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Weak ψ-contractions on partially ordered metric spaces and applications to boundary value problems

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Abstract

A class of weak ψ-contractions satisfying the C-condition is defined on metric spaces. The existence and uniqueness of fixed points of such maps are discussed both on metric spaces and on partially ordered metric spaces. The results are applied to a first order periodic boundary value problem.

MSC: 47H10; 54H25

Keywords: partial order; fixed point; boundary value problem

1 Introduction and preliminaries

Recent developments in fixed point theory have been encouraged by the applicability of the results in the area of boundary value problems for differential and integral equations. Especially in the last few years, a lot of publications in fixed point theory have presented results directly related to specific initial or boundary value problems. These problems include not only ordinary and partial differential equations, but also fractional differential equations.

In 2004 Ran and Reurings [1] investigated the existence of fixed points in partially ordered metric spaces. The importance of this study presented itself in the area of boundary value problems. Nieto and Lopez [2] discussed the applications of the fixed point theorems to the problem of existence and uniqueness of solutions of first order boundary value problems. The results of Ran and Reurings and Nieto and Lopez have been followed soon by numerous studies concerning fixed points on partially ordered metric spaces [3–6]. In the case of partially ordered spaces the continuity condition is no longer needed, however, the map should be nondecreasing.

In a recent paper, Popescu [7] proved two generalizations of a result given by Bogin [8] for a class of non-expansive mappings on complete metric spaces. The idea behind his work was to replace the non-expansiveness condition with the weaker C-condition introduced by Suzuki [9–11]. The existence and uniqueness of fixed points of maps satisfying the C-condition have also been extensively studied; see [12–14]. We state first the definition of a non-expansive map and a map satisfying the C-condition on a metric space.

Definition 1 A mapping T on a metric space (X,d) is called a non-expansive mapping if

$$d(Tx,Ty) \leq d(x,y), \tag{1.1}$$

for all $x,y \in X$.

Definition 2 A mapping T on a metric space (X,d) satisfies the C-condition if

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \Longrightarrow \quad d(Tx,Ty) \leq d(x,y), \tag{1.2}$$

for all $x,y \in X$.

Popescu [7] stated and proved the following fixed point theorem.

Theorem 3 *Let (X,d) be a nonempty complete metric space and $T : X \to X$ be a mapping satisfying*

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \tag{1.3}$$

which implies

$$d(Tx,Ty) \leq ad(x,y) + b\bigl[d(x,Tx) + d(y,Ty)\bigr] + c\bigl[d(x,Ty) + d(y,Tx)\bigr], \tag{1.4}$$

where $a \geq 0$, $b > 0$, $c > 0$ and $a + 2b + 2c = 1$. Then T has a unique fixed point.

In this paper, we investigate the existence and uniqueness of fixed points of maps satisfying the C-condition on metric spaces and on partially ordered metric spaces. As an application, we study the existence and uniqueness of solutions of a first order periodic boundary value problem under certain conditions.

2 Existence and uniqueness of fixed points on metric spaces

Our main results can be considered as a generalization of the result of Popescu [7].

We first prove fixed point theorems on complete metric spaces and then we formulate these results on complete metric spaces endowed with a partial order.

Theorem 4 *Let (X,d) be a complete metric space, $T : X \longrightarrow X$ be a map, and $\psi : [0,\infty) \longrightarrow [0,\infty)$ be a continuous nondecreasing function such that $\psi(0) = 0$ and $\psi(t) > 0$ for $t > 0$. Suppose that*

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \Rightarrow \quad d(Tx,Ty) \leq M(x,y) - \psi\bigl(M(x,y)\bigr), \tag{2.1}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\bigl[d(x,Ty) + d(y,Tx)\bigr]\right\}, \tag{2.2}$$

for all $x,y \in X$. Then the mapping T has a unique fixed point.

Proof Let $x_0 \in X$ and define the sequence $\{x_n\}$ as follows:

$$x_n = Tx_{n-1}, \quad n \in \mathbb{N}.$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is the fixed point of T. Assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$.

