \\ \dot{x}_2(t) &= r_1x_1(t) - d_2x_2(t). \end{aligned} \tag{3.5}$$

Since (H5) holds, by Lemma 2.2 of [10] and a comparison argument, it follows from (3.5) that

$$\begin{aligned} V_{x_1} &= \liminf_{t\to+\infty} x_1(t) \geq \frac{m[rr_1 - d_2(d_1 + r_1)] - a_1d_2}{amd_2} := N_1^{x_1}, \\ V_{x_2} &= \liminf_{t\to+\infty} x_2(t) \geq \frac{mr_1[rr_1 - d_2(d_1 + r_1)] - a_1r_1d_2}{amd_2^2} := N_1^{x_2}. \end{aligned}$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is $T_3 \geq T_2$ such that if $t > T_3$, $x_1(t) \geq N_1^{x_1} - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, it follows from the third and the fourth equations of system (1.2) that for $t > T_3 + \tau$,

$$\begin{aligned} \dot{y}_1(t) &\geq \frac{a_2(N_1^{x_1} - \varepsilon)y_2(t-\tau)}{N_1^{x_1} - \varepsilon + my_2(t-\tau)} - (r_2 + d_3)y_1(t), \\ \dot{y}_2(t) &= r_2y_1(t) - d_4y_2(t). \end{aligned} \tag{3.6}$$

Since $a_2r_2 > d_4(r_2 + d_3)$ holds, by Lemma 2.4 of [6] and a comparison argument, it follows from (3.6) that

$$\begin{aligned} V_{y_1} &= \liminf_{t\to+\infty} y_1(t) \geq \frac{[a_2r_2 - d_4(d_3 + r_2)](N_1^{x_1} - \varepsilon)}{mr_2(d_3 + r_2)}, \\ V_{y_2} &= \liminf_{t\to+\infty} y_2(t) \geq \frac{[a_2r_2 - d_4(d_3 + r_2)](N_1^{x_1} - \varepsilon)}{md_4(d_3 + r_2)}. \end{aligned}$$

Since these two inequalities hold for arbitrary $\varepsilon > 0$ sufficiently small, we conclude that $V_{y_1} \geq N_1^{y_1}$, $V_{y_2} \geq N_1^{y_2}$, where

$$N_1^{y_1} = \frac{a_2r_2 - d_4(d_3 + r_2)}{mr_2(d_3 + r_2)}N_1^{x_1}, \qquad N_1^{y_2} = \frac{a_2r_2 - d_4(d_3 + r_2)}{md_4(d_3 + r_2)}N_1^{x_1}.$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is $T_4 \geq T_3 + \tau$ such that if $t > T_4$, $y_2(t) \geq N_1^{y_2} - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, it follows from the first and the second equations of system (1.2) that for $t > T_4$,

$$\begin{aligned} \dot{x}_1(t) &\leq rx_2(t) - (r_1 + d_1)x_1(t) - ax_1^2(t) - \frac{a_1(N_1^{y_2} - \varepsilon)}{M_1^{x_1} + \varepsilon + m(N_1^{y_2} - \varepsilon)}x_1(t), \\ \dot{x}_2(t) &= r_1x_1(t) - d_2x_2(t). \end{aligned} \tag{3.7}$$

Since (H3) holds, by Lemma 2.2 of [10] and a comparison argument, for arbitrary $\varepsilon > 0$ sufficiently small, it follows from (3.7) that

$$U_{x_1} = \limsup_{t\to+\infty} x_1(t) \le \frac{rr_1 - d_2(r_1 + d_1)}{ad_2} - \frac{a_1 N_1^{y_2}}{a(M_1^{x_1} + mN_1^{y_2})} := M_2^{x_1},$$
$$U_{x_2} = \limsup_{t\to+\infty} x_2(t) \le \frac{r_1}{d_2} M_2^{x_1} := M_2^{x_2}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there is $T_5 \ge T_4$ such that if $t > T_5$, $x_1(t) \le M_2^{x_1} + \varepsilon$. We therefore obtain from the third and the fourth equations of system (1.2) that for $t > T_5 + \tau$,

$$\begin{aligned} \dot{y}_1(t) &\le \frac{a_2(M_2^{x_1} + \varepsilon)y_2(t-\tau)}{M_2^{x_1} + \varepsilon + my_2(t-\tau)} - (r_2 + d_3)y_1(t), \\ \dot{y}_2(t) &= r_2 y_1(t) - d_4 y_2(t). \end{aligned} \tag{3.8}$$

