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Research Article

Approximate Multidegree Reduction of λ-Bézier Curves

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Besides inheriting the properties of classical Bézier curves of degree n, the corresponding λ-Bézier curves have a good performance in adjusting their shapes by changing shape control parameter. In this paper, we derive an approximation algorithm for multidegree reduction of λ-Bézier curves in the L_2-norm. By analysing the properties of λ-Bézier curves of degree n, a method which can deal with approximating λ-Bézier curve of degree $n+1$ by λ-Bézier curve of degree m $(m \le n)$ is presented. Then, in unrestricted and C^0, C^1 constraint conditions, the new control points of approximating λ-Bézier curve can be obtained by solving linear equations, which can minimize the least square error between the approximating curves and the original ones. Finally, several numerical examples of degree reduction are given and the errors are computed in three conditions. The results indicate that the proposed method is effective and easy to implement.

1. Introduction

Bézier curves are one of the main mathematical models in CAD/CAM system [1]. Degree reduction of Bézier curve is an important technique in geometric computation and geometric approximation [2] and has great significance for shape design. Firstly, it is embodied in data transfer and exchange between CAD systems or in CAD system, because the highest allowable degree of Bernstein polynomial for curve is generally different in various CAD systems or models. Next, degree reduction of curve is favorable for data compression. With the popularization of digitized and network product design, data communication between design systems becomes quite frequent [3], and geometric data in design system has come to mass [4]. Therefore, the operation of degree reduction attracts a good deal of attention.

The issue of degree reduction of Bézier curves is concerned with the solution of the following problem: for a given Bézier curve $\mathbf{R}_n(t)$ of degree n with Bézier points $\{\mathbf{r}_i\}_{i=0}^{n}$, find an approximate Bézier curve $\widetilde{\mathbf{R}}_m(t)$ of lower degree m, where $m < n$, with the set of Bézier points $\{\widetilde{\mathbf{r}}_i\}_{i=0}^{m}$, so that $\mathbf{R}_n$ and $\widetilde{\mathbf{R}}_m$ satisfy boundary conditions at the end points, and the error between $\mathbf{R}_n$ and $\widetilde{\mathbf{R}}_m$ is minimum. For degree reduction of Bézier curves, many scholars have done a lot of research that can be classified into three categories: geometry of approximate control point [5–8], algebraic means of basis function transformations [9–14], and B net and constrained optimization [15, 16]. Watkins and Worsey [9] presented an algorithm for generating $(n-1)$st degree approximation to nth degree Bézier curve. Eck [10] investigated a complete algorithm for performing the degree reduction within a prescribed error tolerance by help of subdivision. Chen and Wang [11] investigated the problem of optimal multidegree reduction of Bézier curves with constraints of endpoints continuity. Zheng and Wang [12] proved that the problem of finding a best L_2-approximation over the interval $[0,1]$ for constrained degree reduction is equivalent to that of finding a minimum perturbation vector in a certain weighted Euclidean norm. Using the transformation matrices, Lu and Wang [13] presented a method for the best multidegree reduction with respect to $\sqrt{t-t^2}$-weighted square norm for the unconstrained case. Tan and Fang [14] proposed three methods for degree reduction of interval generalized Ball curves of Wang-Said type. Degree reduction of Bézier curves has been conducted according to different norms, mostly L_2-norms, for both unconstrained and constrained conditions. In general, unconstrained degree reduction gives lower error than the constrained one. However, Bézier curves are often a part of a piecewise curve, so constrained degree reduction is preferred.

Although Bézier curves have now become a powerful tool for constructing free-form curves in CAD/CAM, they have their own disadvantages. Specifically, the shape of a Bézier curve is well-determined by its control points after choosing the basis functions [17]. In recent years, in order to improve the performance of Bézier curves, many scholars have constructed some new curves which are similar to the Bézier ones by introducing parameters into basis functions; see [18–22]. These new curves share many basic properties with the Bézier ones. Furthermore, they hold the property of flexible shape adjustability. Yan and Liang [18] constructed a new kind of basis function by a recursive approach; thus a kind of parametric curves with shape parameter is defined, which are called λ-Bézier curves. These new curves have most properties of the corresponding classical Bézier curves. Moreover, the shape parameter can adjust the shape of the new curves without changing the control points. Focusing on degree reduction of λ-Bézier curves, we study the corresponding problem by the least square method and obtain the new control points as well as the shape parameter of approximating λ-Bézier curves.

The remainder of the paper is organized as follows. The definition and properties of λ-Bézier curves are introduced in Section 2. In Section 3, we give the problem description of approximating degree reduction. In Section 4, we present the least square degree reduction of λ-Bézier curve. Numerical examples are given in Section 5, and we present some applications. At last, a short conclusion is given in Section 6.

2. The Definition and Properties of λ-Bézier Curves

2.1. Extension of Basis Function. The definition of extension Bernstein basis functions is given as follows [18].

Definition 1. Let $\lambda \in [-1, 1]$; for any $t \in [0, 1]$, the polynomial functions

$$\begin{aligned} b_{0,2}(t;\lambda) &= \left(1 - 2\lambda t + \lambda t^2\right)(1-t)^2, \\ b_{1,2}(t;\lambda) &= 2t(1-t)\left(1 + \lambda - \lambda t + \lambda t^2\right), \\ b_{2,2}(t;\lambda) &= \left(1 - \lambda + \lambda t^2\right)t^2 \end{aligned} \tag{1}$$

are called the extension Bernstein basis functions of degree 2 associated with the shape parameter λ.

For any integer n ($n \geq 3$), the functions $b_{i,n}(t;\lambda)$ ($i = 0, 1, \dots, n$) defined recursively by

$$b_{i,n}(t;\lambda) = (1-t)\,b_{i,n-1}(t;\lambda) + t b_{i-1,n-1}(t;\lambda), \quad t \in [0,1], \tag{2}$$

are called the extension Bernstein basis functions of degree n. In the case $k = -1$ or $k > l$, we set $b_{k,l}(t;\ \lambda) = 0$.

Theorem 2. *The extension Bernstein basis functions of degree n can be expressed explicitly as*

$$\begin{aligned} b_{i,n}(t;\lambda) &= \left(1 + \frac{3C_{n-2}^{i-1} + C_{n-1}^{i} - C_{n}^{i}}{C_{n}^{i}}\lambda - \frac{2C_{n-1}^{i}}{C_{n}^{i}}\lambda t + \lambda t^2\right) \\ &\quad \cdot C_n^i t^i (1-t)^{n-i} \quad (i = 0, 1, \dots, n), \end{aligned} \tag{3}$$

where $n \geq 2$, $C_n^i = n!/i!(n-i)!$.

2.2. Construction of λ-Bézier Curves

Definition 3. Given control points $\mathbf{P}_i^*$ $(i = 0, 1, \dots, n; n \geq 2)$ in R^2 or R^3, then

$$\mathbf{p}^*(t;\lambda) = \sum_{i=0}^{n} \mathbf{P}_i^* b_{i,n}(t;\lambda), \quad t \in [0,1], \ \lambda \in [-1,1] \tag{4}$$

is called a λ-Bézier curve of degree n with shape parameter λ, where basis functions $b_{i,n}(t;\lambda)$ $(i = 0, 1, \dots, n; n \geq 2)$ are defined by (3) (see Definition 3.2 in [18]).

