Jiang et al. *Advances in Difference Equations* (2015) 2015:368
DOI 10.1186/s13662-015-0702-5

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence and iterative approximations of nonoscillatory solutions for second order nonlinear neutral delay difference equations

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Abstract

This paper investigates the second order nonlinear neutral delay difference equation

$$\begin{aligned}&\Delta\big[a_n\Delta(x_n+bx_{n-\tau}-d_n)\big]+\Delta f(n,x_{f_1(n)},x_{f_2(n)},\dots,x_{f_k(n)})\\&\quad+g(n,x_{g_1(n)},x_{g_2(n)},\dots,x_{g_k(n)})=c_n,\quad n\ge n_0.\end{aligned}$$

By using the Banach fixed point theorem and some new techniques, we establish the existence results of uncountably many bounded nonoscillatory solutions for the above equation, propose a few Mann type iterative approximation schemes with errors and obtain several errors estimates between the iterative approximations and the nonoscillatory solutions. Examples that cannot be solved by known results are given to illustrate our theorems.

MSC: 39A10; 39A20

Keywords: Second order nonlinear neutral delay difference equation; uncountably many bounded nonoscillatory solutions; Banach fixed point theorem; Mann iterative sequence with errors

1 Introduction

In recent years there has been much research activity concerning the oscillation, nonoscillation and existence of solutions for various second order difference equations, for example, see [1–14] and the references therein.

By using the Z_p geometrical index theory, Guo and Yu [4] obtained some sufficient conditions on the multiplicity results of periodic solutions to the second order difference equation

$$\Delta^2 x_{n-1}+f(x_n)=0,\quad n\in\mathbb{Z}.\tag{1.1}$$

Thandapani *et al.* [12] gave sufficient conditions for the oscillation of bounded solutions for the second order neutral difference equation

$$\Delta^2(x_n-px_{n-k})-q_nf(x_{n-l})=0,\quad n\ge 1.\tag{1.2}$$

By applying the contraction principle, Jinfa [5] discussed the existence of a nonoscillatory solution for the second order neutral delay difference equation with positive and negative coefficients

$$\Delta^2(x_n + px_{n-m}) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad n \geq n_0, \tag{1.3}$$

where $p \in \mathbb{R} \setminus \{-1\}$. Thandapani *et al.* [13] studied the asymptotic behavior of solutions of the second order neutral difference equations of the form

$$\Delta^2(x_n + px_{n-k}) + f(n, x_{n-l}) = 0, \quad n \geq 1, \tag{1.4}$$

and

$$\Delta^2(x_n + px_{n-k}) + f(n, x_{n-l}, \Delta x_{n-l}) = 0, \quad n \geq 1, \tag{1.5}$$

in terms of some difference inequalities. González and Jiménez-Melado [3] used a fixed-point theorem derived from the theory of measures of noncompactness to investigate the existence of solutions for the second order difference equation

$$\Delta(q_n \Delta x_n) + f_n(x_n) = 0, \quad n \geq 0. \tag{1.6}$$

Ma and Guo [9] proved the existence of a nontrivial homoclinic solution for the second order difference equations

$$\Delta(p_n \Delta u_{n-1}) + q_n u_n = f(n, u_n), \quad n \in \mathbb{Z} \tag{1.7}$$

in terms of the Mountain Pass theorem relying on Ekeland's variational principle and the diagonal method. Yu *et al.* [14] established the existence of a periodic solution for equation (1.7) by means of the critical point theory. Utilizing the contraction principle, Liu *et al.* [8] investigated the global existence of solutions for the second order nonlinear neutral delay difference equation

$$\Delta\bigl[a_n \Delta(x_n + bx_{n-\tau})\bigr] + f(n, x_{n-d_{1n}}, x_{n-d_{2n}}, \ldots, x_{n-d_{kn}}) = c_n, \quad n \geq n_0, \tag{1.8}$$

relative to all $b \in \mathbb{R}$.

Inspired and motivated by the work in [1–14], we introduce and study the following more general second order nonlinear neutral difference equation:

$$\begin{aligned}&\Delta\bigl[a_n \Delta(x_n + bx_{n-\tau} - d_n)\bigr] + \Delta f(n, x_{f_1(n)}, x_{f_2(n)}, \ldots, x_{f_k(n)}) \\ &\quad + g(n, x_{g_1(n)}, x_{g_2(n)}, \ldots, x_{g_k(n)}) = c_n, \quad n \geq n_0,\end{aligned} \tag{1.9}$$

where $b \in \mathbb{R}$, $\tau, k \in \mathbb{N}$, $n_0 \in \mathbb{N}_0$, $\{a_n\}_{n \in \mathbb{N}_{n_0}}$ and $\{c_n\}_{n \in \mathbb{N}_{n_0}}$ are real sequences with $a_n \neq 0$ for $n \in \mathbb{N}_{n_0}$, $\{d_n\}_{n \in \mathbb{N}_{n_0}}$ is a bounded sequence, $f, g : \mathbb{N}_{n_0} \times \mathbb{R}^k \to \mathbb{R}$ and $f_l, g_l : \mathbb{N}_{n_0} \to \mathbb{Z}$ with

$$\lim_{n\to\infty} f_l(n) = \lim_{n\to\infty} g_l(n) = +\infty, \quad l \in \{1, 2, \ldots, k\}.$$

Using the Banach fixed point theorem, we obtain sufficient conditions of the existence of uncountably many bounded nonoscillatory solutions for equation (1.9) relative to $b \in \mathbb{R} \setminus \{\pm 1\}$, suggest a few Mann type iterative approximation methods with errors for these bounded nonoscillatory solutions and study error estimates between the approximation sequences and the bounded nonoscillatory solutions. The results obtained in this paper extend and improve the corresponding results in [5, 8]. Four nontrivial examples are given to demonstrate the effectiveness of our results.