Since $a_2 r_2 > d_4(r_2 + d_3)$ holds, by Lemma 2.4 of [6] and a comparison argument, for arbitrary $\varepsilon > 0$ sufficiently small, it follows from (3.8) that

$$U_{y_1} = \limsup_{t\to+\infty} y_1(t) \le \frac{a_2 r_2 - d_4(r_2 + d_3)}{mr_2(r_2 + d_3)} M_2^{x_1} := M_2^{y_1},$$
$$U_{y_2} = \limsup_{t\to+\infty} y_2(t) \le \frac{a_2 r_2 - d_4(r_2 + d_3)}{md_4(r_2 + d_3)} M_2^{x_1} := M_2^{y_2}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there is $T_6 \ge T_5 + \tau$ such that if $t > T_6$, $y_2(t) \le M_2^{y_2} + \varepsilon$.

Again, for $\varepsilon > 0$ sufficiently small, it follows from the first and the second equations of system (1.2) that for $t > T_6$,

$$\begin{aligned} \dot{x}_1(t) &\ge rx_2(t) - (r_1 + d_1)x_1(t) - ax_1^2(t) - \frac{a_1(M_2^{y_2} + \varepsilon)}{N_1^{x_1} - \varepsilon + m(M_2^{y_2} + \varepsilon)} x_1(t), \\ \dot{x}_2(t) &= r_1 x_1(t) - d_2 x_2(t). \end{aligned} \tag{3.9}$$

Since (H3) holds, by Lemma 2.2 of [10] and a comparison argument, for arbitrary $\varepsilon > 0$ sufficiently small, it follows from (3.9) that

$$V_{x_1} = \liminf_{t\to+\infty} x_1(t) \ge \frac{rr_1 - d_2(r_1 + d_1)}{ad_2} - \frac{a_1 M_2^{y_2}}{a(N_1^{x_1} + mM_2^{y_2})} := N_2^{x_1},$$
$$V_{x_2} = \liminf_{t\to+\infty} x_2(t) \ge \frac{r_1}{d_2} N_2^{x_1} := N_2^{x_2}.$$

Therefore, for $\varepsilon > 0$ sufficiently small, there is $T_7 \ge T_6$ such that if $t > T_7$, $x_1(t) \ge N_2^{x_1} - \varepsilon$.

For $\varepsilon > 0$ sufficiently small, we derive from the third and the fourth equations of system (1.2) that for $t > T_7 + \tau$,

$$\begin{aligned} \dot{y}_1(t) &\ge \frac{a_2(N_2^{x_1} - \varepsilon)y_2(t-\tau)}{N_2^{x_2} - \varepsilon + my_2(t-\tau)} - (d_3 + r_2)y_1(t), \\ \dot{y}_2(t) &= r_2 y_1(t) - d_4 y_2(t). \end{aligned} \tag{3.10}$$

Since $a_2r_2 > d_4(d_3 + r_2)$ holds, by Lemma 2.4 of [6] and a comparison argument, for arbitrary $\varepsilon > 0$ sufficiently small, it follows from (3.10) that

$$V_{y_1} = \liminf_{t\to+\infty} y_1(t) \geq \frac{a_2r_2 - d_4(d_3 + r_2)}{mr_2(d_3 + r_2)} N_2^{x_1} := N_2^{y_1},$$
$$V_{y_2} = \liminf_{t\to+\infty} y_2(t) \geq \frac{a_2r_2 - d_4(d_3 + r_2)}{md_4(d_3 + r_2)} N_2^{x_1} := N_2^{y_2}.$$

Continuing this process, we derive eight sequences $M_k^{x_1}$, $M_k^{x_2}$, $M_k^{y_1}$, $M_k^{y_2}$, $N_k^{x_1}$, $N_k^{x_2}$, $N_k^{y_1}$, $N_k^{y_2}$ $(k = 1, 2, \ldots)$ such that for $k \geq 2$,