When the shape parameter λ is equal to zero, λ-Bézier curves degenerate to the classical Bézier curves. λ-Bézier curve inherits most properties of the classical Bézier curve, such as convex hull property, geometric invariance, symmetry, and the following terminal property:

$$\begin{aligned} \mathbf{p}^*(0;\lambda) &= \mathbf{P}_0^*, \\ \mathbf{p}^*(1;\lambda) &= \mathbf{P}_n^*, \\ \left.\frac{d\mathbf{p}^*(t;\lambda)}{dt}\right|_{t=0} &= (n + 2\lambda)\left(\mathbf{P}_1^* - \mathbf{P}_0^*\right), \\ \left.\frac{d\mathbf{p}^*(t;\lambda)}{dt}\right|_{t=1} &= (n + 2\lambda)\left(\mathbf{P}_n^* - \mathbf{P}_{n-1}^*\right). \end{aligned} \tag{5}$$

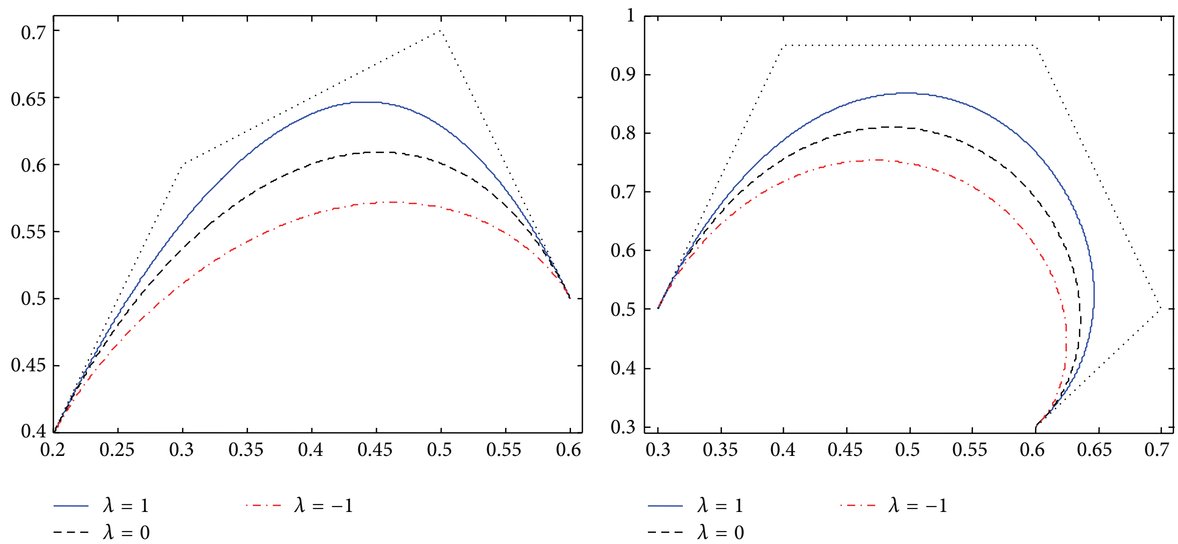
Because of introducing parameter λ, λ-Bézier curves have more powerful expressiveness than the classical Bézier curves.

Figure 1 shows graphs of λ-Bézier curves with the same control polygon but different shape parameters. Figure 1(a) shows the curves generated by the extension Bernstein basis functions with $n = 3$ and $\mathbf{p}^*(t;1)$ (solid lines), $\mathbf{p}^*(t;0)$ (dashed lines), and $\mathbf{p}^*(t;-1)$ (dot-dashed lines), respectively. Figure 1(b) shows the curves generated by the same basis functions as in Figure 1(a) with $n = 4$ and $\mathbf{p}^*(t;1)$ (solid lines), $\mathbf{p}^*(t;0)$ (dashed lines), and $\mathbf{p}^*(t;-1)$ (dot-dashed lines), respectively. From the figures, we can see that λ-Bézier curves approach the control polygon when the shape parameter is increasing.

3. Problem Description

Problem 4. Given control points $\{\mathbf{P}_i^*\}_{i=0}^{n+1} \subset \mathbf{R}^s$ $(s = 2, 3)$, λ-Bézier curve of degree $n+1$ is expressed as follows:

$$\mathbf{p}^*(t;\lambda) = \sum_{i=0}^{n+1} \mathbf{P}_i^* b_{i,n+1}(t;\lambda), \quad t \in [0,1], \tag{6}$$



(a) λ-Bézier curve of degree 3

(b) λ-Bézier curve of degree 4

Figure 1: λ-Bézier curves with the same control polygon but different shape parameters.

where $\{b_{i,n+1}(t;\lambda)\}_{i=0}^{n+1}$ are basis functions of degree $n+1$ and $\lambda \in [-1,1]$ is global shape parameter. Given control points $\{\mathbf{P}_i\}_{i=0}^{m} \subset \mathbf{R}^s$, the corresponding λ-Bézier curves of degree m $(m \le n)$ are

$$\mathbf{p}(t;\lambda) = \sum_{i=0}^{m} \mathbf{P}_i b_{i,m}(t;\lambda), \tag{7}$$

such that the distance minimizes between $\mathbf{p}^*(t;\lambda)$ and $\mathbf{p}(t;\lambda)$ in certain distance function $d(\mathbf{p}^*(t;\lambda), \mathbf{p}(t;\lambda))$.

Here we are interested in obtaining explicit expression of approximating λ-Bézier curves $\mathbf{p}(t;\lambda)$, so we choose the following least square distance function:

$$d^2(\mathbf{p}^*(t;\lambda), \mathbf{p}(t;\lambda)) = \int_0^1 \|\mathbf{p}^*(t;\lambda) - \mathbf{p}(t;\lambda)\|_2^2 \, dt. \tag{8}$$

Then we can convert Problem 4 into s subproblems, and every subproblem leads to a minimized component function:

$$d^2(p_k^*(t;\lambda), p_k(t;\lambda)) = \int_0^1 [p_k^*(t;\lambda) - p_k(t;\lambda)]^2 \, dt \quad (k = 1, 2, \ldots, s). \tag{9}$$

Let $\mathbf{p}^*(t;\lambda) = (p_1^*(t;\lambda), p_2^*(t;\lambda), \ldots, p_s^*(t;\lambda))$ and $\mathbf{p}(t;\lambda) = (p_1(t;\lambda), p_2(t;\lambda), \ldots, p_s(t;\lambda))$; then (8) is determined by the following formula:

$$d(\mathbf{p}^*(t;\lambda), \mathbf{p}(t;\lambda)) = \left[\sum_{k=1}^{s} d^2(p_k^*(t;\lambda), p_k(t;\lambda))\right]^{1/2}. \tag{10}$$

For subdistance function $d^2(p_k^*(t;\lambda), p_k(t;\lambda))$, it is sufficient to minimize $d(\mathbf{p}^*(t;\lambda), \mathbf{p}(t;\lambda))$. Therefore, we can just study the problem of minimum component function in the following.

Problem 5. Given a series of real numbers $\{P_i^*\}_{i=0}^{n+1}$, from which we will determine λ-Bézier functions of degree $n+1$,

$$f^*(t;\lambda) = \sum_{i=0}^{n+1} P_i^* b_{i,n+1}(t;\lambda), \tag{11}$$

where P_i^* denotes a component of vector $\mathbf{P}_i^*$, then it is necessary to find real numbers $\{P_j\}_{j=0}^{m}$ with the corresponding λ-Bézier functions of degree m

$$f(t;\lambda) = \sum_{j=0}^{m} P_j b_{j,m}(t;\lambda) \tag{12}$$

such that $d^2(f^*(t;\lambda), f(t;\lambda)) = \int_0^1 [f^*(t;\lambda) - f(t;\lambda)]^2 dt$ minimizes by least square distance.

In order to determine the approximate function $f(t;\lambda)$, primarily, we aim to obtain the coefficients $\{P_j\}_{j=0}^{m}$.