2 Preliminaries

Throughout this paper, we assume that Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $\Delta^2 x_n = \Delta(\Delta x_n)$, A and B are positive constants with $B > A$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ stand for the sets of all integers, positive integers and nonnegative integers, respectively,

$$\begin{aligned}
&\mathbb{N}_{n_0} = \{n : n \in \mathbb{N}_0 \text{ with } n \geq n_0\}, \quad n_0 \in \mathbb{N}_0, \\
&\alpha = \inf\bigl\{f_l(n), g_l(n) : 1 \leq l \leq k, n \in \mathbb{N}_{n_0}\bigr\}, \\
&\beta = \min\{n_0 - \tau, \alpha\}, \qquad \mathbb{Z}_\beta = \{n : n \in \mathbb{Z} \text{ with } n \geq \beta\}, \\
&A_n = d_n + A > 0, \qquad B_n = d_n + B, \quad n \in \mathbb{Z}_\beta, \\
&d_n = d_{n_0}, \quad \beta \leq n \leq n_0 - 1,
\end{aligned}$$

and $\overline{d}$ and $\underline{d}$ are two constants with

$$\underline{d} \leq \inf_{n \in \mathbb{Z}_\beta} d_n, \qquad \overline{d} \geq \sup_{n \in \mathbb{Z}_\beta} d_n.$$

Let l^∞_β denote the Banach space of all bounded sequences in $\mathbb{Z}_\beta$ with norm

$$\|x\| = \sup_{n \in \mathbb{Z}_\beta} |x_n| \quad \text{for } x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l^\infty_\beta$$

and

$$\Omega\bigl(\{A_n\}_{n \in \mathbb{Z}_\beta}, \{B_n\}_{n \in \mathbb{Z}_\beta}\bigr) = \bigl\{x = \{x_n\}_{n \in \mathbb{Z}_\beta} \in l^\infty_\beta : A_n \leq x_n \leq B_n, \quad n \in \mathbb{Z}_\beta\bigr\}.$$

It is easy to see that $\Omega(\{A_n\}_{n \in \mathbb{Z}_\beta}, \{B_n\}_{n \in \mathbb{Z}_\beta})$ is a bounded closed and convex subset of l^∞_β.

By a solution of equation (1.9), we mean a sequence $\{x_n\}_{n \in Z_\beta}$ with a positive integer $T \geq n_0 + \tau + |\beta|$ such that equation (1.9) is satisfied for all $n \geq T$. As is customary, a solution of equation (1.9) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory.

Lemma 2.1 ([15]) *Let $\{\alpha_n\}_{n \in \mathbb{N}_0}$, $\{\beta_n\}_{n \in \mathbb{N}_0}$, $\{\gamma_n\}_{n \in \mathbb{N}_0}$ and $\{t_n\}_{n \in \mathbb{N}_0}$ be four nonnegative sequences satisfying the inequality*

$$\alpha_{n+1} \leq (1 - t_n)\alpha_n + t_n \beta_n + \gamma_n, \quad n \in \mathbb{N}_0,$$

where $\{t_n\}_{n \in \mathbb{N}_0} \subset [0,1]$, $\sum_{n=0}^\infty t_n = +\infty$, $\lim_{n \to \infty} \beta_n = 0$ and $\sum_{n=0}^\infty \gamma_n < +\infty$. Then $\lim_{n \to \infty} \alpha_n = 0$.

3 Existence of uncountably many bounded nonoscillatory solutions

Now we study the existence of uncountably bounded nonoscillatory solutions for equation (1.9) with respect to $b \in \mathbb{R} \setminus \{\pm 1\}$, suggest a few Mann iterative approximation schemes with errors for these bounded nonoscillatory solutions and discuss the errors estimates between the iterative approximations and the bounded nonoscillatory solutions.

Theorem 3.1 *Let $b \in [0,1)$, A and B be two positive constants with $B > A + \frac{b}{1-b}(\overline{d} - \underline{d})$. Assume that there exist four real sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying*

$$\begin{aligned} &\bigl|f(n,u_1,u_2,\ldots,u_k) - f(n,\bar{u}_1,\bar{u}_2,\ldots,\bar{u}_k)\bigr| \le P_n \max\bigl\{|u_l - \bar{u}_l| : 1 \le l \le k\bigr\}, \\ &\bigl|g(n,u_1,u_2,\ldots,u_k) - g(n,\bar{u}_1,\bar{u}_2,\ldots,\bar{u}_k)\bigr| \le R_n \max\bigl\{|u_l - \bar{u}_l| : 1 \le l \le k\bigr\}, \\ &\quad n \in \mathbb{N}_{n_0}, u_l, \bar{u}_l \in [\underline{d} + A, \overline{d} + B], 1 \le l \le k; \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\bigl|f(n,u_1,u_2,\ldots,u_k)\bigr| \le Q_n, \qquad \bigl|g(n,u_1,u_2,\ldots,u_k)\bigr| \le W_n, \\ &\quad n \in \mathbb{N}_{n_0}, u_l \in [\underline{d} + A, \overline{d} + B], 1 \le l \le k; \end{aligned} \tag{3.2}$$

$$\max\Biggl\{\sum_{i=n_0}^{\infty} \frac{1}{|a_i|} \max\{P_i, Q_i\}, \sum_{i=n_0}^{\infty}\sum_{j=i}^{\infty} \frac{1}{|a_i|} \max\bigl\{R_j, W_j, |c_j|\bigr\}\Biggr\} < +\infty. \tag{3.3}$$

