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Blow-up solutions, global existence, and exponential decay estimates for second order parabolic problems

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Abstract

In this paper, we study the blow-up solutions, global existence, and exponential decay estimates for a class of second order parabolic problems with Dirichlet boundary conditions. By constructing auxiliary functions and using maximum principles, the sufficient conditions for the existence of the blow-up solution, the sufficient conditions for the global existence of the solution, an upper bound for the 'blow-up time', and some explicit exponential decay bounds for the solution and its derivatives are specified.

MSC: 35K60; 35K20

Keywords: blow-up solution; global existence; exponential decay estimate

1 Introduction

Many authors have studied the blow-up solutions, global existence, and exponential decay estimates of nonlinear parabolic problems (see, for instance, [1–14]). In this paper, we investigate the following second order parabolic problems with Dirichlet boundary conditions:

$$\begin{cases} (k(u))_t = \nabla\cdot(g(u)\nabla u) + f(u), & (x,t)\in D\times(0,T), \\ u = 0, & (x,t)\in\partial D\times(0,T), \\ u(x,0) = h(x), & x\in\overline{D}, \end{cases} \tag{1.1}$$

where $D\subset\mathbb{R}^N$ $(N\geq 2)$ is a bounded convex domain with smooth boundary $\partial D\in C^{2,\varepsilon}$, T is the maximal existence time of u, and $\overline{D}$ is the closure of D. Set $\mathbb{R}^+ := (0,+\infty)$. We assume, throughout the paper, that $f(s)$ is a nonnegative $C^1(\mathbb{R}^+)$ function, $f(0)=0$, $g(s)$ is a positive $C^2(\mathbb{R}^+)$ function, $g'(s)\leq 0$ for any $s\in\mathbb{R}^+$, $k(s)$ is a $C^2(\overline{\mathbb{R}^+})$ function, $k'(s)>0$ for any $s\in\overline{\mathbb{R}^+}$, and $h(x)$ is a nonnegative $C^2(\overline{D})$ function, $h(x)\not\equiv 0$ for any $x\in\overline{D}$. Under these assumptions, it follows from the maximum principle [15] that $u(x,t)$ is nonnegative.

Some special cases of the problem (1.1) have been discussed already. Payne *et al.* in [16] dealt with the following problem:

$$\begin{cases} u_t = \Delta u + f(u), & (x,t)\in D\times(0,T), \\ u = 0, & (x,t)\in\partial D\times(0,T), \\ u(x,0) = h(x), & x\in\overline{D}. \end{cases}$$

They established conditions on data sufficient to preclude blow-up and to ensure that the solution and its spatial gradient decay exponentially for all $t > 0$. In [17], Enache researched the following problem:

$$\begin{cases} u_t = \nabla \cdot (g(u)\nabla u) + f(u), & (x,t) \in D \times (0,T), \\ u = 0, & (x,t) \in \partial D \times (0,T), \\ u(x,0) = h(x), & x \in \overline{D}. \end{cases}$$

His purpose was to establish conditions on the data sufficient to guarantee blow-up of solution at some finite time, conditions to ensure that the solution remains bounded as well as conditions to derive some explicit exponential decay bounds for the solution and its derivatives. Some authors also discussed blow-up phenomena for parabolic problems with Dirichlet boundary conditions and obtained a lot of interesting results (see, for instance, [18–24]).

In the process of heat conduction and mass diffusion, many problems can be summarized as the problem (1.1). Therefore, in this paper, we study the problem (1.1). By constructing auxiliary functions and using maximum principles, the sufficient conditions for the existence of the blow-up solution, the sufficient conditions for the global existence of the solution, an upper bound for the 'blow-up time', and some explicit exponential decay bounds for the solution and its derivatives are specified. Our results extend and supplement those obtained in [16, 17].

We proceed as follows. In Section 2 we study the blow-up solution of (1.1). Section 3 is devoted to the global solution of (1.1) and the explicit exponential decay bounds for the solution. The explicit exponential decay bounds for the derivatives of the solution are given in Section 4. A few examples are presented in Section 5 to illustrate the applications of the abstract results.

2 Blow-up solution

In order to get the sufficient conditions for the existence of the blow-up solution, we define the following functions:

$$F(u) := \int_0^u f(s)g(s)\,\mathrm{d}s, \qquad G(u) := 2\int_0^u sg(s)k'(s)\,\mathrm{d}s,$$
$$A(t) := \int_D G\bigl(u(x,t)\bigr)\,\mathrm{d}x, \qquad B(t) := \int_D \left(F(u) - \frac{1}{2}g^2(u)|\nabla u|^2\right)\mathrm{d}x.$$

The following theorem is the main result for the blow-up solution.

Theorem 2.1 *Let u be a classical solution of the problem* (1.1). *Suppose we have the following.*

(i)
$$\bigl(g(s)k'(s)\bigr)' \le 0, \qquad sf(s)g(s) \ge \frac{1}{2}(4+\alpha)F(s), \quad s \in \mathbb{R}^+, \tag{2.1}$$

where α is a positive constant.

(ii)
$$\lim_{s\to 0^+} s^2 g(s) = 0. \tag{2.2}$$

(iii)
$$B(0) = \int_D \left(F(h) - \frac{1}{2} g^2(h) |\nabla h|^2 \right) \mathrm{d}x \geq 0.$$

Then $u(x,t)$ must blow up at some finite time $t^* < T$ and

$$T := \frac{4}{\alpha(\alpha+4)} A(0) B^{-1}(0) \leq +\infty.$$

Proof Making use of the differential equation (1.1), of the divergence theorem, of the fact that $g' \leq 0$ and of the assumption (2.1), we have

$$\begin{aligned} A'(t) &= 2\int_D gk' u u_t \,\mathrm{d}x = 2\int_D fgu \,\mathrm{d}x + 2\int_D gu\big(g\Delta u + g'|\nabla u|^2\big) \,\mathrm{d}x \\ &= 2\int_D fgu \,\mathrm{d}x - 2\int_D gg' u |\nabla u|^2 \,\mathrm{d}x - 2\int_D g^2 |\nabla u|^2 \,\mathrm{d}x \\ &\geq 2\int_D fgu \,\mathrm{d}x - 2\int_D g^2 |\nabla u|^2 \,\mathrm{d}x \\ &\geq (4+\alpha) \int_D \left(F(u) - \frac{1}{2} g^2(u) |\nabla u|^2 \right) \mathrm{d}x + \frac{\alpha}{2} \int_D g^2(u) |\nabla u|^2 \,\mathrm{d}x \\ &\geq (4+\alpha) B(t). \end{aligned} \tag{2.3}$$