Substituting $x = x_n$ and $y = Tx_n = x_{n+1}$ in (2.1) we get

$$\begin{aligned} &\frac{1}{2}d(x_n, Tx_n) = \frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) \\ &\quad \Rightarrow \quad d(Tx_n, Tx_{n+1}) = d(x_{n+1}, x_{n+2}) \leq M(x_n, x_{n+1}) - \psi\big(M(x_n, x_{n+1})\big), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} &M(x_n, x_{n+1}) \\ &\quad = \max\left\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \frac{1}{2}\big[d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)\big]\right\} \\ &\quad = \max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{1}{2}\big[d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})\big]\right\} \\ &\quad = \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{1}{2}d(x_n, x_{n+2})\right\}. \end{aligned} \tag{2.4}$$

From the triangle inequality we have

$$\frac{1}{2}d(x_n, x_{n+2}) \leq \frac{1}{2}\big[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\big] \leq \max\big\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\big\}. \tag{2.5}$$

Therefore, $M(x_n, x_{n+1})$ can be either $d(x_{n+1}, x_{n+2})$ or $d(x_n, x_{n+1})$. If $M(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2})$, then (2.3) implies

$$d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, x_{n+2}) - \psi\big(d(x_{n+1}, x_{n+2})\big), \tag{2.6}$$

so that $\psi(d(x_{n+1}, x_{n+2})) = 0$ and hence, $d(x_{n+1}, x_{n+2}) = 0$ which contradicts the assumption $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Thus, $M(x_n, x_{n+1}) = d(x_n, x_{n+1})$, which results in

$$d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}) - \psi\big(d(x_n, x_{n+1})\big) \leq d(x_n, x_{n+1}). \tag{2.7}$$

Therefore, the sequence $d_n = d(x_n, x_{n+1})$ is non-increasing and bounded below by 0. Hence,

$$\lim_{n\to\infty} d_n = L \geq 0. \tag{2.8}$$

However, letting $n \to \infty$ in (2.7) we get

$$L \leq L - \psi(L), \tag{2.9}$$

and we conclude that $L = 0$, since $\psi(L) = 0$, and therefore

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.10}$$

We shall prove next that the sequence $\{x_n\}$ is a Cauchy sequence. Assume the contrary, that is, $\{x_n\}$ is not Cauchy. Then there exists $\varepsilon > 0$ for which one can find subsequences $\{n(i)\}$ and $\{m(i)\}$ in $\mathbb{N}$ such that

$$d(x_{n(i)}, x_{m(i)}) \geq \varepsilon \tag{2.11}$$

for $m(i) > n(i) > i$ where $m(i)$ is the smallest index satisfying (2.11), that is,

$$d(x_{n(i)}, x_{m(i)-1}) < \varepsilon. \tag{2.12}$$

From the triangle inequality we have

$$\begin{aligned} \varepsilon \leq d(x_{n(i)}, x_{m(i)}) &\leq d(x_{n(i)}, x_{m(i)-1}) + d(x_{m(i)-1}, x_{m(i)}) \\ &< \varepsilon + d(x_{m(i)-1}, x_{m(i)}). \end{aligned} \tag{2.13}$$

Taking the limit as $i \to \infty$ in (2.13) and using (2.10) we get

$$\lim_{i\to\infty} d(x_{n(i)}, x_{m(i)}) = \varepsilon. \tag{2.14}$$

On the other hand, the convergence of $\{d(x_n, x_{n+1})\}$ implies that for this $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \varepsilon$, for all $n \geq N_0$. Let $N_1 = \max\{m(i), N_0\}$. Then, for all $m(k) > n(k) \geq N_1$, we have

$$d(x_{n(k)}, x_{n(k)+1}) < \varepsilon \leq d(x_{n(k)}, x_{m(k)}), \tag{2.15}$$

where $m(k) \geq n(k)$ and, hence,

$$\frac{1}{2} d(x_{n(k)}, x_{n(k)+1}) \leq d(x_{n(k)}, x_{m(k)}). \tag{2.16}$$

Then from (2.1) with $x = x_{n(k)}$ and $y = x_{m(k)}$ we obtain

$$d(Tx_{n(k)}, Tx_{m(k)}) = d(x_{n(k)+1}, x_{m(k)+1}) \leq M(x_{n(k)}, x_{m(k)}) - \psi\big(M(x_{n(k)}, x_{m(k)})\big), \tag{2.17}$$

where

$$\begin{aligned} M(x_{n(k)}, x_{m(k)}) = \max\Big\{ & d(x_{n(k)}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}), \\ & d(x_{m(k)}, x_{m(k)+1}), \frac{1}{2}\big[d(x_{n(k)}, x_{m(k)+1}) + d(x_{n(k)+1}, x_{m(k)})\big]\Big\}. \end{aligned} \tag{2.18}$$

Regarding (2.10) and (2.14), we see that

$$\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}) = \max\{\varepsilon, 0\} = \varepsilon. \tag{2.19}$$

Letting $k \to \infty$ in (2.17) we get

$$\varepsilon \leq \varepsilon - \psi(\varepsilon), \tag{2.20}$$

which implies $\psi(\varepsilon) = 0$ and hence, $\varepsilon = 0$. This contradicts the assumption that $\{x_n\}$ is not a Cauchy sequence. Therefore, $\{x_n\}$ is Cauchy and by the completeness of X it converges to a limit, say $x \in X$.