Then

(a) *for each $L \in (A + b(\overline{d} + B), B + b(\underline{d} + A))$, there exist $\theta \in (0,1)$ and $T \ge n_0 + \tau + |\beta|$ such that for any $z_0 = \{z_{0,n}\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$, the Mann iterative sequence with errors $\{z_m\}_{m\in\mathbb{N}_0}$, where $z_m = \{z_{m,n}\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ for all $m \in \mathbb{N}_0$, generated by the scheme*:

$$z_{m+1,n} = \begin{cases} (1-\alpha_m-\beta_m)z_{m,n} + \alpha_m\{L + d_n - bz_{m,n-\tau} \\ \quad + \sum_{i=n}^{\infty} \frac{1}{a_i}[f(i, z_{m,f_1(i)}, z_{m,f_2(i)}, \ldots, z_{m,f_k(i)}) \\ \quad - \sum_{j=i}^{\infty}(g(j, z_{m,g_1(j)}, z_{m,g_2(j)}, \ldots, z_{m,g_k(j)}) - c_j)]\} \\ \quad + \beta_m\gamma_{m,n}, \quad n \ge T, m \in \mathbb{N}_0, \\ (1-\alpha_m-\beta_m)z_{m,T} + \alpha_m\{L + d_T - bz_{m,T-\tau} \\ \quad + \sum_{i=T}^{\infty} \frac{1}{a_i}[f(i, z_{m,f_1(i)}, z_{m,f_2(i)}, \ldots, z_{m,f_k(i)}) \\ \quad - \sum_{j=i}^{\infty}(g(j, z_{m,g_1(j)}, z_{m,g_2(j)}, \ldots, z_{m,g_k(j)}) - c_j)]\} \\ \quad + \beta_m\gamma_{m,T}, \quad \beta \le n < T, m \in \mathbb{N}_0 \end{cases} \tag{3.4}$$

converges to a bounded nonoscillatory solution $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ of equation (1.9) *and has the following errors estimate*:

$$\|z_{m+1} - x\| \le \bigl(1 - (1-\theta)\alpha_m\bigr)\|z_m - x\| + 2(\overline{d} + B)\beta_m, \quad m \in \mathbb{N}_0, \tag{3.5}$$

where $\{\gamma_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ with $\gamma_m = \{\gamma_{m,n}\}_{n\in\mathbb{Z}_\beta}$ for each $m \in \mathbb{N}_0$, $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0,1]$ *such that*

$$\sum_{m=0}^{\infty} \alpha_m = +\infty \tag{3.6}$$

and

$$\sum_{m=0}^{\infty}\beta_m < +\infty \textit{ or there exists a sequence } \{\xi_m\}_{m\in\mathbb{N}_0}\subset[0,+\infty) \\ \textit{satisfying } \beta_m=\xi_m\alpha_m, m\ge 0 \textit{ and } \lim_{m\to\infty}\xi_m=0; \tag{3.7}$$

(b) *equation* (1.9) *possesses uncountably many bounded nonoscillatory solutions in* $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta},\{B_n\}_{n\in\mathbb{Z}_\beta})$.

Proof Set $L\in(A+b(\overline{d}+B),B+b(\underline{d}+A))$. It follows from (3.3) and $b\in[0,1)$ that there exist $\theta\in(0,1)$ and $T\ge n_0+\tau+|\beta|$ satisfying

$$\theta=b+\sum_{i=T}^{\infty}\frac{1}{|a_i|}\left(P_i+\sum_{j=i}^{\infty}R_j\right) \tag{3.8}$$

and

$$\sum_{i=T}^{\infty}\frac{1}{|a_i|}\left[Q_i+\sum_{j=i}^{\infty}\bigl(W_j+|c_j|\bigr)\right]\le\min\bigl\{L-A-b(\overline{d}+B),B+b(\underline{d}+A)-L\bigr\}. \tag{3.9}$$

In order to prove (i), we now define a mapping $S_L:\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta},\{B_n\}_{n\in\mathbb{Z}_\beta})\to l_\beta^\infty$ by

$$(S_Lx)_n=\begin{cases}L+d_n-bx_{n-\tau}+\sum_{i=n}^{\infty}\frac{1}{a_i}[f(i,x_{f_1(i)},x_{f_2(i)},\dots,x_{f_k(i)}) \\ \quad -\sum_{j=i}^{\infty}(g(j,x_{g_1(j)},x_{g_2(j)},\dots,x_{g_k(j)})-c_j)], \quad n\ge T, \\ (S_Lx)_T, \quad \beta\le n<T,\end{cases} \tag{3.10}$$

for any $x=\{x_n\}_{n\in\mathbb{Z}_\beta}\in\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta},\{B_n\}_{n\in\mathbb{Z}_\beta})$, and show that S_L has a fixed point, which is also a bounded nonoscillatory solution of equation (1.9).