It follows from the divergence theorem that

$$\begin{aligned} B'(t) &= \int_D \big(fgu_t - gg'|\nabla u|^2 u_t - g^2 \nabla u \cdot \nabla u_t\big) \,\mathrm{d}x \\ &= \int_D \big(fgu_t - gg'|\nabla u|^2 u_t\big) \,\mathrm{d}x + 2\int_D gg'|\nabla u|^2 u_t \,\mathrm{d}x - \int_D \nabla\big(g^2 u_t\big) \cdot \nabla u \,\mathrm{d}x \\ &= \int_D fgu_t \,\mathrm{d}x + \int_D gg'|\nabla u|^2 u_t \,\mathrm{d}x + \int_D g^2 u_t \Delta u \,\mathrm{d}x \\ &= \int_D gu_t\big(f + g'|\nabla u|^2 + g\Delta u\big) \,\mathrm{d}x \\ &= \int_D gk'(u_t)^2 \,\mathrm{d}x \geq 0. \end{aligned} \tag{2.4}$$

Consequently, $B(t)$ is a nondecreasing function in t and

$$B(t) \geq B(0) \geq 0.$$

By the Schwarz inequality, (2.3), and (2.4), we have

$$\begin{aligned} \left(1 + \frac{\alpha}{4}\right) A'B \leq \frac{A'^2}{4} &= \left(\int_D gk' u u_t \,\mathrm{d}x \right)^2 \leq \int_D gk'(u_t)^2 \,\mathrm{d}x \int_D gk' u^2 \,\mathrm{d}x \\ &= B'(t) \int_D gk' u^2 \,\mathrm{d}x. \end{aligned}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned} A(t) &= \int_D G(u)\,\mathrm{d}x = 2\int_D \left(\int_0^u sg(s)k'(s)\,\mathrm{d}s\right)\mathrm{d}x \\ &= \int_D \left(g(u)k'(u)u^2 - \int_0^u \big(g(s)k'(s)\big)' s^2\,\mathrm{d}s\right)\mathrm{d}x \\ &\geq \int_D gk'u^2\,\mathrm{d}x. \end{aligned}$$

Thus,

$$\left(1 + \frac{\alpha}{4}\right)A'B \leq AB',$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(BA^{-1-\frac{\alpha}{4}}\big) \geq 0. \tag{2.5}$$

With (2.3) and (2.5), we get

$$-\frac{4}{\alpha(\alpha+4)}\big(A^{-\frac{\alpha}{4}}\big)' = \frac{1}{\alpha+4}A'A^{-1-\frac{\alpha}{4}} \geq BA^{-1-\frac{\alpha}{4}} \geq M := B(0)\big[A(0)\big]^{-1-\frac{\alpha}{4}}. \tag{2.6}$$

Integrate (2.6) over $[0, t]$ to get

$$\big[A(t)\big]^{-\frac{\alpha}{4}} \leq \big[A(0)\big]^{-\frac{\alpha}{4}} - \frac{\alpha(\alpha+4)}{4}Mt,$$

which cannot hold for

$$t \geq T := \frac{4}{\alpha(\alpha+4)}A(0)B^{-1}(0).$$

Hence, $u(x,t)$ must blow up at some finite time $t^* < T$. The proof is complete. □

3 Global solution

In order to get the sufficient conditions for the existence of the global solution and the explicit exponential decay bounds for the solution, we suppose the following:

$$s\big(f(s)g(s)\big)' + s^2 g'(s) \geq f(s)g(s), \quad s \in \mathbb{R}^+, \tag{3.1}$$

$$\frac{f'(s)}{k'(s)} \leq p(s), \qquad \frac{1}{k'(s)}\left(\frac{k'(s)}{g(s)}\right)' \leq q(s), \quad s \in \overline{\mathbb{R}^+}, \tag{3.2}$$

where $p(s)$ and $q(s)$ are nondecreasing functions of s. Since the solution of problem (1.1) might blow up in a finite time t^*, the solution exists in an internal $(0, \gamma)$ with $\gamma < t^*$. Further we define

$$u_m := \max_{D\times(0,\gamma)} u(x,t) \quad (< +\infty).$$

Next, we give two lemmas from which the main results of this section are derived.

Lemma 3.1 *Let u be a classical solution of the problem* (1.1). *Suppose that* (3.2) *holds and*

$$-c \le \frac{g(h)}{k'(h)}\bigl[\nabla \cdot \bigl(g(h)\nabla h\bigr) + f(h)\bigr] \le 0, \quad x \in D, \tag{3.3}$$

where c is a positive constant. Then

$$-c \le g u_t \le 0, \quad (x,t) \in D \times (0,\gamma). \tag{3.4}$$

Proof Construct an auxiliary function

$$z(x,t) := g(u)u_t, \tag{3.5}$$

from which we have

$$\begin{aligned} &\nabla z = g'\nabla u u_t + g\nabla u_t, \\ &\Delta z = g''|\nabla u|^2 u_t + 2g'\nabla u_t \cdot \nabla u + g'\Delta u u_t + g\Delta u_t, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} z_t &= g'(u_t)^2 + g u_{tt} = g'(u_t)^2 + g\left(\frac{g}{k'}\Delta u + \frac{g'}{k'}|\nabla u|^2 + \frac{f}{k'}\right)_t \\ &= g'(u_t)^2 + \left(\frac{gg'}{k'} - \frac{g^2k''}{k'^2}\right)\Delta u u_t + \frac{g^2}{k'}\Delta u_t + \left(\frac{gg''}{k'} - \frac{gg'k''}{k'^2}\right)|\nabla u|^2 u_t + 2\frac{gg'}{k'}\nabla u_t \cdot \nabla u \\ &\quad + \left(\frac{f'g}{k'} - \frac{fgk''}{k'^2}\right)u_t. \end{aligned} \tag{3.7}$$