Assume now that there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} &d(x_n, x) < \frac{1}{2} d(x_n, x_{n+1}) \quad \text{and} \\ &d(x_{n+1}, x) < \frac{1}{2} d(x_{n+1}, x_{n+2}). \end{aligned}$$

Then we have

$$\begin{aligned} d_n &= d(x_n, x_{n+1}) \le d(x_n, x) + d(x_{n+1}, x) < \frac{1}{2}\bigl[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\bigr] \\ &\le \frac{1}{2}(d_n + d_{n+1}) \le d_n, \end{aligned} \tag{2.21}$$

which is a contradiction. Hence, we must have $d(x_n, x) \ge \frac{1}{2} d(x_n, x_{n+1})$ or $d(x_{n+1}, x) \ge \frac{1}{2} d(x_{n+1}, x_{n+2})$, for all $n \in \mathbb{N}$. Therefore, for a subsequence $\{n(k)\}$ of $\mathbb{N}$ we have

$$\frac{1}{2} d(x_{n(k)}, Tx_{n(k)}) = \frac{1}{2} d(x_{n(k)}, x_{n(k)+1}) \le d(x_{n(k)}, x),$$

for all $k \in \mathbb{N}$, which implies

$$d(Tx_{n(k)}, Tx) = d(x_{n(k)+1}, Tx) \le M(x_{n(k)}, x) - \psi\bigl(M(x_{n(k)}, x)\bigr), \tag{2.22}$$

where

$$M(x_{n(k)}, x) = \max\left\{d(x_{n(k)}, x), d(x_{n(k)}, x_{n(k)+1}), d(x, Tx), \frac{1}{2}\bigl[d(x, x_{n(k)+1}) + d(x_{n(k)}, Tx)\bigr]\right\}.$$

Obviously,

$$\lim_{k\to\infty} M(x_{n(k)}, x) = \max\left\{0, d(x, Tx), \frac{1}{2} d(x, Tx)\right\} = d(x, Tx). \tag{2.23}$$

Letting $k \to \infty$ in (2.22) we get

$$d(x, Tx) \le d(x, Tx) - \psi\bigl(d(x, Tx)\bigr) \tag{2.24}$$

and, hence, $d(x, Tx) = 0$, that is, $x = Tx$.

Finally, we prove the uniqueness of the fixed point. Assume that $x \neq y$ and $x = Tx$ and $y = Ty$. Then

$$0 = \frac{1}{2} d(x, Tx) \le d(x, y), \tag{2.25}$$

which implies

$$d(Tx, Ty) = d(x, y) \le M(x, y) - \psi\bigl(M(x, y)\bigr), \tag{2.26}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\big[d(x,Ty) + d(y,Tx)\big]\right\} = d(x,y).$$

Thus, (2.26) becomes

$$d(x,y) \leq d(x,y) - \psi\big(d(x,y)\big),$$

and, clearly, $d(x,y) = 0$, that is, $x = y$. □

We next define a contractive condition similar to that in Theorem 4. The reason for introducing this new contraction is that in the framework of partially ordered metric spaces uniqueness of a fixed point requires an additional condition on the space. However, this condition is not sufficient for the uniqueness of the fixed point for a map satisfying contractive condition defined in Theorem 4.

Theorem 5 *Let (X,d) be a complete metric space, $T : X \longrightarrow X$ be a map, and $\psi : [0,\infty) \longrightarrow [0,\infty)$ be a continuous nondecreasing function such that $\psi(0) = 0$ and $\psi(t) > 0$ for $t > 0$. Suppose that*

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \Rightarrow \quad d(Tx,Ty) \leq N(x,y) - \psi\big(N(x,y)\big), \tag{2.27}$$

where

$$N(x,y) = \max\left\{d(x,y), \frac{1}{2}\big[d(x,Tx) + d(y,Ty)\big], \frac{1}{2}\big[d(x,Ty) + d(y,Tx)\big]\right\}, \tag{2.28}$$

for all $x, y \in X$. Then the mapping T has a unique fixed point.

The proof of Theorem 5 can be done by following the lines of the proof of Theorem 4 and, hence, is omitted.

3 Fixed points on metric spaces with a partial order

In this section the fixed point theorems, Theorems 4 and 5, are formulated in the framework of partially ordered metric spaces. In what follows, we define a partial order $\preceq$ on the metric space (X,d).

Our first result is a counterpart of Theorem 4 on a partially ordered metric space.