$$\begin{aligned}
&M_k^{x_1} = \frac{rr_1 - d_2(r_1 + d_1)}{ad_2} - \frac{a_1 N_{k-1}^{y_2}}{a(M_{k-1}^{x_1} + mN_{k-1}^{y_2})}, \qquad M_k^{x_2} = \frac{r_1}{d_2} M_k^{x_1},\\
&M_k^{y_1} = \frac{a_2r_2 - d_4(r_2 + d_3)}{mr_2(r_2 + d_3)} M_k^{x_1}, \qquad M_k^{y_2} = \frac{a_2r_2 - d_4(r_2 + d_3)}{md_4(r_2 + d_3)} M_k^{x_1},\\
&N_k^{x_1} = \frac{rr_1 - d_2(r_1 + d_1)}{ad_2} - \frac{a_1 M_k^{y_2}}{a(N_{k-1}^{x_1} + mM_k^{y_2})}, \qquad N_k^{x_2} = \frac{r_1}{d_2} N_k^{x_1},\\
&N_k^{y_1} = \frac{a_2r_2 - d_4(r_2 + d_3)}{mr_2(r_2 + d_3)} N_k^{x_1}, \qquad N_k^{y_2} = \frac{a_2r_2 - d_4(d_3 + r_2)}{md_4(r_2 + d_3)} N_k^{x_1}.
\end{aligned} \tag{3.11}$$

It is readily seen that

$$N_k^{x_i} \leq V_{x_i} \leq U_{x_i} \leq M_k^{x_i}, \qquad N_k^{y_i} \leq V_{y_i} \leq U_{y_i} \leq M_k^{y_i} \quad (i = 1, 2). \tag{3.12}$$

It is easy to show that the sequences $M_k^{x_i}$, $M_k^{y_i}$ are nonincreasing and the sequences $N_k^{x_i}$, $N_k^{y_i}$ are nondecreasing. Hence, the limit of each sequence in $M_k^{x_i}$, $M_k^{y_i}$, $N_k^{x_i}$, $N_k^{y_i}$ exists. Denote

$$\bar{x}_i = \lim_{t\to+\infty} M_n^{x_i}, \qquad \underline{x}_i = \lim_{t\to+\infty} N_n^{x_i}, \qquad \bar{y}_i = \lim_{t\to+\infty} M_n^{y_i}, \qquad \underline{y}_i = \lim_{t\to+\infty} N_n^{y_i}, \quad i = 1, 2.$$

We therefore obtain from (3.11) that

$$\begin{aligned}
&\bar{x}_1 = \frac{rr_1 - d_2(r_1 + d_1)}{ad_2} - \frac{a_1 \underline{y}_2}{a(\bar{x}_1 + m\underline{y}_2)}, \qquad \bar{x}_2 = \frac{r_1}{d_2}\bar{x}_1,\\
&\bar{y}_1 = \frac{a_2r_2 - d_4(r_2 + d_3)}{mr_2(r_2 + d_3)}\bar{x}_1, \qquad \bar{y}_2 = \frac{a_2r_2 - d_4(r_2 + d_3)}{md_4(r_2 + d_3)}\bar{x}_1,\\
&\underline{x}_1 = \frac{rr_1 - d_2(r_1 + d_1)}{ad_2} - \frac{a_1 \bar{y}_2}{a(\underline{x}_1 + m\bar{y}_2)}, \qquad \underline{x}_2 = \frac{r_1}{d_2}\underline{x}_1,\\
&\underline{y}_1 = \frac{a_2r_2 - d_4(r_2 + d_3)}{mr_2(r_2 + d_3)}\underline{x}_1, \qquad \underline{y}_2 = \frac{a_2r_2 - d_4(r_2 + d_3)}{md_4(r_2 + d_3)}\underline{x}_1.
\end{aligned} \tag{3.13}$$

To complete the proof, it is sufficient to prove that $\bar{x}_i = \underline{x}_i$, $\bar{y}_i = \underline{y}_i$ $(i = 1, 2)$. It follows from (3.13) that

$$\begin{aligned}
&ad_2\bar{x}_1^2 + ad_2\frac{a_2r_2 - d_4(r_2 + d_3)}{d_4(r_2 + d_3)}\bar{x}_1\underline{x}_1\\
&\quad = \bigl[rr_1 - d_2(r_1 + d_1)\bigr]\bar{x}_1 + \bigl[m\bigl(rr_1 - d_2(r_1 + d_1)\bigr) - a_1d_2\bigr]\frac{a_2r_2 - d_4(r_2 + d_3)}{md_4(r_2 + d_3)}\underline{x}_1,
\end{aligned} \tag{3.14}$$

$$
\begin{aligned}
&ad_2\underline{x}_1^2 + ad_2\frac{a_2r_2 - d_4(r_2+d_3)}{d_4(r_2+d_3)}\bar{x}_1\underline{x}_1 \\
&\quad = \bigl[rr_1 - d_2(r_1+d_1)\bigr]\underline{x}_1 + \bigl[m\bigl(rr_1 - d_2(r_1+d_1)\bigr) - a_1d_2\bigr]\frac{a_2r_2 - d_4(r_2+d_3)}{md_4(r_2+d_3)}\bar{x}_1.
\end{aligned} \tag{3.15}
$$