It follows from (3.5), (3.6), (3.7), and the first equation of (1.1) that

$$\frac{g}{k'}\Delta z - z_t + \frac{f'}{k'}z - \frac{1}{k'}\left(\frac{k'}{g}\right)' z^2 = 0. \tag{3.8}$$

The comparison principle [15], (3.2), (3.3), and (3.8) imply (3.4) holds. The proof is complete. □

In the following, we use the first Dirichlet eigenvalue λ_1 of the Laplacian and the corresponding eigenfunction Φ_1 for a region $\tilde{D} \supseteq D$:

$$\begin{cases} \Delta\Phi_1(x) + \lambda_1\Phi_1(x) = 0, & x \in \tilde{D}, \\ \Phi_1(x) = 0, & x \in \partial\tilde{D}. \end{cases} \tag{3.9}$$

Further since $\Phi_1(x)$ is determined up to an arbitrary multiplicative constant, we can normalize $\Phi_1(x)$ by

$$\max_{\tilde{D}} \Phi_1(x) = 1. \tag{3.10}$$

Lemma 3.2 *Let u be a classical solution of the problem* (1.1). *Suppose that assumptions* (3.1), (3.2), *and* (3.3) *hold and*

$$\left(\frac{k'(s)}{g(s)}\right)' \geq 0, \quad s \in \mathbb{R}^+. \tag{3.11}$$

Then $u(x,t)$ satisfies the following inequality:

$$0 \leq u(x,t) \leq \Gamma_1 \exp\left[-\frac{g(u_m)}{k'(u_m)}\left(\lambda_1 - \frac{f(u_m)}{u_m g(u_m)}\right)t\right], \quad (x,t) \in D \times (0,\gamma), \tag{3.12}$$

where

$$\Gamma_1 := \max_D \frac{h(x)}{\Phi_1(x)} < +\infty. \tag{3.13}$$

Proof Construct the following auxiliary function:

$$v(x,t) := u(x,t) \exp\left(-\frac{f(u_m)}{u_m k'(u_m)} t\right).$$

Here, (3.1) and the fact that $g' \leq 0$ imply

$$\left(\frac{f(s)}{sg(s)}\right)' \geq 0, \quad s \in \mathbb{R}^+. \tag{3.14}$$

It follows from Lemma 3.1, (3.11), and (3.14) that

$$\begin{aligned}\left(\Delta v - \frac{k'(u_m)}{g(u_m)} v_t\right) \exp\left(\frac{f(u_m)}{u_m k'(u_m)} t\right) &= \Delta u - \frac{k'(u_m)}{g(u_m)} u_t + \frac{f(u_m)}{u_m g(u_m)} u \\ &\geq \Delta u - \frac{k'(u)}{g(u)} u_t + \frac{f(u)}{g(u)} = -\frac{g'(u)}{g(u)} |\nabla u|^2 \geq 0.\end{aligned}$$

Thus, we have

$$\begin{cases} \Delta v - \frac{k'(u_m)}{g(u_m)} v_t \geq 0, & (x,t) \in D \times (0,\gamma), \\ v(x,t) = 0, & (x,t) \in \partial D \times (0,\gamma), \\ v(x,0) = h(x), & x \in \overline{D}. \end{cases} \tag{3.15}$$

Let

$$w(x,t) := \Gamma_1 \Phi_1(x) \exp\left(-\frac{\lambda_1 g(u_m)}{k'(u_m)} t\right). \tag{3.16}$$

With (3.9), (3.13), and (3.16), we have

$$\begin{cases} \Delta w - \frac{k'(u_m)}{g(u_m)} w_t = 0, & (x,t) \in D \times (0,\gamma), \\ w(x,t) \geq 0, & (x,t) \in \partial D \times (0,\gamma), \\ w(x,0) = \Gamma_1 \Phi_1(x) \geq h(x), & x \in \overline{D}. \end{cases} \tag{3.17}$$

It follows from (3.15), (3.17), and the comparison principle that

$$v(x,t) \leq w(x,t),$$

which implies

$$u(x,t) \leq \Gamma_1 \Phi_1(x) \exp\left[-\frac{g(u_m)}{k'(u_m)}\left(\lambda_1 - \frac{f(u_m)}{u_m g(u_m)}\right)t\right]. \tag{3.18}$$

With (3.10) and (3.18), we derive (3.12). The proof is complete. □

Next, we can get Theorem 3.1 from Lemmas 3.1-3.2.

Theorem 3.1 *Let u be a classical solution of the problem* (1.1). *Suppose that* (3.1), (3.2), (3.3), *and* (3.11) *hold and*

$$\frac{f(\Gamma_1)}{\Gamma_1 g(\Gamma_1)} < \lambda_1. \tag{3.19}$$

Then we have

$$t^* = \infty$$

and

$$\max_D \frac{f(u(x,t))}{u(x,t)g(u(x,t))} < \lambda_1, \quad t \in [0,\infty). \tag{3.20}$$

Proof We assume that (3.20) cannot hold. There exists a first time $\tilde{t} < \infty$ for which $\frac{f(u)}{ug(u)}$ reaches the value λ_1. Thus, we have

$$\max_D \frac{f(u(x,\tilde{t}))}{u(x,\tilde{t})g(u(x,\tilde{t}))} = \lambda_1. \tag{3.21}$$