Theorem 6 *Let $(X,d,\preceq)$ be a partially ordered complete metric space, $T : X \longrightarrow X$ be a nondecreasing map, and $\psi : [0,\infty) \longrightarrow [0,\infty)$ be a continuous nondecreasing function such that $\psi(0) = 0$ and $\psi(t) > 0$ for $t > 0$. Suppose that*

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad \Rightarrow \quad d(Tx,Ty) \leq M(x,y) - \psi\big(M(x,y)\big), \tag{3.1}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\big[d(x,Ty) + d(y,Tx)\big]\right\}, \tag{3.2}$$

for all $x, y \in X$ with $x \preceq y$. If there exists $x_0 \in X$ satisfying $x_0 \preceq Tx_0$, then T has a fixed point in X.

Proof Let $x_0 \in X$ satisfy $x_0 \preceq Tx_0$. Define the sequence $\{x_n\}$ as follows:

$$x_n = Tx_{n-1}, \quad n \in \mathbb{N}.$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is the fixed point of T. Assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Since $x_0 \preceq Tx_0 = x_1$ and T is nondecreasing, then obviously

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots. \tag{3.3}$$

Substituting $x = x_n$ and $y = Tx_n = x_{n+1}$ in (3.1) we get

$$\begin{aligned} &\frac{1}{2}d(x_n, Tx_n) = \frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) \\ &\quad \Rightarrow \quad d(Tx_n, Tx_{n+1}) = d(x_{n+1}, x_{n+2}) \leq M(x_n, x_{n+1}) - \psi\bigl(M(x_n, x_{n+1})\bigr), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} &M(x_n, x_{n+1}) \\ &\quad = \max\left\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \frac{1}{2}\bigl[d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)\bigr]\right\} \\ &\quad = \max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{1}{2}\bigl[d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})\bigr]\right\} \\ &\quad = \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{1}{2}d(x_n, x_{n+2})\right\}. \end{aligned} \tag{3.5}$$

For the rest of the existence proof one can follow the lines of the proof of Theorem 4, since they are similar. □

Assume now that the space $(X, d, \preceq)$ satisfies the condition

$$\text{(U)} \quad \text{For all } x, y \in X, \text{ there exists } z \in X \text{ such that } x \preceq z \text{ and } y \preceq z. \tag{3.6}$$

Our last result shows that the map given in Theorem 5 has a unique fixed point whenever it is defined on a partially ordered space $(X, d, \preceq)$, satisfying the condition (U).

Theorem 7 *Let $(X, d, \preceq)$ be a partially ordered complete metric space satisfying the condition* (U), *$T : X \longrightarrow X$ be a nondecreasing map, and $\psi : [0, \infty) \longrightarrow [0, \infty)$ be a continuous nondecreasing function such that $\psi(0) = 0$ and $\psi(t) > 0$ for $t > 0$. Suppose that*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \quad \Rightarrow \quad d(Tx, Ty) \leq N(x, y) - \psi\bigl(N(x, y)\bigr), \tag{3.7}$$

where

$$N(x, y) = \max\left\{d(x, y), \frac{1}{2}\bigl[d(x, Tx) + d(y, Ty)\bigr], \frac{1}{2}\bigl[d(x, Ty) + d(y, Tx)\bigr]\right\}, \tag{3.8}$$

for all $x, y \in X$ with $x \preceq y$. If there exists $x_0 \in X$ satisfying $x_0 \preceq Tx_0$, then T has a unique fixed point in X.

Proof The existence proof is done by mimicking the proofs of Theorem 6 and Theorem 4. To prove the uniqueness we assume that there are two different fixed points, x and y, that is, $x \neq y$ and $x = Tx$ and $y = Ty$. We consider the following cases:

Case 1. Suppose that x and y are comparable and, without loss of generality, that $x \preceq y$. Then

$$0 = \frac{1}{2}d(x, Tx) \leq d(x, y), \tag{3.9}$$

which implies

$$d(Tx, Ty) = d(x, y) \leq N(x, y) - \psi\big(N(x, y)\big), \tag{3.10}$$

where

$$N(x, y) = \max\left\{d(x, y), \frac{1}{2}\big[d(x, Tx) + d(y, Ty)\big], \frac{1}{2}\big[d(x, Ty) + d(y, Tx)\big]\right\} = d(x, y).$$

Thus, (3.10) becomes

$$d(x, y) \leq d(x, y) - \psi\big(d(x, y)\big),$$

and, clearly, $d(x, y) = 0$, that is, $x = y$.