Let $x=\{x_n\}_{n\in\mathbb{Z}_\beta},y=\{y_n\}_{n\in\mathbb{Z}_\beta}\in\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta},\{B_n\}_{n\in\mathbb{Z}_\beta})$. In view of (3.1), (3.8), and (3.10), we get for any $n\ge T$

$$\begin{aligned}
&\bigl|(S_Lx)_n-(S_Ly)_n\bigr| \\
&\quad=\Biggl|bx_{n-\tau}-by_{n-\tau}+\sum_{i=n}^{\infty}\frac{1}{a_i}\Biggl[f(i,x_{f_1(i)},x_{f_2(i)},\dots,x_{f_k(i)})-f(i,y_{f_1(i)},y_{f_2(i)},\dots,y_{f_k(i)}) \\
&\qquad-\sum_{j=i}^{\infty}\bigl(g(j,x_{g_1(j)},x_{g_2(j)},\dots,x_{g_k(j)})-g(j,y_{g_1(j)},y_{g_2(j)},\dots,y_{g_k(j)})\bigr)\Biggr]\Biggr| \\
&\quad\le b|x_{n-\tau}-y_{n-\tau}|+\sum_{i=n}^{\infty}\frac{1}{|a_i|}\Biggl[P_i\max\bigl\{|x_{f_l(i)}-y_{f_l(i)}|:1\le l\le k\bigr\} \\
&\qquad+\sum_{j=i}^{\infty}R_j\max\bigl\{|x_{g_l(j)}-y_{g_l(j)}|:1\le l\le k\bigr\}\Biggr] \\
&\quad\le b\|x-y\|+\sum_{i=n}^{\infty}\frac{1}{|a_i|}\left(P_i+\sum_{j=i}^{\infty}R_j\right)\|x-y\| \\
&\quad\le\theta\|x-y\|,
\end{aligned}$$

which leads that

$$\|S_L x - S_L y\| \le \theta \|x - y\|, \quad x, y \in \Omega\big(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta}\big). \tag{3.11}$$

By (3.2), (3.9), and (3.10), we infer that for each $n \ge T$

$$\begin{aligned}(S_L x)_n &= L + d_n - bx_{n-\tau} + \sum_{i=n}^{\infty} \frac{1}{a_i}\Bigg[f(i, x_{f_1(i)}, x_{f_2(i)}, \ldots, x_{f_k(i)}) \\ &\quad - \sum_{j=i}^{\infty}\big(g(j, x_{g_1(j)}, x_{g_2(j)}, \ldots, x_{g_k(j)}) - c_j\big)\Bigg] \\ &\le L + d_n - b(\underline{d} + A) + \sum_{i=T}^{\infty} \frac{1}{|a_i|}\Bigg[Q_i + \sum_{j=i}^{\infty}\big(W_j + |c_j|\big)\Bigg] \\ &\le L + d_n - b(\underline{d} + A) + \min\big\{L - A - b(\overline{d} + B), B + b(\underline{d} + A) - L\big\} \\ &\le B_n\end{aligned}$$

and

$$\begin{aligned}(S_L x)_n &= L + d_n - bx_{n-\tau} + \sum_{i=n}^{\infty} \frac{1}{a_i}\Bigg[f(i, x_{f_1(i)}, x_{f_2(i)}, \ldots, x_{f_k(i)}) \\ &\quad - \sum_{j=i}^{\infty}\big(g(j, x_{g_1(j)}, x_{g_2(j)}, \ldots, x_{g_k(j)}) - c_j\big)\Bigg] \\ &\ge L + d_n - b(\overline{d} + B) - \sum_{i=T}^{\infty} \frac{1}{|a_i|}\Bigg[Q_i + \sum_{j=i}^{\infty}\big(W_j + |c_j|\big)\Bigg] \\ &\ge L + d_n - b(\overline{d} + B) - \min\big\{L - A - b(\overline{d} + B), B + b(\underline{d} + A) - L\big\} \\ &\ge A_n,\end{aligned}$$

which yield

$$S_L\big(\Omega\big(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta}\big)\big) \subseteq \Omega\big(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta}\big). \tag{3.12}$$

Hence (3.11) and (3.12) mean that S_L is a contraction mapping in $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ and it has a unique fixed point $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$. It follows from (3.10) that

$$\begin{aligned}x_n &= L + d_n - bx_{n-\tau} + \sum_{i=n}^{\infty} \frac{1}{a_i}\Bigg[f(i, x_{f_1(i)}, x_{f_2(i)}, \ldots, x_{f_k(i)}) \\ &\quad - \sum_{j=i}^{\infty}\big(g(j, x_{g_1(j)}, x_{g_2(j)}, \ldots, x_{g_k(j)}) - c_j\big)\Bigg], \quad n \ge T,\end{aligned}$$

which gives

$$\Delta(x_n + bx_{n-\tau} - d_n) = -\frac{1}{a_n}\Bigg[f(n, x_{f_1(n)}, x_{f_2(n)}, \ldots, x_{f_k(n)}) \\ - \sum_{j=n}^{\infty}\big(g(j, x_{g_1(j)}, x_{g_2(j)}, \ldots, x_{g_k(j)}) - c_j\big)\Bigg], \quad n \geq T,$$

and

$$\Delta\big(a_n\Delta(x_n + bx_{n-\tau} - d_n)\big) + \Delta f(n, x_{f_1(n)}, x_{f_2(n)}, \ldots, x_{f_k(n)}) \\ + g(n, x_{g_1(n)}, x_{g_2(n)}, \ldots, x_{g_k(n)}) = c_n, \quad n \geq T.$$

That is, the fixed point $x = \{x_n\}_{n\in\mathbb{Z}_\beta}$ of S_L in $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ is a bounded nonoscillatory solution of equation (1.9).