The fact that $\frac{f(s)}{sg(s)}$ is a nondecreasing functions in s, (3.19), and Lemma 3.2 imply

$$u(x,t) \leq \Gamma_1, \quad (x,t) \in D \times [0,\tilde{t}]$$

and

$$\frac{f(u(x,t))}{u(x,t)g(u(x,t))} \leq \frac{f(\Gamma_1)}{\Gamma_1 g(\Gamma_1)} < \lambda_1, \quad (x,t) \in D \times [0,\tilde{t}].$$

Hence, we have

$$\max_D \frac{f(u(x,\tilde{t}))}{u(x,\tilde{t})g(u(x,\tilde{t}))} < \lambda_1,$$

which contradicts with the inequality (3.21). So we conclude that $\tilde{t} = \infty$ and (3.20) holds. The proof is complete. □

4 Exponential decay estimate

In this section, we will use a comma to denote partial differentiation and adopt the summation convection, *i.e.*, if an index is repeated, summation from 1 to N is understood, for example,

$$u_{,i}u_{,k}u_{,ik} = \sum_{i,k=1}^{N} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_i \partial x_k}.$$

Hence, the differentiated form of the first equation of (1.1) is

$$k'(u)u_t = \bigl(g(u)u_{,i}\bigr)_{,i} + f(u). \tag{4.1}$$

In order to get the exponential decay bounds for the derivatives of the solution, we consider

$$\Psi(x,t) := \left(g^2(u)|\nabla u|^2 + 2\int_0^u f(s)g(s)\,\mathrm{d}s + 2a\int_0^u sg(s)\,\mathrm{d}s\right)e^{2\beta t}, \tag{4.2}$$

where $a \geq 1$ and $0 < \beta \leq 1$ are some positive constants to be determined. Our main result is Theorem 4.1.

Theorem 4.1 *Let u be the classical solution of the problem* (1.1). *Suppose the following.*

(i) *The inequalities* (3.1), (3.2), (3.3), *and* (3.11) *hold and*

$$0 < k'(s) \leq b \leq 1, \quad s \in \mathbb{R}^+, \tag{4.3}$$

where b is a positive constant.

(ii)

$$\lim_{s\to 0^+} sg(s) = 0. \tag{4.4}$$

(iii)

$$\frac{a}{b} := M + \beta < \frac{\pi^2}{4d^2}g(\Gamma_1) - \frac{f(\Gamma_1)}{\Gamma_1}, \tag{4.5}$$

where d is the in-radius of D and

$$M := c \max_{s\in[0,\Gamma_1]}\left\{\frac{1}{k'(s)}\left(\frac{k'(s)}{g(s)}\right)'\right\}$$

with c given in Lemma 3.2. *Thus,* $\Psi(x,t)$ *takes its maximum value at* $t = 0$, *i.e.*,

$$g^2(u)|\nabla u|^2 + 2\int_0^u f(s)g(s)\,\mathrm{d}s + 2a\int_0^u sg(s)\,\mathrm{d}s \leq H^2 e^{-2\beta t}, \quad (x,t) \in D \times (0,\infty),$$

with

$$H^2 := \max_D\left\{g^2(h)|\nabla h|^2 + 2\int_0^h f(s)g(s)\,\mathrm{d}s + 2a\int_0^h sg(s)\,\mathrm{d}s\right\}.$$

Proof The theorem will be proved in three steps.

Step 1. Differentiating (4.2), we get

$$\Psi_{,k} = 2\big(gg'|\nabla u|^2 u_{,k} + g^2 u_{,i}u_{,ik} + fgu_{,k} + augu_{,k}\big)\mathrm{e}^{2\beta t}, \tag{4.6}$$

$$\begin{aligned}(g\Psi_{,k})_{,k} &= \big[2\big(g^2g'|\nabla u|^2 u_{,k} + g^3 u_{,i}u_{,ik} + fg^2 u_{,k} + aug^2 u_{,k}\big)\mathrm{e}^{2\beta t}\big]_{,k} \\ &= 2\mathrm{e}^{2\beta t}\big(2g\big(g'\big)^2|\nabla u|^4 + g^2g''|\nabla u|^4 + 4g^2g' u_{,i}u_{,k}u_{,ik} \\ &\quad + g^2g'|\nabla u|^2\Delta u + g^2(gu_{,ik})_{,k}u_{,i} \\ &\quad + g^3 u_{,ik}u_{,ik} + f'g^2|\nabla u|^2 + 2fgg'|\nabla u|^2 + fg^2\Delta u + ag^2|\nabla u|^2 \\ &\quad + 2augg'|\nabla u|^2 + aug^2\Delta u\big),\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\Psi_t &= 2\mathrm{e}^{2\beta t}\bigg(gg'|\nabla u|^2 u_t + g^2 u_{,i}u_{t,i} + fgu_t + augu_t + \beta g^2|\nabla u|^2 + 2\beta\int_0^u f(s)g(s)\,\mathrm{d}s \\ &\quad + 2a\beta\int_0^u sg(s)\,\mathrm{d}s\bigg).\end{aligned} \tag{4.8}$$

It follows from the first equation of (1.1) that

$$\Delta u = -\frac{g'}{g}|\nabla u|^2 - \frac{f}{g} + \frac{k'}{g}u_t. \tag{4.9}$$

Next, substituting (4.9) into (4.7), we have

$$\begin{aligned}(g\Psi_{,k})_{,k} &= 2\mathrm{e}^{2\beta t}\big(g\big(g'\big)^2|\nabla u|^4 + g^2g''|\nabla u|^4 + 4g^2g' u_{,i}u_{,k}u_{,ik} + gg'k'|\nabla u|^2 u_t \\ &\quad + g^2(gu_{,ik})_{,k}u_{,i} + g^3 u_{,ik}u_{,ik} + f'g^2|\nabla u|^2 + fgk'u_t - f^2g + ag^2|\nabla u|^2 \\ &\quad + augg'|\nabla u|^2 + augk'u_t - aufg\big).\end{aligned} \tag{4.10}$$

Differentiating (4.1), we have

$$(gu_{,k})_{,ki} = \big(k'u_t - f\big)_{,i},$$

i.e.,

$$\big(g'u_{,i}u_{,k} + gu_{,ik}\big)_{,k} = k''u_{,i}u_t + k'u_{t,i} - f'u_{,i}. \tag{4.11}$$