Case 2. Assume that x and y are not comparable. From the condition (U) there exists $z \in X$ satisfying $x \preceq z$ and $y \preceq z$. Define the sequence $\{z_n\}$ as

$$z_0 = z, \qquad z_{n+1} = Tz_n, \quad n \in \mathbb{N}. \tag{3.11}$$

Notice that since T is nondecreasing and $x \preceq z$, we have

$$x \preceq z \quad \Longrightarrow \quad x = Tx \preceq Tz = z_1 \quad \Longrightarrow \quad \cdots \quad \Longrightarrow \quad x = Tx \preceq Tz_{n+1} = z_n, \quad n \in \mathbb{N}.$$

If $x = z_{n_0}$ for some $n_0 \in \mathbb{N}$, then $x = Tx = Tz_{n_0} = z_{n_0}$ and, hence, $x = T^k z = z_k$, for all $k \geq n_0$. Thus, the sequence z_n converges to the fixed point x, that is, $\lim_{n\to\infty} d(x, z_n) = 0$. Assume that $x \neq z_n$, for all $n \in \mathbb{N}$. Then we have

$$d(x, z_n) > \frac{1}{2}d(x, Tx) = 0, \quad \text{for all } n \in \mathbb{N},$$

which implies that the contractive condition

$$d(Tx, Tz_n) \leq N(x, z_n) - \psi\big(N(x, z_n)\big), \tag{3.12}$$

where

$$N(x, z_n) = \max\left\{d(x, z_n), \frac{1}{2}\big[d(x, Tx) + d(z_n, z_{n+1})\big], \frac{1}{2}\big[d(x, Tz_n) + d(z_n, Tx)\big]\right\},$$

holds for all $n \in \mathbb{N}$. Observe that

$$N(x,z_n) = \max\left\{d(x,z_n), \frac{d(z_n,z_{n+1})}{2}, \frac{1}{2}\left[d(x,Tz_n) + d(z_n,Tx)\right]\right\}$$

can be either $d(x,z_n)$ or $\frac{1}{2}[d(x,Tz_n) + d(z_n,Tx)]$ due to the fact that

$$\frac{d(z_n,z_{n+1})}{2} \le \frac{1}{2}\left[d(x,Tz_n) + d(z_n,Tx)\right]$$

by the triangle inequality. If $N(x,z_n) = \frac{1}{2}[d(x,Tz_n) + d(z_n,Tx)]$, then we have $d(x,z_n) \le N(x,z_n) \le d(x,z_{n+1})$ for some $n \in \mathbb{N}$. In this case, since $\psi(t) > 0$ for $t > 0$, the inequality (3.12) implies

$$d(x,z_{n+1}) \le N(x,z_n) - \psi\big(N(x,z_n)\big) < N(x,z_n) \le d(x,z_{n+1}), \tag{3.13}$$

which is not possible. Then we must have $N(x,z_n) = d(x,z_n)$, for all $n \in \mathbb{N}$, and, thus, the inequality (3.12) implies

$$d(x,z_{n+1}) \le d(x,z_n) - \psi\big(d(x,z_n)\big) < d(x,z_n), \tag{3.14}$$

that is, the sequence $\{d(x,z_n)\}$ is positive and decreasing and, therefore, convergent. Let $\lim_{n\to\infty} d(x,z_n) = L \ge 0$. Taking the limit as $n \to \infty$ in (3.14) we get

$$L \le L - \psi(L), \tag{3.15}$$

from which it follows that $\psi(L) = 0$, and, thus, we deduce

$$L = \lim_{n\to\infty} d(x,z_n) = 0. \tag{3.16}$$

In a similar way we obtain

$$\lim_{n\to\infty} d(y,z_n) = 0. \tag{3.17}$$

From (3.16) and (3.17) it follows that $x = y$, which completes the proof. □

Some consequences of Theorem 7 are given next. If we choose ψ as a specific function we get the following result.

Corollary 8 *Let $(X,d,\preceq)$ be a partially ordered complete metric space satisfying the condition* (U) *and $T : X \longrightarrow X$ be a nondecreasing map. Suppose that the condition*

$$\frac{1}{2}d(x,Tx) \le d(x,y) \quad \Rightarrow \quad d(Tx,Ty) \le kN(x,y), \tag{3.18}$$

where

$$N(x,y) = \max\left\{d(x,y), \frac{1}{2}\left[d(x,Tx) + d(y,Ty)\right], \frac{1}{2}\left[d(x,Ty) + d(y,Tx)\right]\right\} \tag{3.19}$$

holds, for all $x, y \in X$ with $x \preceq y$ and some constant $0 < k < 1$. If there exists $x_0 \in X$ satisfying $x_0 \preceq Tx_0$, then T has a unique fixed point in X.

Proof Choose $\psi(t) = (1-k)t$. Then the maps ψ and T satisfy the conditions of Theorem 7 and, thus, T has a unique fixed point in X. □

The next result is the analog of Theorem 2.1 in [7] on partially ordered metric spaces.