4. Least Square Degree Reduction of λ-Bézier Curves

4.1. The Approximate Degree Reduction of λ-Bézier Curves under Unrestricted Condition

Theorem 6. *If coefficients $\{P_j\}_{j=0}^{m}$ of approximating functions $f(t;\lambda)$ are solutions of Problem 5, the vector $\mathbf{P} = (P_0,$*

$P_1, \ldots, P_m)^T$ *satisfies linear systems* $\mathbf{AP} = \mathbf{b}$, *where*

$$\begin{aligned}
\mathbf{A} &= \left(a_{i,j}\right)_{m+1,m+1}, \\
\mathbf{b} &= \left(b_1, b_2, \ldots, b_{m+1}\right)^T, \\
a_{i+1,j+1} &= \int_0^1 b_{i,m}(t;\lambda)\, b_{j,m}(t;\lambda)\, dt, \\
b_{j+1} &= \int_0^1 \left[\sum_{i=0}^{n+1} P_i^* b_{i,n+1}(t;\lambda)\right] b_{j,m}(t;\lambda)\, dt. \\
&\qquad (i, j = 0, 1, \ldots, m).
\end{aligned} \tag{13}$$

Proof. By Problem 5, we obtain

$$\begin{aligned}
S &= d^2\left(f^*(t;\lambda), f(t;\lambda)\right) \\
&= \int_0^1 \left[f^*(t;\lambda) - f(t;\lambda)\right]^2 dt \\
&= \int_0^1 \left[\sum_{i=0}^{n+1} P_i^* b_{i,n+1}(t;\lambda) - \sum_{j=0}^{m} P_j b_{j,m}(t;\lambda)\right]^2 dt.
\end{aligned} \tag{14}$$

Let $\partial S/\partial P_j = 0$ $(j = 0, 1, \ldots, m)$; then the above equations can be simplified to the following ones:

$$\begin{aligned}
&\sum_{i=0}^{m} P_i \int_0^1 b_{i,m}(t;\lambda)\, b_{j,m}(t;\lambda)\, dt \\
&\quad = \int_0^1 \left[\sum_{i=0}^{n+1} P_i^* b_{i,n+1}(t;\lambda)\right] b_{j,m}(t;\lambda)\, dt \\
&\qquad (j = 0, 1, \ldots, m).
\end{aligned} \tag{15}$$

Furthermore, (15) can be represented in matrix form by calculation, which is described as follows:

$$\mathbf{AP} = \mathbf{b}, \tag{16}$$

where

$$\begin{aligned}
\mathbf{A} &= \left(a_{i,j}\right)_{m+1,m+1}, \\
\mathbf{b} &= \left(b_1, b_2, \ldots, b_{m+1}\right)^T, \\
a_{i+1,j+1} &= \int_0^1 b_{i,m}(t;\lambda)\, b_{j,m}(t;\lambda)\, dt, \\
b_{j+1} &= \int_0^1 \left[\sum_{i=0}^{n+1} P_i^* b_{i,n+1}(t;\lambda)\right] b_{j,m}(t;\lambda)\, dt. \\
&\qquad (i, j = 0, 1, \ldots, m).
\end{aligned} \tag{17}$$

Let $\mathbf{e}_{j+1} = (a_{1,j+1}, a_{2,j+1}, \ldots, a_{m+1,j+1})^T$ $(j = 0, 1, \ldots, m)$, and suppose

$$\begin{aligned}
\sum_{j=0}^{m} c_{j+1}\mathbf{e}_{j+1} &= c_1 \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m+1,1} \end{bmatrix} + c_2 \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{m+1,2} \end{bmatrix} + \cdots \\
&\quad + c_{m+1} \begin{bmatrix} a_{1,m+1} \\ a_{2,m+1} \\ \vdots \\ a_{m+1,m+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.
\end{aligned} \tag{18}$$

That is,

$$\begin{aligned}
\sum_{j=0}^{m} c_{j+1} a_{i+1,j+1} &= \int_0^1 \left[\sum_{j=0}^{m} c_{j+1} b_{j,m}(t;\lambda)\right] b_{i,m}(t;\lambda)\, dt \\
&= 0 \quad (i = 0, 1, \ldots, m).
\end{aligned} \tag{19}$$

We then get the following formula:

$$\begin{aligned}
&\int_0^1 \left[\sum_{j=0}^{m} c_{j+1} b_{j,m}(t;\lambda)\right]^2 dt \\
&\quad = \int_0^1 \left[\sum_{j=0}^{m} c_{j+1} b_{j,m}(t;\lambda)\right] \left[\sum_{i=0}^{m} c_{i+1} b_{i,m}(t;\lambda)\right] dt \\
&\quad = \sum_{i=0}^{m} c_{i+1} \int_0^1 \left[\sum_{j=0}^{m} c_{j+1} b_{j,m}(t;\lambda)\right] b_{i,m}(t;\lambda)\, dt = 0;
\end{aligned} \tag{20}$$

thus $\sum_{j=0}^{m} c_{j+1} b_{j,m}(t;\lambda) \equiv 0$. Since $\{b_{0,m}(t;\lambda), b_{1,m}(t;\lambda), \ldots, b_{m,m}(t;\lambda)\}$ are linearly independent in interval $t \in [0,1]$, we have $c_{j+1} \equiv 0$ $(j = 0, 1, \ldots, m)$, which means vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{m+1}\}$ are linearly independent, and then solutions of linear systems (16) are uniquely determined. □

4.2. The Approximate Degree Reduction of λ-Bézier Curves under C^0 Constraint Condition. When approximating degree reduction, we expect to satisfy C^0 continuity, that is, maintaining interpolation of terminal points, so two equations $P_0 = P_0^*$ and $P_m = P_{n+1}^*$ are determined. The remaining $m - 1$ control points are determined by the following theorem.

Theorem 7. *If coefficients* $\{P_i\}_{i=0}^{m}$ *of approximating functions* $f(t;\lambda)$ *are solutions of Problem 5 and maintain* C^0 *continuity, the vector* $\mathbf{P} = (P_1, P_2, \ldots, P_{m-1})^T$ *satisfies linear systems*

$\mathbf{AP} = \mathbf{b}$ *except for two equations* $P_0 = P_0^*$ *and* $P_m = P_{n+1}^*$ *for terminal points, where*

$$\begin{gathered}\mathbf{A} = \left(a_{i,j}\right)_{m-1,m-1}, \\ \mathbf{b} = \left(b_1, b_2, \ldots, b_{m-1}\right)^T, \\ a_{i,j} = \int_0^1 b_{i,m}(t;\lambda)\, b_{j,m}(t;\lambda)\, dt, \\ b_j = \int_0^1 \left[\sum_{i=0}^{n+1} P_i^* b_{i,n+1}(t;\lambda)\right] b_{j,m}(t;\lambda)\, dt - \int_0^1 \left(P_0^* b_{0,m}(t;\lambda) + P_{n+1}^* b_{m,m}(t;\lambda)\right) b_{j,m}(t;\lambda)\, dt. \\ (i, j = 1, 2, \ldots, m-1).\end{gathered} \tag{21}$$

Proof. According to the condition of C^0 continuity, that is, $f^*(0;\lambda) = f(0;\lambda)$ and $f^*(1;\lambda) = f(1;\lambda)$, it is easy to obtain two equations $P_0 = P_0^*$ and $P_m = P_{n+1}^*$. Then by applying Problem 5, we get

$$\begin{aligned} S = d^2(f^*, f) &= \int_0^1 \left[f^*(t;\lambda) - f(t;\lambda)\right]^2 dt \\ &= \int_0^1 \left[\sum_{i=0}^{n+1} P_i^* b_{i,n+1}(t;\lambda) - \sum_{j=0}^{m} P_j b_{j,m}(t;\lambda)\right]^2 dt. \end{aligned} \tag{22}$$