It follows from (4.11) that

$$\begin{aligned}(gu_{,ik})_{,k} &= k''u_{,i}u_t + k'u_{t,i} - f'u_{,i} - \big(g'u_{,i}u_{,k}\big)_{,k} \\ &= k''u_{,i}u_t + k'u_{t,i} - f'u_{,i} - g''|\nabla u|^2 u_{,i} - g'u_{,k}u_{,ik} - g'u_{,i}\Delta u.\end{aligned} \tag{4.12}$$

Multiplied by $g^2 u_{,i}$ from (4.12), we have

$$\begin{aligned}g^2(gu_{,ik})_{,k}u_{,i} &= g^2k''|\nabla u|^2 u_t + g^2k'u_{,i}u_{t,i} - f'g^2|\nabla u|^2 \\ &\quad - g^2g''|\nabla u|^4 - g^2g'u_{,i}u_{,k}u_{,ik} - g^2g'|\nabla u|^2\Delta u.\end{aligned} \tag{4.13}$$

Substituting (4.9) into (4.13), we get

$$g^2(gu_{,ik})_{,k}u_{,i} = g^2k''|\nabla u|^2u_t + g^2k'u_{,i}u_{t,i} - f'g^2|\nabla u|^2 - g^2g''|\nabla u|^4 - g^2g'u_{,i}u_{,k}u_{,ik} - gg'k'|\nabla u|^2u_t + fgg'|\nabla u|^2 + g(g')^2|\nabla u|^4. \tag{4.14}$$

We substitute (4.14) into (4.10) to obtain

$$\begin{aligned}(g\Psi_{,k})_{,k} = 2\mathrm{e}^{2\beta t}\big(&2g(g')^2|\nabla u|^4 + 3g^2g'u_{,i}u_{,k}u_{,ik} + g^2k''|\nabla u|^2u_t \\ &+ g^2k'u_{,i}u_{t,i} + fgg'|\nabla u|^2 + g^3u_{,ik}u_{,ik} + fgk'u_t \\ &- f^2g + ag^2|\nabla u|^2 + augg'|\nabla u|^2 + augk'u_t - aufg\big).\end{aligned} \tag{4.15}$$

It follows from (4.6) that

$$g^2u_{,i}u_{,ik} = \frac{1}{2}\mathrm{e}^{-2\beta t}\Psi_{,k} - gg'|\nabla u|^2u_{,k} - fgu_{,k} - augu_{,k}. \tag{4.16}$$

Substituting (4.16) into (4.15), we get

$$\begin{aligned}(g\Psi_{,k})_{,k} = 2\mathrm{e}^{2\beta t}\bigg(&-g(g')^2|\nabla u|^4 + \frac{3}{2}\mathrm{e}^{-2\beta t}g'u_{,k}\Psi_{,k} - 2fgg'|\nabla u|^2 \\ &- 2augg'|\nabla u|^2 + g^2k''|\nabla u|^2u_t + g^2k'u_{,i}u_{t,i} + g^3u_{,ik}u_{,ik} + fgk'u_t \\ &- f^2g + ag^2|\nabla u|^2 + augk'u_t - aufg\bigg).\end{aligned} \tag{4.17}$$

It follows from (4.8) and (4.17) that

$$\begin{aligned}(g\Psi_{,k})_{,k} - k'\Psi_t = 2\mathrm{e}^{2\beta t}\bigg(&-g(g')^2|\nabla u|^4 + \frac{3}{2}\mathrm{e}^{-2\beta t}g'u_{,k}\Psi_{,k} - 2fgg'|\nabla u|^2 - 2augg'|\nabla u|^2 \\ &+ g^2k''|\nabla u|^2u_t + g^3u_{,ik}u_{,ik} - f^2g + ag^2|\nabla u|^2 - aufg - gg'k'|\nabla u|^2u_t \\ &- \beta g^2k'|\nabla u|^2 - 2\beta k'\int_0^u f(s)g(s)\,\mathrm{d}s - 2a\beta k'\int_0^u sg(s)\,\mathrm{d}s\bigg).\end{aligned} \tag{4.18}$$

Next, we use the Cauchy-Schwarz inequality in the following form:

$$|\nabla u|^2u_{,ik}u_{,ik} \geq u_{,k}u_{,ik}u_{,j}u_{,ij}. \tag{4.19}$$

It follows from (4.6) that

$$u_{,k}u_{,ik} = u_{,j}u_{,ij} = \frac{1}{g^2}\bigg(\frac{1}{2}\mathrm{e}^{-2\beta t}\Psi_{,i} - gg'|\nabla u|^2u_{,i} - fgu_{,i} - augu_{,i}\bigg). \tag{4.20}$$

Further, with (4.19) and (4.20), we obtain

$$\begin{aligned}u_{,ik}u_{,ik} &\geq \frac{1}{g^4|\nabla u|^2}\bigg(\frac{1}{2}\mathrm{e}^{-2\beta t}\Psi_{,i} - gg'|\nabla u|^2u_{,i} - fgu_{,i} - augu_{,i}\bigg)^2 \\ &= \frac{1}{4g^4|\nabla u|^2}\mathrm{e}^{-4\beta t}(\Psi_{,i})^2 - \frac{1}{g^3|\nabla u|^2}\big(g'|\nabla u|^2u_{,i} + auu_{,i} + fu_{,i}\big)\mathrm{e}^{-2\beta t}\Psi_{,i}\end{aligned}$$

$$+\frac{1}{g^2}\left((g')^2|\nabla u|^4 + a^2u^2 + f^2 + 2aug'|\nabla u|^2 + 2auf + 2fg'|\nabla u|^2\right),$$
$$x \in D \setminus W, \tag{4.21}$$

where $W := \{x \in D : \nabla u(x,t) = 0\}$ is the set of critical points of u. Substituting (4.21) into (4.18), we have

$$\begin{aligned}
&\frac{g}{k'}\Delta\Psi + \frac{1}{k'}\left\{\left(g' - \frac{\mathrm{e}^{-2\beta t}}{2g|\nabla u|^2}\right)\nabla\Psi - \left[g' - \frac{2}{|\nabla u|^2}(f+au)\right]\nabla u\right\}\cdot\nabla\Psi - \Psi_t \\
&\quad\geq 2\mathrm{e}^{2\beta t}\left\{a\frac{fgu}{k'} - 2\beta\int_0^u f(s)g(s)\,\mathrm{d}s + a^2\frac{gu^2}{k'} - 2a\beta\int_0^u sg(s)\,\mathrm{d}s \right.\\
&\qquad\left. + g^2|\nabla u|^2\left[\frac{a}{k'} - \beta + \frac{g}{k'}\left(\frac{k'}{g}\right)' u_t\right]\right\}.
\end{aligned}\tag{4.22}$$