In light of (3.4), (3.8), (3.10), and (3.11), we deduce that for all $n \geq T$ and $m \in \mathbb{N}_0$

$$\begin{aligned} &|z_{m+1,n} - x_n| \\ &\quad = \Bigg|(1 - \alpha_m - \beta_m)z_{m,n} + \alpha_m\Bigg\{L + d_n - bz_{m,n-\tau} \\ &\qquad + \sum_{i=n}^{\infty}\frac{1}{a_i}\Bigg[f(i, z_{m,f_1(i)}, z_{m,f_2(i)}, \ldots, z_{m,f_k(i)}) \\ &\qquad - \sum_{j=i}^{\infty}\big(g(j, z_{m,g_1(j)}, z_{m,g_2(j)}, \ldots, z_{m,g_k(j)}) - c_j\big)\Bigg]\Bigg\} + \beta_m\gamma_{m,n} - x_n\Bigg| \\ &\quad \leq (1 - \alpha_m - \beta_m)|z_{m,n} - x_n| + \alpha_m\big|(S_Lz_m)_n - (S_Lx)_n\big| + \beta_m|\gamma_{m,n} - x_n| \\ &\quad \leq (1 - \alpha_m - \beta_m)\|z_m - x\| + \alpha_m\theta\|z_m - x\| + 2(\overline{d} + B)\beta_m \\ &\quad \leq \big(1 - (1 - \theta)\alpha_m\big)\|z_m - x\| + 2(\overline{d} + B)\beta_m, \end{aligned}$$

which implies that (3.5) holds. It follows from (3.6), (3.7), and Lemma 2.1 that $\lim_{m\to\infty} z_m = x$.

Next we prove (ii). It follows from (i) that for any distinct $L_1, L_2 \in (A + b(\overline{d} + B), B + b(\underline{d} + A))$, there exist $\theta_1, \theta_2 \in (0, 1)$ and $T_1, T_2 \geq n_0 + \tau + |\beta|$ satisfying (3.8)-(3.10), where θ, T, L and S_L are replaced by θ_j, T_j, L_j and S_{T_j}, $j \in \{1, 2\}$, respectively. In view of (3.1) there exists $T_3 > \max\{T_1, T_2\}$ satisfying

$$\sum_{i=T_3}^{\infty}\frac{1}{|a_i|}\Bigg(P_i + \sum_{j=i}^{\infty}R_j\Bigg) < \frac{|L_1 - L_2|}{4(\overline{d} + B)}. \tag{3.13}$$

Obviously, the contraction mappings S_{L_1} and S_{L_2} have the unique fixed points $x = \{x_n\}_{n\in\mathbb{Z}_\beta}, y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$, respectively. That is, x and y are bounded nonoscillatory solutions of equation (1.9) in $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$. In the following, we show only that $x \neq y$. In view of (3.10), we arrive at

$$x_n = L_1 + d_n - bx_{n-\tau} + \sum_{i=n}^{\infty} \frac{1}{a_i} \Bigg[f(i, x_{f_1(i)}, x_{f_2(i)}, \dots, x_{f_k(i)}) - \sum_{j=i}^{\infty} \big(g(j, x_{g_1(j)}, x_{g_2(j)}, \dots, x_{g_k(j)}) - c_j \big) \Bigg], \quad n \geq T_3$$

and

$$y_n = L_2 + d_n - by_{n-\tau} + \sum_{i=n}^{\infty} \frac{1}{a_i} \Bigg[f(i, y_{f_1(i)}, y_{f_2(i)}, \dots, y_{f_k(i)}) - \sum_{j=i}^{\infty} \big(g(j, y_{g_1(j)}, y_{g_2(j)}, \dots, y_{g_k(j)}) - c_j \big) \Bigg], \quad n \geq T_3,$$

which together with (3.13) yield

$$\begin{aligned} &\big| (x_n - y_n) + b(x_{n-\tau} - y_{n-\tau}) \big| \\ &= \Bigg| L_1 - L_2 + \sum_{i=n}^{\infty} \frac{1}{a_i} \Bigg\{ f(i, x_{f_1(i)}, x_{f_2(i)}, \dots, x_{f_k(i)}) - f(i, y_{f_1(i)}, y_{f_2(i)}, \dots, y_{f_k(i)}) \\ &\quad - \sum_{j=i}^{\infty} \big[g(j, x_{g_1(j)}, x_{g_2(j)}, \dots, x_{g_k(j)}) - g(j, y_{g_1(j)}, y_{g_2(j)}, \dots, y_{g_k(j)}) \big] \Bigg\} \Bigg| \\ &\geq |L_1 - L_2| - \sum_{i=n}^{\infty} \frac{1}{|a_i|} \Bigg[P_i \max \big\{ |x_{f_l(i)} - y_{f_l(i)}| : 1 \leq l \leq k \big\} \\ &\quad + \sum_{j=i}^{\infty} R_j \max \big\{ |x_{g_l(j)} - y_{g_l(j)}| : 1 \leq l \leq k \big\} \Bigg] \\ &\geq |L_1 - L_2| - \sum_{i=n}^{\infty} \frac{1}{|a_i|} \Bigg(P_i + \sum_{j=i}^{\infty} R_j \Bigg) \|x - y\| \\ &\geq |L_1 - L_2| - 2(\overline{d} + B) \sum_{i=T_3}^{\infty} \frac{1}{|a_i|} \Bigg(P_i + \sum_{j=i}^{+\infty} R_j \Bigg) \\ &\geq \frac{|L_1 - L_2|}{2} \\ &> 0, \quad n \geq T_3, \end{aligned}$$

that is, $x \neq y$. This completes the proof. □

Theorem 3.2 *Let* $b \in (-1, 0]$, *A and B be two positive constants with* $B > A + \frac{b}{1+b}(\underline{d} - \overline{d})$. *Assume that there exist four real sequences* $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ *and* $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ *satisfying* (3.1)-(3.3). *Then*