Equation (3.14) minus (3.15),

$$
\begin{aligned}
ad_2\bigl(\bar{x}_1^2 - \underline{x}_1^2\bigr) &= \bigl[rr_1 - d_2(r_1+d_1)\bigr](\bar{x}_1 - \underline{x}_1) \\
&\quad - \frac{[a_2r_2 - d_4(r_2+d_3)][m(rr_1 - d_2(r_1+d_1)) - a_1(r_1+d_1)]}{md_4(r_2+d_3)} \\
&\quad \times (\bar{x}_1 - \underline{x}_1).
\end{aligned} \tag{3.16}
$$

Assume that $\bar{x}_1 \neq \underline{x}_1$, then we derive from (3.16) that

$$
\begin{aligned}
ad_2(\bar{x}_1 + \underline{x}_1) &= rr_1 - d_2(r_1+d_1) \\
&\quad - \bigl[m\bigl(rr_1 - d_2(r_1+d_1)\bigr) - a_1d_2\bigr]\frac{a_2r_2 - d_4(r_2+d_3)}{md_4(r_2+d_3)}.
\end{aligned} \tag{3.17}
$$

Equation (3.14) plus (3.15),

$$
\begin{aligned}
&ad_2(\bar{x}_1 + \underline{x}_1)^2 + 2ad_2\frac{a_2r_2 - 2d_4(r_2+d_3)}{d_4(r_2+d_3)}\bar{x}_1\underline{x} \\
&\quad = \left[rr_1 - d_2(r_1+d_1) + \bigl(m\bigl(rr_1 - d_2(r_1+d_1)\bigr) - a_1d_2\bigr)\frac{a_2r_2 - d_4(r_2+d_3)}{md_4(r_2+d_3)}\right] \\
&\qquad \times (\bar{x}_1 + \underline{x}_1).
\end{aligned} \tag{3.18}
$$

Substituting (3.17) into (3.18), it follows that

$$
\begin{aligned}
&ad_2\bigl[a_2r_2 - 2d_4(r_2+d_3)\bigr]\bar{x}_1\underline{x} \\
&\quad = \bigl[a_2r_2 - d_4(r_2+d_3)\bigr]\left[rr_1 - d_2(r_1+d_1) - \frac{a_1d_2}{m}\right](\bar{x}_1 + \underline{x}_1).
\end{aligned} \tag{3.19}
$$

Note that $\bar{x}_1 > 0$, $\underline{x}_1 > 0$ and (H5) holds. This is a contradiction. Accordingly, we have $\bar{x}_1 = \underline{x}_1$. It therefore follows from (3.13) that $\bar{x}_2 = \underline{x}_2$, $\bar{y}_1 = \underline{y}_1$ and $\bar{y}_2 = \underline{y}_2$. Hence, the positive equilibrium E^* is global stability. The proof is complete. □

Theorem 3.2 *The predator-extinction equilibrium* $E_1(x_1^+, x_2^+, 0, 0)$ *of system* (1.2) *is globally stable provided that*

(H6) $a_2r_2 < d_4(r_2+d_3)$, $\frac{rr_1 - d_2(r_1+d_1)}{a_1d_2} > \frac{1}{m}$.