Let $\partial S/\partial P_j = 0$ $(j = 1, 2, \ldots, m-1)$. Equation (22) can be simplified to the following equations:

$$\begin{aligned} &\sum_{i=1}^{m-1} P_i \int_0^1 b_{i,m}(t;\lambda)\, b_{j,m}(t;\lambda)\, dt \\ &\quad = \int_0^1 \left[\sum_{i=0}^{n+1} P_i^* b_{i,n+1}(t;\lambda) - P_0^* b_{0,m}(t;\lambda) - P_{n+1}^* b_{m,m}(t;\lambda)\right] b_{j,m}(t;\lambda)\, dt. \end{aligned} \tag{23}$$

Furthermore, these equations can be represented in matrix form as follows:

$$\mathbf{AP} = \mathbf{b}, \tag{24}$$

where

$$\begin{gathered}\mathbf{A} = \left(a_{i,j}\right)_{m-1,m-1}, \\ \mathbf{b} = \left(b_1, b_2, \ldots, b_{m-1}\right)^T, \\ a_{i,j} = \int_0^1 b_{i,m}(t;\lambda)\, b_{j,m}(t;\lambda)\, dt, \\ b_j = \int_0^1 \left[\sum_{i=0}^{n+1} P_i^* b_{i,n+1}(t;\lambda)\right] b_{j,m}(t;\lambda)\, dt - \int_0^1 \left(P_0^* b_{0,m}(t;\lambda) + P_{n+1}^* b_{m,m}(t;\lambda)\right) b_{j,m}(t;\lambda)\, dt. \\ (i, j = 1, 2, \ldots, m-1).\end{gathered} \tag{25}$$

Let $\mathbf{e}_j = (a_{1,j}, a_{2,j}, \ldots, a_{m-1,j})^T$ $(j = 1, 2, \ldots, m-1)$, and suppose

$$\sum_{j=1}^{m-1} c_j \mathbf{e}_j = c_1 \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m-1,1} \end{bmatrix} + c_2 \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{m-1,2} \end{bmatrix} + \cdots + c_{m-1} \begin{bmatrix} a_{1,m-1} \\ a_{2,m-1} \\ \vdots \\ a_{m-1,m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{26}$$

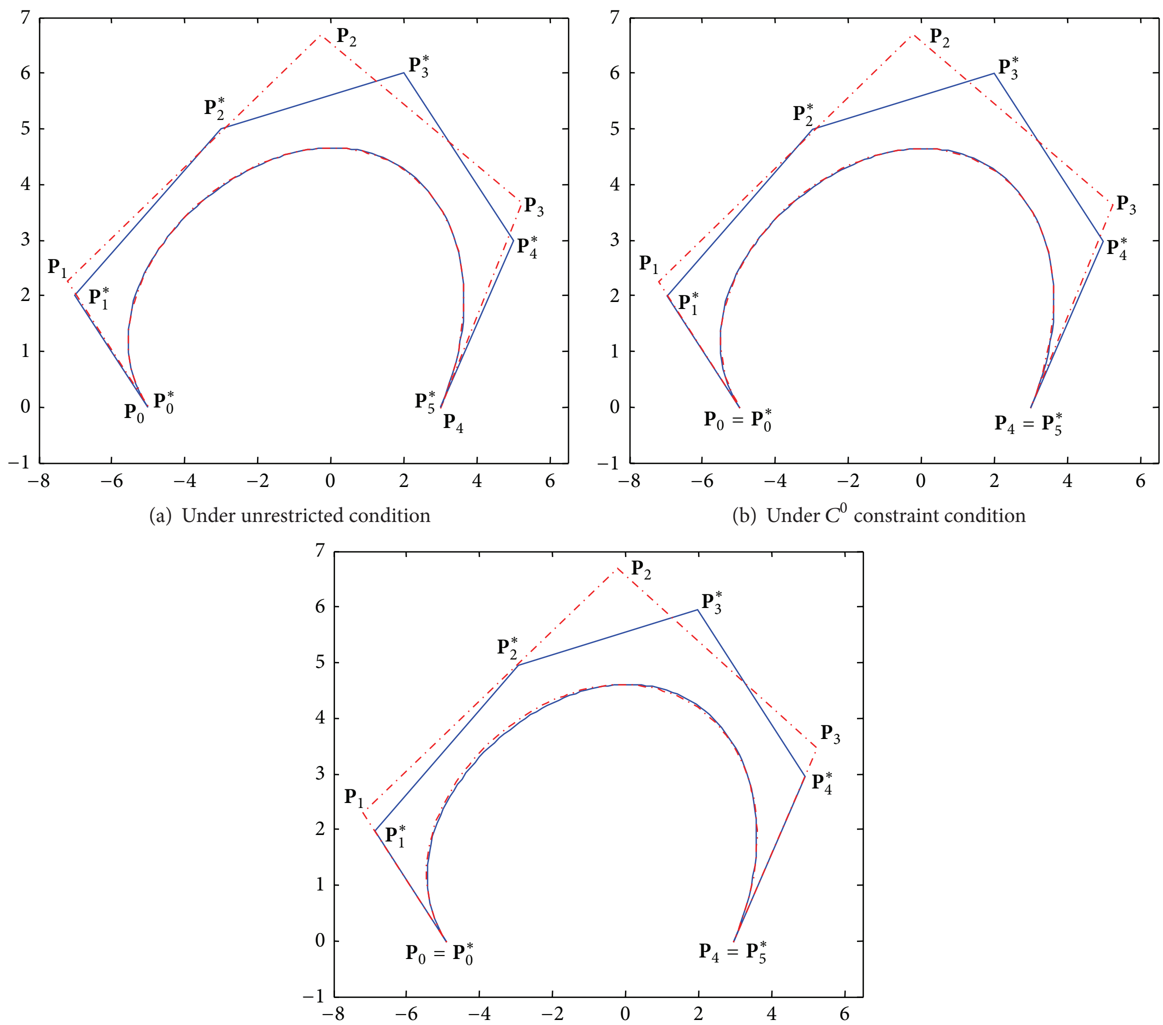
That is,

$$\sum_{j=1}^{m-1} c_j a_{i,j} = \int_0^1 \left[\sum_{j=1}^{m-1} c_j b_{j,m}(t;\lambda)\right] b_{i,m}(t;\lambda)\, dt = 0 \quad (i = 1, 2, \ldots, m-1). \tag{27}$$

Because $\{b_{1,m}(t;\lambda), b_{2,m}(t;\lambda), \ldots, b_{m-1,m}(t;\lambda)\}$ are linearly independent in interval $t \in [0,1]$, the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{m-1}\}$, as in Theorem 6, are linearly independent. Thus solutions of linear systems (24) are uniquely determined and maintain C^0 continuity. □

4.3. The Approximate Degree Reduction of λ-Bézier Curves under C^1 Constraint Condition. When approximating degree reduction, if C^1 continuity is maintained (i.e., four equations $P_0 = P_0^*$, $P_m = P_{n+1}^*$, $P_1 = P_0^* + ((n+1+2\lambda)/(m+2\lambda))(P_1^* - P_0^*)$, and $P_{m-1} = P_{n+1}^* - ((n+1+2\lambda)/(m+2\lambda))(P_{n+1}^* - P_n^*)$ are specified), the remaining $m-3$ control points are determined by the following theorem.