(a) *for any* $L \in (A + b(\underline{d} + A), B + b(\overline{d} + B))$, *there exist* $\theta \in (0,1)$ *and* $T \geq n_0 + \tau + |\beta|$ *such that for each* $z_0 = \{z_{0,n}\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$, *the Mann iterative sequence with errors* $\{z_m\}_{m\in\mathbb{N}_0}$ *generated by* (3.4) *with* $z_m = \{z_{m,n}\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ *for all* $m \in$

$\mathbb{N}_0$ *converges to a bounded nonoscillatory solution* $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ *of equation* (1.9) *and satisfies* (3.5), *where* $\{\gamma_m\}_{m\in\mathbb{N}_0}$ *is an arbitrary sequence in* $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ *with* $\gamma_m = \{\gamma_{m,n}\}_{n\in\mathbb{Z}_\beta}$ *for each* $m \in \mathbb{N}_0$, $\{\alpha_m\}_{m\in\mathbb{N}_0}$ *and* $\{\beta_m\}_{m\in\mathbb{N}_0}$ *are any sequences in* $[0,1]$ *satisfying* (3.6) *and* (3.7);

(b) *equation* (1.9) *possesses uncountably many bounded nonoscillatory solutions in* $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$.

Proof Let $L \in (A + b(\underline{d} + A), B + b(\overline{d} + B))$. On account of (3.3) and $b \in (-1, 0]$, there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + |\beta|$ satisfying

$$\theta = -b + \sum_{i=T}^{\infty} \frac{1}{|a_i|} \left(P_i + \sum_{j=i}^{\infty} R_j \right) \tag{3.14}$$

and

$$\sum_{i=T}^{\infty} \frac{1}{|a_i|} \left[Q_i + \sum_{j=i}^{\infty} \left(W_j + |c_j| \right) \right] \leq \min\left\{ L - A - b(\underline{d} + A), B + b(\overline{d} + B) - L \right\}. \tag{3.15}$$

Let the mapping $S_L : \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta}) \to l_\beta^\infty$ be defined by (3.10). The rest of the proof is similar to that of Theorem 3.1 and is omitted. This completes the proof. □

Theorem 3.3 *Let* $b \in (1, +\infty)$, *A and B be two positive constants with* $B > A + \frac{b+2}{b-1}(\overline{d} - \underline{d})$. *Assume that there exist four real sequences* $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ *and* $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ *satisfying* (3.1)-(3.3). *Then*

(a) *for any* $L \in (B + b(\overline{d} + A) + \overline{d} - \underline{d}, A + b(\underline{d} + B) + \underline{d} - \overline{d})$, *there exist* $\theta \in (0,1)$ *and* $T \geq n_0 + \tau + |\beta|$ *such that for each* $z_0 = \{z_{0,n}\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$, *the Mann iterative sequence with errors* $\{z_m\}_{m\in\mathbb{N}_0}$, *where* $z_m = \{z_{m,n}\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ *for all* $m \in \mathbb{N}_0$, *generated by the schemes*:

$$z_{m+1,n} = \begin{cases} (1 - \alpha_m - \beta_m) z_{m,n} + \alpha_m \{ \frac{L}{b} + \frac{d_{n+\tau}}{b} - \frac{z_{m,n+\tau}}{b} \\ \quad + \frac{1}{b} \sum_{i=n+\tau}^{\infty} \frac{1}{a_i} [f(i, z_{m,f_1(i)}, z_{m,f_2(i)}, \ldots, z_{m,f_k(i)}) \\ \quad - \sum_{j=i}^{\infty} (g(j, z_{m,g_1(j)}, z_{m,g_2(j)}, \ldots, z_{m,g_k(j)}) - c_j)]\} \\ \quad + \beta_m \gamma_{m,n}, \quad n \geq T, m \in \mathbb{N}_0, \\ (1 - \alpha_m - \beta_m) z_{m,T} + \alpha_m \{ \frac{L}{b} + \frac{d_{T+\tau}}{b} - \frac{z_{m,T+\tau}}{b} \\ \quad + \frac{1}{b} \sum_{i=T+\tau}^{+\infty} \frac{1}{a_i} [f(i, z_{m,f_1(i)}, z_{m,f_2(i)}, \ldots, z_{m,f_k(i)}) \\ \quad - \sum_{j=i}^{\infty} (g(j, z_{m,g_1(j)}, z_{m,g_2(j)}, \ldots, z_{m,g_k(j)}) - c_j)]\} \\ \quad + \beta_m \gamma_{m,T}, \quad \beta \leq n < T, m \in \mathbb{N}_0, \end{cases} \tag{3.16}$$

converges to a bounded nonoscillatory solution $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ *of equation* (1.9) *and satisfies* (3.5), *where* $\{\gamma_m\}_{m\in\mathbb{N}_0}$ *is an arbitrary sequence in* $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ *with* $\gamma_m = \{\gamma_{m,n}\}_{n\in\mathbb{Z}_\beta}$ *for each* $m \in \mathbb{N}_0$, $\{\alpha_m\}_{m\in\mathbb{N}_0}$ *and* $\{\beta_m\}_{m\in\mathbb{N}_0}$ *are any sequences in* $[0,1]$ *satisfying* (3.6) *and* (3.7);

(b) *equation* (1.9) *possesses uncountably many bounded nonoscillatory solutions in* $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$.