Proof Let $(x_1(t), x_2(t), y_1(t), y_2(t))$ be any positive solution of system (1.2) with initial conditions (1.3). It follows from the first and the second equations of (1.2) that

$$
\begin{aligned}
&\dot{x}_1(t) \le rx_2(t) - (r_1+d_1)x_1(t) - ax_1^2(t), \\
&\dot{x}_2(t) = r_1x_1(t) - d_2x_2(t).
\end{aligned}
$$

Consider the following auxiliary equations:

$$\begin{aligned}
&\dot{z}_1(t) = rz_2(t) - (r_1 + d_1)z_1(t) - az_1^2(t), \\
&\dot{z}_2(t) = r_1 z_1(t) - d_2 z_2(t).
\end{aligned} \tag{3.20}$$

Since $rr_1 > d_2(r_1 + d_1)$, by Lemma 2.2 of [10], it follows from (3.20) that

$$\lim_{t\to+\infty} z_1(t) = \frac{rr_1 - d_2(r_1 + d_1)}{ad_2}, \qquad \lim_{t\to+\infty} z_2(t) = \frac{r_1[rr_1 - d_2(r_1 + d_1)]}{ad_2^2}.$$

By comparison, we obtain that

$$\limsup_{t\to+\infty} x_1(t) \le \frac{rr_1 - d_2(r_1 + d_1)}{ad_2}, \qquad \limsup_{t\to+\infty} x_2(t) \le \frac{r_1[rr_1 - d_2(r_1 + d_1)]}{ad_2^2}. \tag{3.21}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is $T_1 > 0$ such that if $t > T_1$, then $x_1(t) \le x_1^+ + \varepsilon$. We therefore derive from the third and the fourth equations of system (1.2) that for $t > T_1 + \tau$,

$$\begin{aligned}
&\dot{y}_1(t) \le \frac{a_2(x_1^+ + \varepsilon)y_2(t-\tau)}{x_1^+ + \varepsilon + my_2(t-\tau)} - (r_2 + d_3)y_1(t), \\
&\dot{y}_2(t) = r_2 y_1(t) - d_4 y_2(t).
\end{aligned}$$

Consider the following auxiliary equations:

$$\begin{aligned}
&\dot{u}_1(t) = \frac{a_2(x_1^+ + \varepsilon)u_2(t-\tau)}{x_1^+ + \varepsilon + mu_2(t-\tau)} - (r_2 + d_3)u_1(t), \\
&\dot{u}_2(t) = r_2 u_1(t) - d_4 u_2(t).
\end{aligned} \tag{3.22}$$

Since (H6) holds, by Lemma 2.4 of [6], it follows from (3.22) that

$$\lim_{t\to+\infty} u_1(t) = 0, \qquad \lim_{t\to+\infty} u_2(t) = 0.$$

A comparison argument shows that

$$\lim_{t\to+\infty} y_1(t) = 0, \qquad \lim_{t\to+\infty} y_2(t) = 0.$$

Hence, for $\varepsilon > 0$ sufficiently small, there is $T_2 > T_1$ such that if $t > T_2$, $y_2(t) < \varepsilon$.

It follows from the first and the second equations of system (1.2) that for $t > T_2$,

$$\begin{aligned}
&\dot{x}_1(t) \ge rx_2(t) - (r_1 + d_1)x_1(t) - ax_1^2(t) - \frac{a_1 \varepsilon x_1(t)}{m\varepsilon + x_1(t)}, \\
&\dot{x}_2(t) = r_1 x_1(t) - d_2 x_2(t).
\end{aligned}$$

Consider the following auxiliary equations:

$$\begin{aligned}
&\dot{z}_1(t) = rz_2(t) - (r_1 + d_1)z_1(t) - az_1^2(t) - \frac{a_1 \varepsilon z_1(t)}{m\varepsilon + z_1(t)}, \\
&\dot{z}_2(t) = r_1 z_1(t) - d_2 z_2(t).
\end{aligned} \tag{3.23}$$

Since (H6) holds, by Lemma 2.2 of [10], we can obtain

$$\lim_{t\to+\infty} z_1(t) = \sqrt{\left(\frac{rr_1 - d_2(r_1+d_1)}{2ad_2} - \frac{m}{2}\varepsilon\right)^2 + \varepsilon\left[m\frac{rr_1 - d_2(r_1+d_1)}{ad_2} - \frac{a_1}{a}\right]} + \frac{rr_1 - d_2(r_1+d_1)}{2ad_2} - \frac{m}{2}\varepsilon := z_1^*,$$

$$\lim_{t\to+\infty} z_2(t) = \lim_{t\to+\infty} \frac{r_1}{d_2} z_1^*.$$

For $\varepsilon > 0$ is arbitrary small, by comparison, we derive that

$$\liminf_{t\to+\infty} x_1(t) \ge \frac{rr_1 - d_2(r_1+d_1)}{ad_2}, \qquad \liminf_{t\to+\infty} x_2(t) \ge \frac{r_1[rr_1 - d_2(r_1+d_1)]}{ad_2^2},$$

which, together with (3.21), yields

$$\lim_{t\to+\infty} x_1(t) = \frac{rr_1 - d_2(r_1+d_1)}{ad_2}, \qquad \lim_{t\to+\infty} x_2(t) = \frac{r_1[rr_1 - d_2(r_1+d_1)]}{ad_2^2}.$$

This completes the proof. □

We now give some examples to illustrate the main results above.