Corollary 9 *Let $(X, d, \preceq)$ be a partially ordered complete metric space satisfying the condition* (U) *and $T : X \longrightarrow X$ be a nondecreasing map. Suppose that*

$$\begin{aligned} &\frac{1}{2}d(x, Tx) \leq d(x,y) \\ &\quad \Rightarrow \quad d(Tx, Ty) \leq ad(x,y) + \frac{b}{2}\bigl[d(x, Tx) + d(y, Ty)\bigr] + \frac{c}{2}\bigl[d(x, Ty) + d(y, Tx)\bigr], \end{aligned} \tag{3.20}$$

where

$$a, b, c > 0, \qquad 0 < a + b + c = r < 1, \tag{3.21}$$

for all $x, y \in X$ with $x \preceq y$. If there exists $x_0 \in X$ satisfying $x_0 \preceq Tx_0$, then T has a unique fixed point in X.

Proof Define

$$N(x,y) = \max\left\{d(x,y), \frac{1}{2}\bigl[d(x, Tx) + d(y, Ty)\bigr], \frac{1}{2}\bigl[d(x, Ty) + d(y, Tx)\bigr]\right\}. \tag{3.22}$$

Then, clearly,

$$\begin{aligned} &ad(x,y) + \frac{b}{2}\bigl[d(x, Tx) + d(y, Ty)\bigr] + \frac{c}{2}\bigl[d(x, Ty) + d(y, Tx)\bigr] \\ &\quad \leq (a + b + c)N(x,y) \leq rN(x,y). \end{aligned} \tag{3.23}$$

Then the map T satisfies the conditions of the Corollary 8 and, thus, T has a unique fixed point in X. □

4 Applications

In this section we investigate the existence and uniqueness of solutions of periodic boundary value problems of first order. These problems have been studied under different conditions in [2, 15–18]. However, the existence and uniqueness conditions obtained here are weaker than those in the previous studies.

Define the partial ordering and the metric in $X = C[0, T]$ as follows:

$$\begin{aligned} &u \preceq v \quad \Rightarrow \quad u(t) \leq v(t), \quad \text{for all } t \in [0, T], \\ &d(u, v) = \sup\bigl\{\bigl|u(t) - v(t)\bigr|, t \in [0, T]\bigr\}. \end{aligned} \tag{4.1}$$

The space $(X, d, \preceq)$ satisfies the condition (U). Indeed, it is obvious that for every pair $u(t)$, $v(t)$ in X, we have $u(t) \preceq \max\{u(t), v(t)\}$ and $v(t) \preceq \max\{u(t), v(t)\}$. We will consider the following first order periodic boundary value problem:

$$\begin{cases} u'(t) = f(t, u(t)), & t \in [0, T], \\ u(0) = u(T). \end{cases} \tag{4.2}$$

Definition 10 A lower solution of the problem (4.2) is a function $u(t) \in C[0, T]$ satisfying

$$\begin{cases} u'(t) \le f(t, u(t)), & t \in [0, T], \\ u(0) \le u(T). \end{cases} \tag{4.3}$$

An upper solution to the problem (4.2) is a function $u(t) \in C[0, T]$ satisfying

$$\begin{cases} u'(t) \ge f(t, u(t)), & t \in [0, T], \\ u(0) \ge u(T). \end{cases} \tag{4.4}$$

Observe that the problem (4.2) can be written as

$$\begin{cases} u'(t) + \lambda u(t) = f(t, u(t)) + \lambda u(t), & t \in [0, T], \\ u(0) = u(T). \end{cases} \tag{4.5}$$

This problem is equivalent to the integral equation

$$u(t) = \int_0^T G(t,s)\big[f\big(s, u(s)\big) + \lambda u(s)\big]\, ds, \tag{4.6}$$

where $G(t,s)$ is the Green function defined by

$$G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \le s < t \le T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \le t < s \le T. \end{cases} \tag{4.7}$$

In what follows, we give a theorem for the existence and uniqueness of a solution of the problem (4.3).

Theorem 11 *Consider the periodic boundary value problem* (4.2). *Assume that* f *is continuous and that there exists* $\lambda > 0$ *such that, for all* $u, v \in C[0, T]$ *satisfying* $u \le v$, *the following condition holds*:

$$\left.\begin{cases} v'(t) \ge f(t, u(t)), & t \in [0, T], \\ v(0) \ge v(T) \end{cases}\right\} \\ \implies \quad 0 \le f\big(t, v(t)\big) + \lambda v(t) - f\big(t, u(t)\big) - \lambda u(t) \le k(v - u), \tag{4.8}$$

for some $k, \lambda \in [0, \infty)$, *such that* $0 < k < \lambda$. *If the problem* (4.2) *has a lower solution, then it has a unique solution*.