Proof Set $L \in (B + b(\overline{d} + A) + \overline{d} - \underline{d}, A + b(\underline{d} + B) + \underline{d} - \overline{d})$. In view of (3.3) and $b \in (1, +\infty)$, there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + |\beta|$ such that

$$\theta = \frac{1}{b} + \frac{1}{b}\sum_{i=T}^{\infty}\frac{1}{|a_i|}\left(P_i + \sum_{j=i}^{\infty}R_j\right) \tag{3.17}$$

and

$$\begin{aligned}&\sum_{i=T}^{\infty}\frac{1}{|a_i|}\left[Q_i + \sum_{j=i}^{\infty}\bigl(W_j + |c_j|\bigr)\right]\\&\quad\leq \min\bigl\{b(\underline{d} + B) + \underline{d} + A - \overline{d} - L, L - b(\overline{d} + A) - \overline{d} - B + \underline{d}\bigr\}.\end{aligned} \tag{3.18}$$

Define a mapping $S_L : \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta}) \to l_\beta^\infty$ by

$$(S_Lx)_n = \begin{cases}\frac{L}{b} + \frac{d_{n+\tau}}{b} - \frac{x_{n+\tau}}{b} + \frac{1}{b}\sum_{i=n+\tau}^{\infty}\frac{1}{a_i}[f(i, x_{f_1(i)}, x_{f_2(i)}, \ldots, x_{f_k(i)})\\ \quad - \sum_{j=i}^{\infty}(g(j, x_{g_1(j)}, x_{g_2(j)}, \ldots, x_{g_k(j)}) - c_j)], \quad n \geq T,\\ (S_Lx)_T, \quad \beta \leq n < T,\end{cases} \tag{3.19}$$

for any $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$.

Let $x = \{x_n\}_{n\in\mathbb{Z}_\beta}, y = \{y_n\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$. Using (3.1), (3.17), and (3.19), we deduce that for any $n \geq T$

$$\begin{aligned}\bigl|(S_Lx)_n - (S_Ly)_n\bigr| &\leq \frac{1}{b}|x_{n+\tau} - y_{n+\tau}|\\&\quad + \frac{1}{b}\sum_{i=n+\tau}^{\infty}\frac{1}{|a_i|}\Biggl[\bigl|f(i, x_{f_1(i)}, x_{f_2(i)}, \ldots, x_{f_k(i)}) - f(i, y_{f_1(i)}, y_{f_2(i)}, \ldots, y_{f_k(i)})\bigr|\\&\quad + \sum_{j=i}^{\infty}\bigl|g(j, x_{g_1(j)}, x_{g_2(j)}, \ldots, x_{g_k(j)}) - g(j, y_{g_1(j)}, y_{g_2(j)}, \ldots, y_{g_k(j)})\bigr|\Biggr]\\&\leq \frac{1}{b}\|x - y\| + \frac{1}{b}\sum_{i=n+\tau}^{\infty}\frac{1}{|a_i|}\left(P_i + \sum_{j=i}^{\infty}R_j\right)\|x - y\|\\&\leq \theta\|x - y\|,\end{aligned}$$

which means (3.11).

By (3.2), (3.18), and (3.19), we infer that for each $n \geq T$

$$\begin{aligned}(S_Lx)_n &= \frac{L}{b} + \frac{d_{n+\tau}}{b} - \frac{x_{n+\tau}}{b} + \frac{1}{b}\sum_{i=n+\tau}^{\infty}\frac{1}{a_i}\Biggl[f(i, x_{f_1(i)}, x_{f_2(i)}, \ldots, x_{f_k(i)})\\&\quad - \sum_{j=i}^{\infty}\bigl(g(j, x_{g_1(j)}, x_{g_2(j)}, \ldots, x_{g_k(j)}) - c_j\bigr)\Biggr]\\&\leq \frac{L}{b} + \frac{\overline{d}}{b} - \frac{\underline{d} + A}{b} + \frac{1}{b}\sum_{i=n+\tau}^{\infty}\frac{1}{|a_i|}\left[Q_i + \sum_{j=i}^{\infty}\bigl(W_j + |c_j|\bigr)\right]\end{aligned}$$

$$
\begin{aligned}
&\le \frac{L}{b} + \frac{\overline{d}}{b} - \frac{\underline{d} + A}{b} \\
&\quad + \frac{1}{b} \min\bigl\{b(\underline{d} + B) + \underline{d} + A - \overline{d} - L, L - b(\overline{d} + A) - \overline{d} - B + \underline{d}\bigr\} \\
&\le B_n
\end{aligned}
$$

and

$$
\begin{aligned}
(S_L x)_n &= \frac{L}{b} + \frac{d_{n+\tau}}{b} - \frac{x_{n+\tau}}{b} + \frac{1}{b}\sum_{i=n+\tau}^{\infty} \frac{1}{a_i}\Biggl[f(i, x_{f_1(i)}, x_{f_2(i)}, \ldots, x_{f_k(i)}) \\
&\quad - \sum_{j=i}^{\infty}\bigl(g(j, x_{g_1(j)}, x_{g_2(j)}, \ldots, x_{g_k(j)}) - c_j\bigr)\Biggr] \\
&\ge \frac{L}{b} + \frac{\underline{d}}{b} - \frac{\overline{d} + B}{b} - \frac{1}{b}\sum_{i=T}^{\infty}\frac{1}{|a_i|}\Biggl[Q_i + \sum_{j=i}^{\infty}\bigl(W_j + |c_j|\bigr)\Biggr] \\
&\ge \frac{L}{b} + \frac{\underline{d}}{b} - \frac{\overline{d} + B}{b} \\
&\quad - \frac{1}{b} \min\bigl\{b(\underline{d} + B) + \underline{d} + A - \overline{d} - L, L - b(\overline{d} + A) - \overline{d} - B + \underline{d}\bigr\} \\
&\ge A_n,
\end{aligned}
$$