Example 1 In (1.2), let $a = 1$, $a_1 = 2$, $a_2 = 1$, $d_1 = d_2 = 1$, $d_3 = d_4 = 1/2$, $r = 3$, $r_1 = 2$, $r_2 = 1$ and $m = 2$. With above coefficients, system (1.2) has a unique coexistence equilibrium $E^*(2.75, 5.5, 0.2292, 0.4583)$. It is easy to show that $\frac{rr_1-d_2(r_1+d_1)}{a_1d_2} - \frac{a_2r_2-d_4(r_2+d_3)}{a_2r_2m}(1 + \frac{d_4(r_2+d_3)}{a_2r_2}) \approx 1.2813$, $a_2r_2 - 2d_4(r_2 + d_3) = -0.5$, $\frac{rr_1-d_2(r_1+d_1)}{a_1d_2} - \frac{1}{m} = 1$. By Theorem 3.1, E^* is globally asymptotically stable. Numerical simulation illustrates this fact (see Figure 1).

Example 2 In (1.2), we let $a = 1$, $a_1 = 2$, $a_2 = 2$, $d_1 = d_2 = 2$, $d_3 = d_4 = 1$, $r = 1$, $r_1 = 1$, $r_2 = 1/3$ and $m = 2$. System (1.2) with the above coefficients has a unique coexistence equilibrium $E^*(0.8889, 2.6667, 2.0741, 6.2222)$. Clearly, $\frac{rr_1-d_2(r_1+d_1)}{a_1d_2} - \frac{a_2r_2-d_4(r_2+d_3)}{a_2r_2m}(1 + \frac{d_4(r_2+d_3)}{a_2r_2}) \approx 0.0988$, $\frac{rr_1-d_2(r_1+d_1)}{a_1d_2} - \frac{a_2r_2-d_4(r_2+d_3)}{a_2r_2m}(1 + 2\frac{d_4(r_2+d_3)}{a_2r_2+d_4(r_2+d_3)}) \approx -0.1212$. By Theorem 2.1, there is $\tau_0 > 0$ such that for $\tau < \tau_0$, the coexistence equilibrium E^* is locally stable; for $\tau > \tau_0$, the coexistence equilibrium E^* is unstable. Numerical simulation illustrates this result (see Figure 2).

Figure 1 The temporal solution found by numerical integration of system (1.2) with $\tau = 3$; $(\phi_1, \phi_2, \varphi_1, \varphi_2) = (1, 1, 1, 1)$.

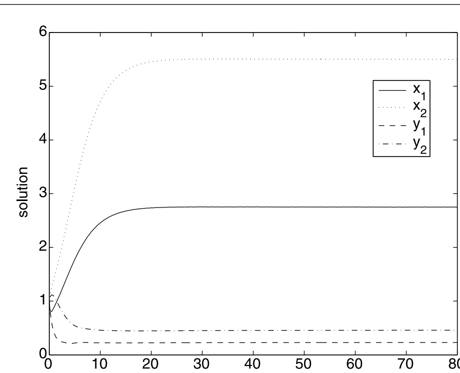


Figure 2 **The temporal solution found by numerical integration of system (1.2) with $\tau = 128$; $(\phi_1, \phi_2, \varphi_1, \varphi_2) = (1, 1, 1, 1)$.**