Proof Define the map $F : C[0, T] \to \mathbb{R}$ as follows:

$$Fu(t) = \int_0^T G(t,s)\big[f\big(s, u(s)\big) + \lambda u(s)\big]\, ds, \tag{4.9}$$

where $G(t,s)$ is the Green's function given in (4.7). Then the solution of the problem (4.2) is the fixed point of F. Assume that $u \leq v$ are functions in $C[0,T]$ satisfying (4.8). Rewrite the inequality $v'(t) \geq f(t,u(t))$ as

$$v'(t) + \lambda v(t) \geq f\bigl(t,u(t)\bigr) + \lambda u(t).$$

Multiplying both sides by $e^{\lambda t}$ and integrating from 0 to t we obtain

$$e^{\lambda t} v(t) \geq v(0) + \int_0^t e^{\lambda s}\bigl[f\bigl(s,u(s)\bigr) + \lambda u(s)\bigr]\,ds, \tag{4.10}$$

which, due to the condition $v(0) \geq v(T)$, gives

$$e^{\lambda T} v(0) \geq e^{\lambda T} v(T) \geq v(0) + \int_0^T e^{\lambda s}\bigl[f\bigl(s,u(s)\bigr) + \lambda u(s)\bigr]\,ds.$$

Hence,

$$v(0) \geq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T} - 1}\bigl[f\bigl(s,u(s)\bigr) + \lambda u(s)\bigr]\,ds.$$

Employing this inequality and (4.10) we get

$$e^{\lambda t} v(t) \geq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T} - 1}\bigl[f\bigl(s,u(s)\bigr) + \lambda u(s)\bigr]\,ds + \int_0^t e^{\lambda s}\bigl[f\bigl(s,u(s)\bigr) + \lambda u(s)\bigr]\,ds. \tag{4.11}$$

Hence, we obtain

$$v(t) \geq \int_0^t \frac{e^{\lambda(s-t+T)}}{e^{\lambda T} - 1}\bigl[f\bigl(s,u(s)\bigr) + \lambda u(s)\bigr]\,ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}\bigl[f\bigl(s,u(s)\bigr) + \lambda u(s)\bigr]\,ds, \tag{4.12}$$

which can be written as

$$v(t) \geq \int_0^T G(t,s)\bigl[f\bigl(s,u(s)\bigr) + \lambda u(s)\bigr]\,ds. \tag{4.13}$$

This implies

$$\sup\bigl\{\bigl|v(t) - u(t)\bigr|, t \in [0,T]\bigr\} \geq \sup\bigl\{\bigl|Fu(t) - u(t)\bigr|, t \in [0,T]\bigr\}, \tag{4.14}$$

or, in terms of the metric,

$$d(u,v) \geq d(Fu,u) \geq \frac{1}{2} d(Fu,u). \tag{4.15}$$

Moreover, since f satisfies (4.8), we have

$$\begin{aligned} Fu(t) &= \int_0^T G(t,s)\bigl[f\bigl(s,u(s)\bigr) + \lambda u(s)\bigr]\,ds \\ &\leq \int_0^T G(t,s)\bigl[f\bigl(s,v(s)\bigr) + \lambda v(s)\bigr]\,ds = Fv(t), \end{aligned} \tag{4.16}$$

that is, F is nondecreasing. Consider now

$$\begin{aligned} d(Fv, Fu) &= \sup\left|\int_0^T G(t,s)\bigl[f\bigl(s, v(s)\bigr) + \lambda v(s) - f\bigl(s, u(s)\bigr) - \lambda u(s)\bigr]\,ds\right| \\ &\leq \sup\int_0^T G(t,s)k\bigl|v(s) - u(s)\bigr|\,ds \\ &\leq kd(u,v)\int_0^T G(t,s)\,ds \\ &= \frac{k}{\lambda}d(u,v) \leq \frac{k}{\lambda}N(u,v), \end{aligned}$$

where $N(u,v) = \max\{d(x,y), \frac{1}{2}[d(x,Tx) + d(y,Ty)], \frac{1}{2}[d(x,Ty) + d(y,Tx)]\}$. Choosing λ in a way that $0 < k < \lambda$ we see that the nondecreasing map F satisfies the condition (2.21) of Corollary 8. We next show that $u_0 \leq Fu_0$ for some $u_0 \in X$. Since the problem (4.2) has a lower solution, there exists $u_0 \in X$ satisfying (4.3). Hence, we have

$$\begin{aligned} &u_0'(t) + \lambda u_0(t) \leq f\bigl(t, u_0(t)\bigr) + \lambda u_0(t), \quad t \in [0,T], \\ &u_0(0) \leq u_0(T). \end{aligned} \tag{4.17}$$