Theorem 8. *If coefficients* $\{P_i\}_{i=0}^m$ *of approximate functions* $f(t;\lambda)$ *are solutions of Problem 5 and maintain* C^1 *continuity,*



(a) Under unrestricted condition

(b) Under C^0 constraint condition

(c) Under C^1 constraint condition

Figure 2: Degree reduction with various constraint conditions (from degree 5 to degree 4). Blue solid: the given curve of degree 5; red dot-dashed line: the degree-reduced curve of degree 4.

the vector $\mathbf{P} = (P_2, P_3, \ldots, P_{m-2})^T$ *satisfies linear systems* $\mathbf{AP} = \mathbf{b}$ *except for four equations* $P_0 = P_0^*$, $P_m = P_{n+1}^*$, $P_1 = P_0^* + ((n+1+2\lambda)/(m+2\lambda))(P_1^* - P_0^*)$, *and* $P_{m-1} = P_{n+1}^* - ((n+1+2\lambda)/(m+2\lambda))(P_{n+1}^* - P_n^*)$ *for terminal points, where*

$$\mathbf{A} = \left(a_{i-1,j-1}\right)_{m-3,m-3},$$

$$\mathbf{b} = (b_1, b_2, \ldots, b_{m-3})^T,$$

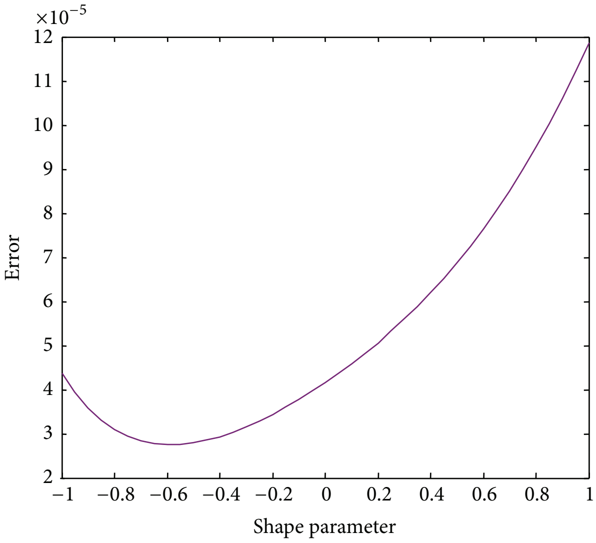
$$a_{i-1,j-1} = \int_0^1 b_{i,m}(t;\lambda)\, b_{j,m}(t;\lambda)\, dt,$$

$$b_{j-1} = \int_0^1 \left[\sum_{i=0}^{n+1} P_i^* b_{i,n+1}(t;\lambda) - P_0 b_{0,m}(t;\lambda) - P_1 b_{1,m}(t;\lambda)\right] b_{j,m}(t;\lambda)\, dt - \int_0^1 \left(P_{m-1} b_{m-1,m}(t;\lambda) + P_m b_{m,m}(t;\lambda)\right) \cdot b_{j,m}(t;\lambda)\, dt.$$

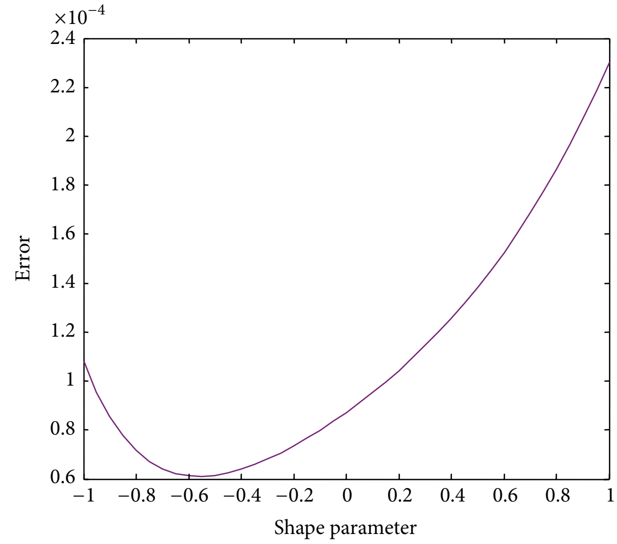
$$(i, j = 2, 3, \ldots, m-2). \tag{28}$$

Proof. According to the condition of C^1 continuity, we get

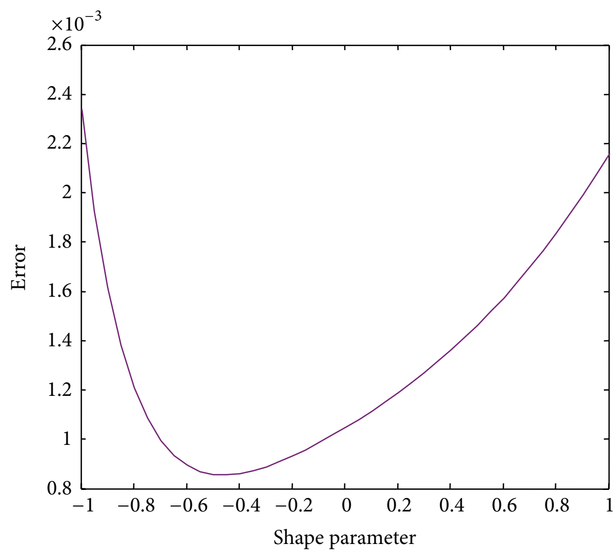
$$f^*(0;\lambda) = f(0;\lambda),$$
$$f^*(1;\lambda) = f(1;\lambda),$$
$$\left.\frac{d[f^*(t;\lambda)]}{dt}\right|_{t=0} = \left.\frac{d[f(t;\lambda)]}{dt}\right|_{t=0},$$
$$\left.\frac{d[f^*(t;\lambda)]}{dt}\right|_{t=1} = \left.\frac{d[f(t;\lambda)]}{dt}\right|_{t=1}. \tag{29}$$



(a) Under unrestricted condition



(b) Under C^0 constraint condition



(c) Under C^1 constraint condition

FIGURE 3: Error graph of degree reduction of λ-Bézier of degree 5.

It is easy to obtain the following four equations:

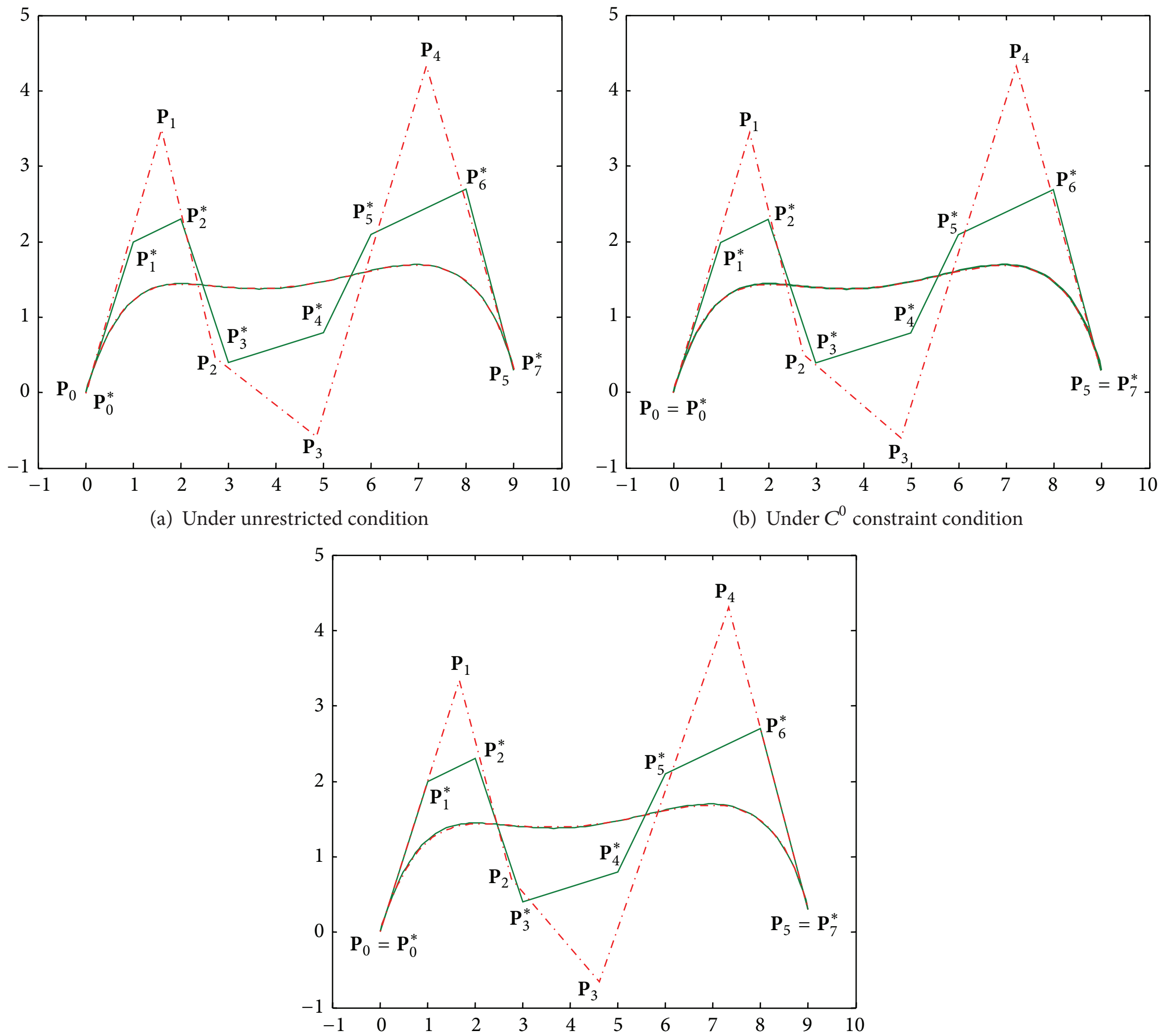
$$\begin{aligned} P_0 &= P_0^*, \\ P_m &= P_{n+1}^*, \\ P_1 &= P_0^* + \frac{n+1+2\lambda}{m+2\lambda}\left(P_1^* - P_0^*\right), \\ P_{m-1} &= P_{n+1}^* - \frac{n+1+2\lambda}{m+2\lambda}\left(P_{n+1}^* - P_n^*\right). \end{aligned} \tag{30}$$

Then by Problem 5, we obtain

$$\begin{aligned} S = d^2\left(f^*, f\right) &= \int_0^1 \left[f^*(t;\lambda) - f(t;\lambda)\right]^2 dt \\ &= \int_0^1 \left[\sum_{i=0}^{n+1} P_i^* b_{i,n+1}(t;\lambda) - \sum_{j=0}^{m} P_j b_{j,m}(t;\lambda)\right]^2 dt. \end{aligned} \tag{31}$$

Let $\partial S/\partial P_j = 0$ $(j = 2, 3, \ldots, m-2)$. Equation (31) can be simplified to the following form:

$$\begin{aligned} &\sum_{i=2}^{m-2} P_i \int_0^1 b_{i,m}(t;\lambda)\, b_{j,m}(t;\lambda)\, dt \\ &\quad = \int_0^1 \left[\sum_{i=0}^{n+1} P_i^* b_{i,n+1}(t;\lambda)\right] b_{j,m}(t;\lambda)\, dt \\ &\quad - \int_0^1 \left(P_0 b_{0,m}(t;\lambda) + P_1 b_{1,m}(t;\lambda)\right. \\ &\quad \left. + P_{m-1} b_{m-1,m}(t;\lambda) + P_m b_{m,m}(t;\lambda)\right) b_{j,m}(t;\lambda)\, dt. \end{aligned} \tag{32}$$



(a) Under unrestricted condition

(b) Under C^0 constraint condition

(c) Under C^1 constraint condition

FIGURE 4: Degree reduction with various constraint conditions (from degree 7 to degree 5). Green solid: the given curve of degree 7; red dot-dashed line: the degree-reduced curve of degree 5.

Furthermore, this equation can be represented in matrix form as follows:

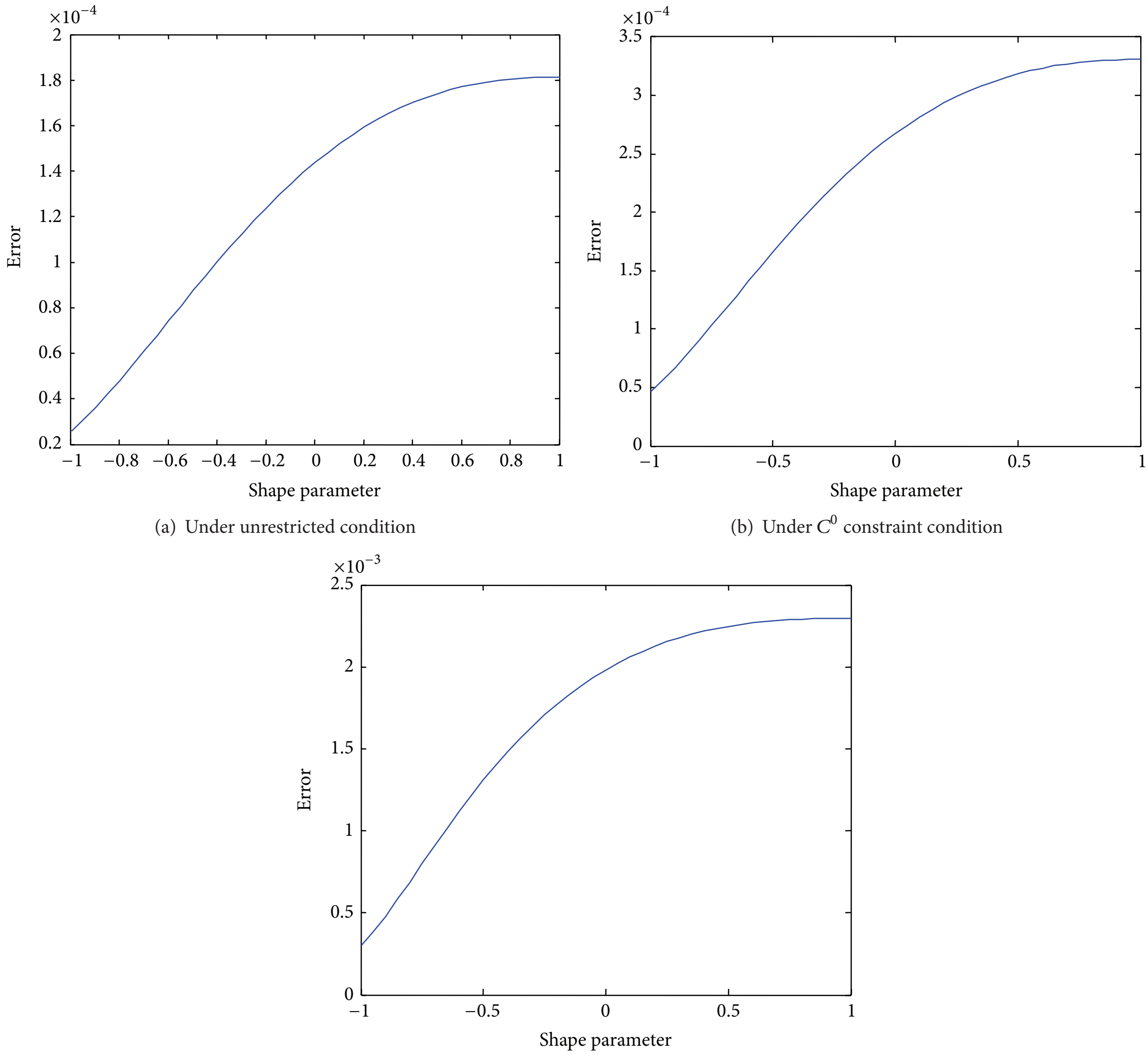
$$\mathbf{AP} = \mathbf{b}, \tag{33}$$

where

$$\begin{gathered}
\mathbf{A} = \left(a_{i-1,j-1}\right)_{m-3,m-3}, \\
\mathbf{b} = \left(b_1, b_2, \ldots, b_{m-3}\right)^T, \\
a_{i-1,j-1} = \int_0^1 b_{i,m}(t;\lambda)\, b_{j,m}(t;\lambda)\, dt, \\
b_{j-1} = \int_0^1 \left[\sum_{i=0}^{n+1} P_i^* b_{i,n+1}(t;\lambda) - P_0 b_{0,m}(t;\lambda) - P_1 b_{1,m}(t;\lambda)\right] b_{j,m}(t;\lambda)\, dt \\
- \int_0^1 \left(P_{m-1} b_{m-1,m}(t;\lambda) + P_m b_{m,m}(t;\lambda)\right) \cdot b_{j,m}(t;\lambda)\, dt. \\
(i, j = 2, 3 \ldots, m-2).
\end{gathered} \tag{34}$$

Let $\mathbf{e}_j = (a_{1,j}, a_{2,j}, \ldots, a_{m-3,j})^T$ $(j = 1, 2, \ldots, m-3)$, and suppose

$$\sum_{j=1}^{m-3} c_j \mathbf{e}_j = c_1 \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m-3,1} \end{bmatrix} + c_2 \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{m-3,2} \end{bmatrix} + \cdots + c_{m-3} \begin{bmatrix} a_{1,m-3} \\ a_{2,m-3} \\ \vdots \\ a_{m-3,m-3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{35}$$



(a) Under unrestricted condition

(b) Under C^0 constraint condition

(c) Under C^1 constraint condition

FIGURE 5: Error graph of degree reduction of λ-Bézier of degree 7.

That is,

$$\sum_{j=1}^{m-3} c_j a_{i,j} = \int_0^1 \left[\sum_{j=1}^{m-3} c_j b_{j+1,m}(t;\lambda) \right] b_{i+1,m}(t;\lambda)\,dt = 0 \qquad (i = 1, 2, \ldots, m-3). \tag{36}$$

Because $\{b_{2,m}(t;\lambda), \ldots, b_{m-2,m}(t;\lambda)\}$ are linearly independent in interval $t \in [0,1]$, the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{m-3}\}$, as in Theorem 6, are linearly independent. Therefore, solutions of linear systems (33) exist uniquely and maintain C^1 continuity. □

5. Numerical Examples

Example 1. Given the shape parameter $\lambda = 1$ and the coordinates of control points $\{\mathbf{P}_0^* = (-5, 0)$, $\mathbf{P}_1^* = (-7, 2)$, $\mathbf{P}_2^* = (-3, 5)$, $\mathbf{P}_3^* = (2, 6)$, $\mathbf{P}_4^* = (5, 3)$, $\mathbf{P}_5^* = (3, 0)\}$, we can construct λ-Bézier curve of degree 5. Then this curve can be separately reduced to λ-Bézier curve of degree 4, respectively, under unrestricted and C^0, C^1 constraint condition. Control points and errors for approximating λ-Bézier curve of degree 4 to a λ-Bézier curve of degree 5 are shown in Table 1. Degree reduction with various constraint conditions from degree 5 to degree 4 is shown in Figure 2.

We give the approximation error graphs of degree reduction of degree 5 in three conditions with different shape parameter, as shown in Figure 3. From Figure 3, the approximation error value of degree reduction decreases at first and then increases when increasing the shape parameter. The range of error value is $[0.27488 \times 10^{-4}, 1.1886 \times 10^{-4}]$ in (a), and those in (b) and (c) are $[0.60937 \times 10^{-4}, 2.3013 \times 10^{-4}]$ and $[0.85492 \times 10^{-3}, 2.3364 \times 10^{-3}]$, respectively.

Table 1: Control points and approximation errors with different constraint conditions in Example 1 (from degree 5 to degree 4).

Constraint condition	Control points	Errors
Under unrestricted condition	$\mathbf{P}_0 = (-5.0241, -0.01868)$, $\mathbf{P}_1 = (-7.2251, 2.266)$, $\mathbf{P}_2 = (-0.24919, 6.690)$, $\mathbf{P}_3 = (5.2589, 3.672)$, $\mathbf{P}_4 = (3.0186, -0.03573)$	$d^2(\mathbf{p}_5^*(t;1), \mathbf{p}_4(t;1)) = 0.11866 \times 10^{-3}$
Under C^0 constraint condition	$\mathbf{P}_0 = (-5, 0)$, $\mathbf{P}_1 = (-7.2449, 2.273)$, $\mathbf{P}_2 = (-0.24428, 6.705)$, $\mathbf{P}_3 = (5.2705, 3.639)$, $\mathbf{P}_4 = (3, 0)$	$d^2(\mathbf{p}_5^*(t;1), \mathbf{p}_4(t;1)) = 0.23013 \times 10^{-3}$
Under C^1 constraint condition	$\mathbf{P}_0 = (-5, 0)$, $\mathbf{P}_1 = (-7.3333, 2.3333)$, $\mathbf{P}_2 = (-0.22601, 6.7616)$, $\mathbf{P}_3 = (5.3333, 3.5000)$, $\mathbf{P}_4 = (3, 0)$	$d^2(\mathbf{p}_5^*(t;1), \mathbf{p}_4(t;1)) = 0.21581 \times 10^{-2}$

Table 2: Control points and approximation errors with different constraint conditions in Example 2 (from degree 7 to degree 5).

Constraint condition	Control points	Error
Under unrestricted condition	$\mathbf{P}_0 = (-0.0073157, -0.01488)$, $\mathbf{P}_1 = (1.6003, 3.489)$, $\mathbf{P}_2 = (2.7337, 0.4590)$, $\mathbf{P}_3 = (4.8462, -0.5802)$, $\mathbf{P}_4 = (7.1691, 4.336)$, $\mathbf{P}_5 = (9.0139, 0.2929)$	$d^2(\mathbf{p}_7^*(t;1), \mathbf{p}_5(t;1)) = 0.25675 \times 10^{-4}$
Under C^0 constraint condition	$\mathbf{P}_0 = (0, 0)$, $\mathbf{P}_1 = (1.6128, 3.443)$, $\mathbf{P}_2 = (2.7475, 0.5180)$, $\mathbf{P}_3 = (4.7940, -0.5994)$, $\mathbf{P}_4 = (7.2115, 4.325)$, $\mathbf{P}_5 = (9, 0.3)$	$d^2(\mathbf{p}_7^*(t;1), \mathbf{p}_5(t;1)) = 0.45964 \times 10^{-4}$
Under C^1 constraint condition	$\mathbf{P}_0 = (0, 0)$, $\mathbf{P}_1 = (1.6667, 3.3333)$, $\mathbf{P}_2 = (2.7655, 0.7045)$, $\mathbf{P}_3 = (4.6134, -0.6613)$, $\mathbf{P}_4 = (7.3333, 4.3)$, $\mathbf{P}_5 = (9, 0.3)$	$d^2(\mathbf{p}_7^*(t;1), \mathbf{p}_5(t;1)) = 0.30212 \times 10^{-3}$

Example 2. Given the shape parameter $\lambda = -1$ and the coordinates of control points $\{\mathbf{P}_0^* = (0,0)$, $\mathbf{P}_1^* = (1,2)$, $\mathbf{P}_2^* = (2,2.3)$, $\mathbf{P}_3^* = (3,0.4)$, $\mathbf{P}_4^* = (5,0.8)$, $\mathbf{P}_5^* = (6,2.1)$, $\mathbf{P}_6^* = (8,2.7)$, $\mathbf{P}_7^* = (9,0.3)\}$, we can construct a λ-Bézier curve of degree 7. Then this curve will be separately reduced to λ-Bézier curves of degree 5 under three conditions. Control points and errors for approximating λ-Bézier curve of degree 5 to a λ-Bézier curve of degree 7 are shown in Table 2. Degree reductions with various constraint conditions from degree 7 to degree 5 are shown in Figure 4.