Integrating (3.1) from 0 to $u(x,t)$ and using (4.4), we get

$$fgu - 2\int_0^u f(s)g(s)\,\mathrm{d}s + gu^2 - 2\int_0^u sg(s)\,\mathrm{d}s \geq 0. \tag{4.23}$$

Making use of the fact that $a \geq 1$, (4.3), and (4.23), we have

$$\begin{aligned}
&a\frac{fgu}{k'} - 2\beta\int_0^u f(s)g(s)\,\mathrm{d}s + a^2\frac{gu^2}{k'} - 2a\beta\int_0^u sg(s)\,\mathrm{d}s \\
&\quad\geq a\frac{fgu}{b} - 2a\beta\int_0^u f(s)g(s)\,\mathrm{d}s + a^2\frac{gu^2}{b} - 2a\beta\int_0^u sg(s)\,\mathrm{d}s \\
&\quad\geq afgu - 2a\beta\int_0^u f(s)g(s)\,\mathrm{d}s + a^2gu^2 - 2a\beta\int_0^u sg(s)\,\mathrm{d}s \\
&\quad\geq a\left(fgu - 2\int_0^u f(s)g(s)\,\mathrm{d}s + agu^2 - 2\int_0^u sg(s)\,\mathrm{d}s\right) \\
&\quad\geq fgu - 2\int_0^u f(s)g(s)\,\mathrm{d}s + gu^2 - 2\int_0^u sg(s)\,\mathrm{d}s \geq 0.
\end{aligned}\tag{4.24}$$

Moreover, by (4.5), it is easy to see

$$\frac{f(\Gamma_1)}{\Gamma_1} < \frac{\pi^2}{4d^2}g(\Gamma_1) \leq \lambda_1 g(\Gamma_1).$$

It follows from Theorem 3.1 that

$$\frac{f(u_m)}{u_m g(u_m)} < \lambda_1,$$

which implies

$$u_m \leq \Gamma_1. \tag{4.25}$$

Next, it follows from (4.25), (4.3), (4.5), and Lemma 3.1 that

$$\frac{a}{k'} - \beta + \frac{g}{k'}\left(\frac{k}{g}\right)' u_t \geq \frac{a}{b} - \beta - \frac{c}{k'}\left(\frac{k}{g}\right)' \geq \frac{a}{b} - \beta - M = 0. \tag{4.26}$$

Consequently, (4.22), (4.24), and (4.26) imply

$$\frac{g}{k'}\Delta\Psi + \frac{1}{k'}\left\{\left(g' - \frac{e^{-2\beta t}}{2g|\nabla u|^2}\right)\nabla\Psi - \left[g' - \frac{2}{|\nabla u|^2}(f+au)\right]\nabla u\right\}\cdot\nabla\Psi - \Psi_t \geq 0,$$
$$x \in D\setminus W.$$

By means of the maximum principle, we have the following possible cases where Ψ may take its maximum value:

(a) on the boundary $\partial D\times(0,T)$,
(b) at a point where $\nabla u = 0$,
(c) for $t = 0$.

Step 2. We first exclude the case (a). Assume $\Psi(x,t)$ takes its maximum value at $\hat{Q} = (\hat{x},\hat{t})$ on ∂D. Since $u = 0$ on ∂D, we have

$$\frac{\partial\Psi}{\partial n} = 2\left[gg'\left(\frac{\partial u}{\partial n}\right)^3 + g^2\frac{\partial^2 u}{\partial n^2}\frac{\partial u}{\partial n}\right]e^{2\beta t}. \tag{4.27}$$

With (1.1) and $f(0) = 0$, evaluated on $\partial D \in C^{2,\varepsilon}$, we get

$$g'\left(\frac{\partial u}{\partial n}\right)^2 + g\left[\frac{\partial^2 u}{\partial n^2} + (N-1)K\frac{\partial u}{\partial n}\right] = 0, \tag{4.28}$$

where K is the average curvature of ∂D. By (4.27) and (4.28), we are led to

$$\frac{\partial\Psi}{\partial n} = \left[-2(N-1)kg^2\left(\frac{\partial u}{\partial n}\right)^2\right]e^{2\beta t} \leq 0, \quad x\in\partial D.$$

Hence, we have

$$\left.\frac{\partial\Psi}{\partial n}\right|_{\hat{Q}=(\hat{x},\hat{t})} \leq 0,$$

which contradicts with the maximum principle. Hence, Ψ cannot take its maximum value on ∂D.

Step 3. In the following, we exclude the case (b). Assume $\Psi(x,t)$ takes its maximum value at a critical point $\bar{Q} = (\bar{x},\bar{t})$.

Thus we have

$$\Psi(x,t) \leq \Psi(\bar{x},\bar{t}), \quad (x,t)\in D\times(0,+\infty). \tag{4.29}$$

Replacing t with $\bar{t}$ in (4.29), we obtain

$$g^2\big(u(x,\bar{t})\big)\big|\nabla u(x,\bar{t})\big|^2 \leq 2\int_{u(x,\bar{t})}^{u(\bar{x},\bar{t})} f(s)g(s)\,ds + 2a\int_{u(x,\bar{t})}^{u(\bar{x},\bar{t})} sg(s)\,ds,$$

from which we have

$$g^2\big(u(x,\bar{t})\big)\big|\nabla u(x,\bar{t})\big|^2 \leq 2\int_{u(x,\bar{t})}^{u_M} f(s)g(s)\,ds + 2a\int_{u(x,\bar{t})}^{u_M} sg(s)\,ds, \quad x\in D, \tag{4.30}$$

where $u_M = \max_D u(x,\bar{t})$.