Multiplying both sides by $e^{\lambda t}$ and then integrating from 0 to t we obtain

$$u_0(t)e^{\lambda t} \leq u_0(0) + \int_0^t e^{\lambda s}\bigl[u_0(s) + f\bigl(s, u_0(s)\bigr)\bigr]\,ds. \tag{4.18}$$

Employing the inequality $u_0(0) \leq u_0(T)$ we get

$$u_0(0)e^{\lambda T} \leq u_0(T)e^{\lambda T} \leq u_0(0) + \int_0^T e^{\lambda s}\bigl[u_0(s) + f\bigl(s, u_0(s)\bigr)\bigr]\,ds,$$

or equivalently,

$$u_0(0) \leq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T} - 1}\bigl[u_0(s) + f\bigl(s, u_0(s)\bigr)\bigr]\,ds. \tag{4.19}$$

Combining (4.18) and (4.19) we get

$$\begin{aligned} u_0(t) &\leq \int_0^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}\bigl[u_0(s) + f\bigl(s, u_0(s)\bigr)\bigr]\,ds + \int_0^t e^{\lambda(s-t)}\bigl[u_0(s) + f\bigl(s, u_0(s)\bigr)\bigr]\,ds \\ &= \int_0^T G(t,s)\bigl[u_0(s) + f\bigl(s, u_0(s)\bigr)\bigr]\,ds, \end{aligned} \tag{4.20}$$

where $G(s,t)$ is the Green's function given in (4.8). Hence, we have

$$u_0(t) \leq Fu_0(t)$$

for the lower solution $u_0(t)$ of (4.2). Then, by the Corollary 8, the map F has a unique fixed point; thus, the boundary value problem (4.2) has a unique solution. □

Example 12 Consider the BVP

$$u'(t) = \frac{1}{2}\sin\frac{t}{4} - \frac{1}{4}u, \quad t \in [0, 8\pi],$$
$$u(0) = u(8\pi).$$

It can easily be verified by direct calculation that the unique solution is

$$u(t) = \left[\sin\frac{t}{4} - \cos\frac{t}{4}\right].$$

For this specific example the function $f(t,u) = \frac{1}{2}\sin\frac{t}{4} - \frac{1}{4}u$ satisfies the condition

$$0 \leq f\bigl(t, v(t)\bigr) + \lambda v(t) - f\bigl(t, u(t)\bigr) - \lambda u(t) \leq k\bigl(v(t) - u(t)\bigr),$$

not only for $u \leq v$, with

$$\begin{Bmatrix} v'(t) \geq \frac{1}{2}\sin\frac{t}{4} - \frac{1}{4}u, & t \in [0, 8\pi], \\ v(0) \geq v(8\pi) & \end{Bmatrix}$$

but for all $u \leq v$, where $\lambda - \frac{1}{4} = k < 1$. Indeed,

$$f\bigl(t, v(t)\bigr) + \lambda v(t) - f\bigl(t, u(t)\bigr) - \lambda u(t) = \left(\lambda - \frac{1}{4}\right)\bigl(v(t) - u(t)\bigr) \leq k\bigl(v(t) - u(t)\bigr),$$

for $\lambda - \frac{1}{4} = k < 1$. Observe that $u_0(t) = -2$ is a lower solution of the BVP. Clearly,

$$u_0'(t) = 0 \leq \frac{1}{2}\left(\sin\frac{t}{4} + 1\right), \quad t \in [0, 8\pi],$$

and

$$u_0(0) = -2 = u_0(8\pi).$$

By Theorem 11, the BVP has a unique solution.

Next, we give the following example of a nonlinear equation.

Example 13 Consider the BVP

$$u'(t) = (t+1)^2 - u^2, \quad t \in [0, 1],$$
$$u(0) = u(1).$$

The function $f(t,u) = (t+1)^2 - u^2$ satisfies the condition

$$\begin{aligned} 0 &\leq f(t, v) + \lambda v - f(t, u) - \lambda u \\ &= -v^2 + \lambda v + u^2 - \lambda u \leq (\lambda - M)(v - u) \\ &= k(v - u), \end{aligned}$$

for $u \leq v$ where u and v are nonnegative functions continuous on $[0,1]$, the positive constant M is defined as $M = \max_{t\in[0,1]}(u(t) + v(t))$, and λ is chosen such that $\lambda - M = k < 1$. The existence of $M > 0$ is verified by the fact that u and v are continuous on the closed interval $[0,1]$. Observe that $u_0(t) = 0$ is a lower solution of the BVP. Clearly,

$$u_0'(t) = 0 \leq (t+1)^2, \quad \text{and} \quad u_0(0) = 0 \leq u_0(1).$$

By Theorem 11, the BVP has a unique solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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