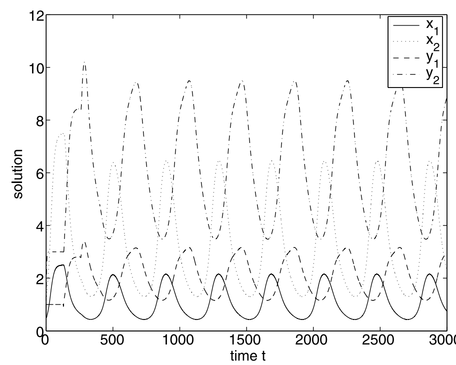
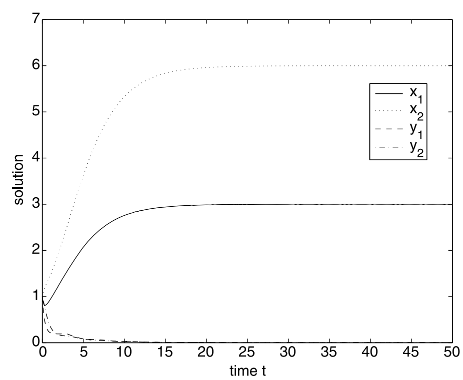


Figure 3 **The temporal solution found by numerical integration of system (1.2) with $\tau = 3$; $(\phi_1, \phi_2, \varphi_1, \varphi_2) = (1, 1, 1, 1)$.**



Example 3 In (1.2), let $a = 1$, $a_1 = 2$, $a_2 = 2$, $d_1 = d_2 = 1$, $d_3 = d_4 = 1$, $r = 3$, $r_1 = 1$, $r_2 = 3/2$ and $m = 2$. System (1.2) with the above coefficients has a predator-extinction equilibrium $E_1(3, 6, 0, 0)$. It is easy to show that $a_2 r_2 - d_4(r_2 + d_3) = -4$, $\frac{rr_1 - d_2(r_1+d_1)}{a_1 d_2} - \frac{1}{m} = 1$. By Theorem 3.2, E_1 is globally asymptotically stable. Numerical simulation illustrates our result (see Figure 3).

4 Discussion

In this paper, we have incorporated stage structure for both the predator and the prey into a predator-prey model with time delay due to the gestation of the predator. By using the iteration technique and comparison arguments, we have established sufficient conditions for the global stability of the coexistence equilibrium and the predator-extinction equilibrium. As a result, we have shown the threshold for the permanence and extinction of the system. By Theorems 3.1 and 3.2, we see that: (i) if (H6) holds, the predator population will go to extinction; (ii) if (H3) and (H5) hold, then both the prey and the predator species of system (1.2) are permanent.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

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References

1. Arditi, R, Perrin, N, Saiah, H: Functional response and heterogeneities: an experiment test with cladocerans. Oikos **60**, 69-75 (1991)
2. Arditi, R, Saiah, H: Empirical evidence of the role of heterogeneity in ratio-dependent consumption. Ecology **73**, 1544-1551 (1992)
3. Beretta, E, Kuang, Y: Global analysis in some delayed ratio-dependent predator-prey systems. Nonlinear Anal. TMA **32**, 381-408 (1998)
4. Kuang, Y, Beretta, E: Global qualitative analysis of a ratio-dependent predator-prey system. J. Math. Biol. **36**, 389-406 (1998)
5. Shi, R, Chen, L: The study of a ratio-dependent predator-prey model with stage structure in the prey. Nonlinear Dyn. **58**, 443-451 (2009)
6. Xu, R, Ma, Z: Stability and Hopf bifurcation in a ratio-dependent predator-prey system with stage structure. Chaos Solitons Fractals **38**, 669-684 (2008)
7. Xu, R, Gan, Q, Ma, Z: Stability and bifurcation analysis on a ratio-dependent predator-prey model with time delay. J. Comput. Appl. Math. **230**, 187-203 (2009)
8. Wang, W, Chen, L: A predator-prey system with stage-structure for predator. Comput. Math. Appl. **33**, 83-91 (1997)
9. Xiao, Y, Chen, L: Global stability of a predator-prey system with stage structure for the predator. Acta Math. Sin. **19**, 1-11 (2003)
10. Xu, R, Ma, Z: The effect of stage-structure on the permanence of a predator-prey system with time delay. Appl. Math. Comput. **189**, 1164-1177 (2007)
11. Hale, J: Theory of Functional Differential Equation. Springer, Heidelberg (1977)
12. Kuang, Y, So, JWH: Analysis of a delayed two-stage population with space-limited recruitment. SIAM J. Appl. Math. **55**, 1675-1696 (1995)
13. Kuang, Y: Delay Differential Equations: With Applications in Population Dynamics. Academic Press, New York (1993)
14. Hassard, B, Kazarinoff, N, Wan, YH: Theory and Applications of Hopf Bifurcation. London Math Soc. Lect. Notes Series, vol. 41. Cambridge University Press, Cambridge (1981)