Here, (3.1) and the fact that $g'(s) \leq 0$ imply

$$\left(\frac{f(s)g(s)}{s}\right)' \geq 0, \quad s \in \mathbb{R}^+. \tag{4.31}$$

Next, making use of Cauchy's mean value theorem and of (4.31), we get

$$\begin{aligned} 2\int_{u(x,\bar{t})}^{u_M} f(s)g(s)\,\mathrm{d}s &= \frac{f(\xi)g(\xi)}{\xi}\big(u_M^2 - u^2(x,\bar{t})\big) \leq \frac{f(u_M)g(u_M)}{u_M}\big(u_M^2 - u^2(x,\bar{t})\big) \\ &\leq \frac{f(u_M)g(u(x,\bar{t}))}{u_M}\big(u_M^2 - u^2(x,\bar{t})\big), \end{aligned} \tag{4.32}$$

where ξ is some intermediate value between $u(x,\bar{t})$ and u_M. The fact that $g'(s) \leq 0$ implies

$$2\int_{u(x,\bar{t})}^{u_M} sg(s)\,\mathrm{d}s \leq g\big(u(x,\bar{t})\big)\big(u_M^2 - u^2(x,\bar{t})\big). \tag{4.33}$$

Hence, inserting (4.32) and (4.33) in (4.30), we get

$$\big|\nabla u(x,\bar{t})\big|^2 \leq \left(\frac{f(u_M)}{u_M} + a\right)\left(\frac{u_M^2 - u^2(x,\bar{t})}{g(u_M)}\right), \quad x \in D. \tag{4.34}$$

With (4.34), we have

$$\frac{du}{\sqrt{u_M^2 - u^2(x,\bar{t})}} \leq \sqrt{\left(\frac{f(u_M)}{u_M} + a\right)\frac{1}{g(u_M)}}d\tau. \tag{4.35}$$

Integrate (4.35) on a straight line from $\bar{x}$ to the nearest point $x_0 \in \partial D$ to obtain

$$\frac{\pi}{2} \leq \sqrt{\left(\frac{f(u_M)}{u_M} + a\right)\frac{1}{g(u_M)}}|\bar{x}x_0| \leq \sqrt{\left(\frac{f(u_M)}{u_M} + a\right)\frac{1}{g(u_M)}}d,$$

from which we have

$$\left(\frac{f(u_M)}{u_M} + a\right)\frac{1}{g(u_M)} \geq \frac{\pi^2}{4d^2}. \tag{4.36}$$

We note that (3.1) and the fact that $g'(s) \leq 0$ ensure $\frac{f(s)}{s}$ is a nondecreasing function. It follows from (4.25) and (4.5) that

$$\frac{f(u_m)}{u_m} \leq \frac{f(\Gamma_1)}{\Gamma_1} < \frac{\pi^2}{4d^2}g(\Gamma_1) - \frac{a}{b} \leq \frac{\pi^2}{4d^2}g(u_m) - \frac{a}{b} \leq \frac{\pi^2}{4d^2}g(u_m) - a,$$

which with $u_M \leq u_m$ implies

$$\left(\frac{f(u_M)}{u_M} + a\right)\frac{1}{g(u_M)} \leq \left(\frac{f(u_m)}{u_m} + a\right)\frac{1}{g(u_m)} < \frac{\pi^2}{4d^2}.$$

which contradicts with (4.36). The proof is complete. □

5 Applications

When $k(u) \equiv u$ and $g(u) \equiv 1$ or $k(u) \equiv u$, the conclusions of Theorems 2.1, 3.1 and 4.1 still hold true. In this sense, our results extend and supplement those of [16, 17]

In what follows, as applications of the obtained results, two examples are presented.

Example 5.1 Let u be a classical solution of the following problem:

$$\begin{cases} u_t = \sqrt{1+\frac{1}{u}}\Delta u - \frac{1}{2u}\sqrt{1+\frac{1}{u}}|\nabla u|^2 + u^2\sqrt{u(u+1)}, & (x,t) \in D \times (0,T), \\ u = 0, & (x,t) \in \partial D \times (0,T), \\ u(x,0) = (16-|x|^2)^2, & x \in \overline{D}, \end{cases}$$

where $D = \{x = (x_1, x_2, x_3) \mid |x| = (\sum_{i=1}^3 x_i^2)^{1/2} < 4\}$ is the ball of $\mathbb{R}^3$. The above problem can be transformed into the following problem:

$$\begin{cases} (2\sqrt{u+1})_t = \nabla \cdot (\frac{1}{\sqrt{u}}\nabla u) + u^{\frac{5}{2}}, & (x,t) \in D \times (0,T), \\ u = 0, & (x,t) \in \partial D \times (0,T), \\ u(x,0) = (16-|x|^2)^2, & x \in \overline{D}. \end{cases}$$

Now,

$$k(u) = 2\sqrt{u+1}, \qquad g(u) = \frac{1}{\sqrt{u}}, \qquad f(u) = u^{\frac{5}{2}}, \qquad h(x) = \left(16-|x|^2\right)^2.$$

We have

$$\begin{aligned} F(u) &= \int_0^u f(s)g(s)\,\mathrm{d}s = \int_0^u s^2\,\mathrm{d}s = \frac{u^3}{3}, \\ G(u) &= 2\int_0^u sk'(s)g(s)\,\mathrm{d}s = 2\int_0^u \sqrt{\frac{s}{s+1}}\,\mathrm{d}s = 2\ln(\sqrt{u+1}-\sqrt{u}) + 2\sqrt{u(u+1)}. \end{aligned}$$