We give the approximation error graphs of degree reduction of λ-Bézier of degree 7 in three conditions with different shape parameter, as shown in Figure 5. From Figure 5, the approximation error value of degree reduction increases and slope decreases by increasing the shape parameter. The range of error value is $[0.25675 \times 10^{-4}, 1.812 \times 10^{-4}]$ in (a), and those in (b) and (c) are $[0.45964 \times 10^{-4}, 3.3081 \times 10^{-4}]$ and $[0.30212 \times 10^{-3}, 2.2919 \times 10^{-3}]$, respectively.

Example 3. Given shape parameter $\lambda = -1$, and two segments of λ-Bézier curves of degree 7 expressing patterned vase, then they will be separately reduced to two segments of λ-Bézier curve of degree 5 under unrestricted condition. Control points and error for approximating λ-Bézier curve of degree 5 to a λ-Bézier curve of degree 7 are shown in Table 3. Degree reductions of these two segments are shown in Figure 6. In addition, approximation errors with C^0 and C^1 constraint conditions in Example 3 are 0.81951×10^{-3} and 0.50732×10^{-2}, respectively.

With the change of shape parameter, we present error graph of degree reduction of λ-Bézier of degree 7 which expresses patterned vase in unrestricted condition, as is shown in Figure 7. From Figure 7, the error value increases with that of shape parameter. The range of error value is $[0.45528 \times 10^{-3}, 1.6712 \times 10^{-3}]$.

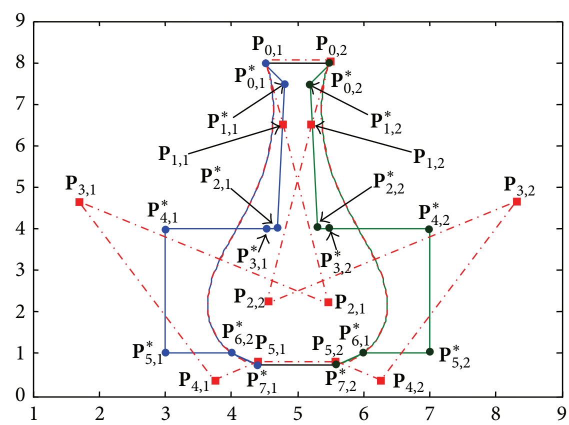


Figure 6: Degree reduction of λ-Bézier curve of degree 7 which expresses patterned vase in unrestricted condition. Green solid and blue solid: the given curve of degree 7; red dot-dashed line: the degree-reduced curve of degree 5.

6. Concluding Remarks

λ-Bézier curves of degree n have the same properties as Bézier curves. In addition, they have better performance when adjusting their shapes by changing the shape parameter, which includes shape adjustability and better approximation to control polygon as shown in Figure 1.

Furthermore, the problem of degree reduction for λ-Bézier curves is studied by least squared approximation. An

TABLE 3: Control points and approximation error under unrestricted condition in Example 3 (from degree 7 to degree 5).

		Control points	Error
Before degree reduction	First segment	$\mathbf{P}^*_{0,1} = (4.5, 8)$, $\mathbf{P}^*_{1,1} = (4.8, 7.5)$, $\mathbf{P}^*_{2,1} = (4.7, 4)$, $\mathbf{P}^*_{3,1} = (4.6, 4)$, $\mathbf{P}^*_{4,1} = (3, 4)$, $\mathbf{P}^*_{5,1} = (3, 1)$, $\mathbf{P}^*_{6,1} = (4, 1)$, $\mathbf{P}^*_{7,1} = (4.4, 0.7)$	$d(\mathbf{p}^*_7(t;1), \mathbf{p}_5(t;1)) = 0.45528 \times 10^{-3}$
	Second segment	$\mathbf{P}^*_{0,2} = (5.5, 8)$, $\mathbf{P}^*_{1,2} = (5.2, 7.5)$, $\mathbf{P}^*_{2,2} = (5.3, 4)$, $\mathbf{P}^*_{3,2} = (5.4, 4)$, $\mathbf{P}^*_{4,2} = (7, 4)$, $\mathbf{P}^*_{5,2} = (7, 1)$, $\mathbf{P}^*_{6,2} = (6, 1)$, $\mathbf{P}^*_{7,2} = (5.6, 0.7)$	
After degree reduction	First segment	$\mathbf{P}_{0,1} = (4.5165, 8.067)$, $\mathbf{P}_{1,1} = (4.7885, 6.478)$, $\mathbf{P}_{2,1} = (5.4628, 2.165)$, $\mathbf{P}_{3,1} = (1.6892, 4.636)$, $\mathbf{P}_{4,1} = (3.7304, 0.3241)$, $\mathbf{P}_{5,1} = (4.4033, 0.7784)$	
	Second segment	$\mathbf{P}_{0,2} = (5.4835, 8.067)$, $\mathbf{P}_{1,2} = (5.2115, 6.478)$, $\mathbf{P}_{2,2} = (4.5372, 2.165)$, $\mathbf{P}_{3,2} = (8.3108, 4.636)$, $\mathbf{P}_{4,2} = (6.2696, 0.3241)$, $\mathbf{P}_{5,2} = (5.5967, 0.7784)$	

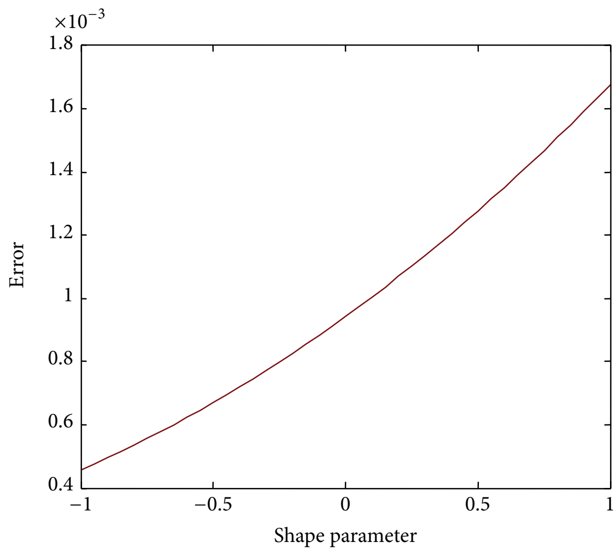


Figure 7: Error graph of degree reduction of λ-Bézier of degree 7 which expresses patterned vase in unrestricted condition.

algorithm for approximating degree reduction of λ-Bézier curves of degree n is provided by adjusting control points under three conditions, which can minimize the least square error between the approximating curves and the original ones. Three practical examples show that the method is applicable for CAD/CAM modeling systems. We will focus on studying the degree reduction for λ-Bézier surfaces in future work.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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