By choosing $\alpha = 1$, it is easy to check that (2.1) and (2.2) hold with

$$\begin{aligned} A(0) &= \int_D G\big(h(x)\big)\,\mathrm{d}x = 2\int_D \ln(\sqrt{h+1}-\sqrt{h}) + \sqrt{h(h+1)}\,\mathrm{d}x \\ &= 2\int_D \ln\Big(\sqrt{\left(16-|x|^2\right)^2+1} - 16 + |x|^2\Big) + \left(16-|x|^2\right)\sqrt{\left(16-|x|^2\right)^2+1}\,\mathrm{d}x \\ &= 8\pi\int_0^4 \Big[\ln\Big(\sqrt{\left(16-r^2\right)^2+1} - 16 + r^2\Big) + \left(16-r^2\right)\sqrt{\left(16-r^2\right)^2+1}\Big]r^2\,\mathrm{d}r \\ &= 3.040559 \times 10^4 \end{aligned}$$

and

$$\begin{aligned} B(0) &= \int_D \left(F(h) - \frac{1}{2}g^2(h)|\nabla h|^2\right)\mathrm{d}x = \int_D \left(\frac{1}{3}h^3 - \frac{|\nabla h|^2}{2h}\right)\mathrm{d}x \\ &= \int_D \left(\frac{1}{3}\left(16-|x|^2\right)^6 - 8|x|^2\right)\mathrm{d}x = 4\pi\int_0^4 \left(\frac{1}{3}\left(16-r^2\right)^6 - 8r^2\right)r^2\,\mathrm{d}r \\ &= 1.022244 \times 10^8. \end{aligned}$$

It follows from Theorem 2.1 that u blows up in a finite time t^* and

$$t^* < T = \frac{4A(0)}{\alpha(\alpha+4)B(0)} = 2.38 \times 10^{-4}.$$

Example 5.2 Let u be a classical solution of the following problem:

$$\begin{cases} u_t = \Delta u - \frac{1}{u+1}|\nabla u|^2 + u^2(u+1), & (x,t) \in D \times (0,T), \\ u = 0, & (x,t) \in \partial D \times (0,T), \\ u(x,0) = \Phi_1(x) = \frac{\sin(8|x|)}{8|x|}, & x \in \overline{D}, \end{cases}$$

where $D = \{x = (x_1, x_2, x_3) \mid |x| = (\sum_{i=1}^3 x_i^2)^{1/2} < \frac{\pi}{8}\}$ is the ball of $\mathbb{R}^3$, $\Phi_1(x)$ is the first eigenfunction of $\tilde{D} = D$ and $\max_D \Phi_1(x) = 1$. The above problem may be turned into the following problem:

$$\begin{cases} (\ln(u+1))_t = \nabla \cdot (\frac{1}{u+1}\nabla u) + u^2, & (x,t) \in D \times (0,T), \\ u = 0, & (x,t) \in \partial D \times (0,T), \\ u(x,0) = \Phi_1(x) = \frac{\sin(8|x|)}{8|x|}, & x \in \overline{D}. \end{cases}$$

Now we have

$$k(u) = \ln(u+1), \qquad g(u) = \frac{1}{u+1}, \qquad f(u) = u^2, \qquad h(x) = \Phi_1(x) = \frac{\sin(8|x|)}{8|x|}.$$

Here,

$$\Gamma_1 = 1, \qquad \frac{f(\Gamma_1)}{\Gamma_1 g(\Gamma_1)} = 2, \qquad \lambda_1 = \frac{\pi^2}{R^2} = 64.$$

By choosing $c = 31$, it is easy to check that (3.1), (3.2), (3.3), (3.11), and (3.19) hold. It follows from Lemma 3.2 and Theorem 3.1 that $u(x,t)$ is a global solution and

$$\begin{aligned} u(x,t) &\leq \Gamma_1 \exp\left[-\frac{g(u_m)}{k'(u_m)}\left(\lambda_1 - \frac{f(u_m)}{g(u_m)u_m}\right)t\right] = \exp\left[-\frac{g(u_m)}{k'(u_m)}\left(64 - \frac{f(u_m)}{g(u_m)u_m}\right)t\right] \\ &\leq \exp\left[-\frac{g(\Gamma_1)}{k'(\Gamma_1)}\left(64 - \frac{f(\Gamma_1)}{\Gamma_1 g(\Gamma_1)}\right)t\right] = \mathrm{e}^{-62t}, \end{aligned}$$

which is the exponential decay estimate of the solution. By taking $a = b = \beta = 1$, it is also easy to check that (4.3), (4.4), and (4.5) hold. It follows from Theorem 4.1 that

$$g^2(u)|\nabla u|^2 + 2\int_0^u f(s)g(s)\,\mathrm{d}s + 2a\int_0^u sg(s)\,\mathrm{d}s \leq H^2 e^{-2\beta t}, \quad (x,t) \in D \times (0,\infty),$$

with

$$\begin{aligned} H^2 &= \max_D\left\{g^2(h)|\nabla h|^2 + 2\int_0^h f(s)g(s)\,\mathrm{d}s + 2a\int_0^h sg(s)\,\mathrm{d}s\right\} = \max_D\left\{\frac{|\nabla h|^2}{(h+1)^2} + h^2\right\} \\ &= \max_D\left\{\frac{(\frac{\cos 8|x|}{|x|} - \frac{\sin 8|x|}{8|x|^2})^2}{(\frac{\sin(8|x|)}{8|x|} + 1)^2} + \left(\frac{\sin(8|x|)}{8|x|}\right)^2\right\} = 7.4. \end{aligned}$$

Hence, we have

$$|\nabla u|^2 \leq \frac{1}{g^2(u)} H^2 \mathrm{e}^{-2\beta t} \leq \frac{1}{g^2(\Gamma_1)} H^2 \mathrm{e}^{-2\beta t} = 29.6\mathrm{e}^{-2t},$$

which is the exponential decay estimate of the gradient for the solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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