which imply (3.12). Consequently S_L is a contraction mapping in $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ and it has a unique fixed point $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$. It follows from (3.19) that

$$
\begin{aligned}
x_n &= \frac{L}{b} + \frac{d_{n+\tau}}{b} - \frac{x_{n+\tau}}{b} + \frac{1}{b}\sum_{i=n+\tau}^{\infty}\frac{1}{a_i}\Biggl[f(i, x_{f_1(i)}, x_{f_2(i)}, \ldots, x_{f_k(i)}) \\
&\quad - \sum_{j=i}^{\infty}\bigl(g(j, x_{g_1(j)}, x_{g_2(j)}, \ldots, x_{g_k(j)}) - c_j\bigr)\Biggr], \quad n \ge T,
\end{aligned}
$$

which gives

$$
\begin{aligned}
\Delta(x_n + bx_{n-\tau} - d_n) &= -\frac{1}{a_n}\Biggl[f(n, x_{f_1(n)}, x_{f_2(n)}, \ldots, x_{f_k(n)}) \\
&\quad - \sum_{j=n}^{\infty}\bigl(g(j, x_{g_1(j)}, x_{g_2(j)}, \ldots, x_{g_k(j)}) - c_j\bigr)\Biggr], \quad n \ge T + \tau
\end{aligned}
$$

and

$$
\begin{aligned}
&\Delta\bigl(a_n\Delta(x_n + bx_{n-\tau} - d_n)\bigr) + \Delta f(n, x_{f_1(n)}, x_{f_2(n)}, \ldots, x_{f_k(n)}) \\
&\quad + g(n, x_{g_1(n)}, x_{g_2(n)}, \ldots, x_{g_k(n)}) = c_n, \quad n \ge T + \tau.
\end{aligned}
$$

That is, $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ is a bounded nonoscillatory solution of equation (1.9).

It follows from (3.11), (3.16), (3.17), and (3.19) that for any $n \geq T$ and $m \in \mathbb{N}_0$

$$\begin{aligned}|z_{m+1} - x_n| &= \left|(1-\alpha_m-\beta_m)z_{m,n} + \alpha_m\left\{\frac{L}{b} + \frac{d_{n+\tau}}{b} - \frac{z_{m,n+\tau}}{b}\right.\right.\\ &\quad + \frac{1}{b}\sum_{i=n+\tau}^{\infty}\frac{1}{a_i}\left[f(i, z_{m,f_1(i)}, z_{m,f_2(i)}, \ldots, z_{m,f_k(i)})\right.\\ &\quad \left.\left.\left. - \sum_{j=i}^{\infty}\big(g(j, z_{m,g_1(j)}, z_{m,g_2(j)}, \ldots, z_{m,g_k(j)}) - c_j\big)\right]\right\} + \beta_m\gamma_{m,n} - x_n\right|\\ &\leq (1-\alpha_m-\beta_m)|z_{m,n} - x_n| + \alpha_m\big|(S_L z_m)_n - (S_L x)_n\big| + \beta_m|\gamma_{m,n} - x_n|\\ &\leq (1-\alpha_m-\beta_m)\|z_m - x\| + \alpha_m\theta\|z_m - x\| + 2(\overline{d} + B)\beta_m\\ &\leq \big(1-(1-\theta)\alpha_m\big)\|z_m - x\| + 2(\overline{d} + B)\beta_m,\end{aligned}$$

which yields (3.5). Thus Lemma 2.1, (3.6), and (3.7) ensure that $\lim_{m\to\infty} z_m = x$. The rest of the proof is similar to that of Theorem 3.1 and is omitted. This completes the proof. □

Theorem 3.4 *Let $b \in (-\infty, -1)$, A and B be two positive constants with $B > A + \frac{b-2}{b+1}(\overline{d} - \underline{d})$. Assume that there exist four real sequences $\{P_n\}_{n\in\mathbb{N}_{n_0}}$, $\{Q_n\}_{n\in\mathbb{N}_{n_0}}$, $\{R_n\}_{n\in\mathbb{N}_{n_0}}$ and $\{W_n\}_{n\in\mathbb{N}_{n_0}}$ satisfying* (3.1)-(3.3). *Then*

(a) *for each $L \in (B + b(\underline{d} + B) + \overline{d} - \underline{d}, A + b(\overline{d} + A) + \underline{d} - \overline{d})$, there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + |\beta|$ such that for each $z_0 = \{z_{0,n}\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$, the Mann iterative sequence with errors $\{z_m\}_{m\in\mathbb{N}_0}$ generated by the schemes* (3.16) *with $z_m = \{z_{m,n}\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ for all $m \in \mathbb{N}_0$ converges to a bounded nonoscillatory solution $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ of equation* (1.9) *and satisfies* (3.5), *where $\{\gamma_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ with $\gamma_m = \{\gamma_{m,n}\}_{n\in\mathbb{Z}_\beta}$ for each $m \in \mathbb{N}_0$, $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0,1]$ satisfying* (3.6) *and* (3.7);

(b) *equation* (1.9) *possesses uncountably many bounded nonoscillatory solutions in $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$.*

Proof Let $L \in (B + b(\underline{d} + B) + \overline{d} - \underline{d}, A + b(\overline{d} + A) + \underline{d} - \overline{d})$. It follows from (3.3) and $b \in (-\infty, -1)$ that there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + |\beta|$ satisfying

$$\theta = -\frac{1}{b} - \frac{1}{b}\sum_{i=T}^{\infty}\frac{1}{|a_i|}\left(P_i + \sum_{j=i}^{\infty}R_j\right) \tag{3.20}$$

and

$$\begin{aligned}&\sum_{i=T}^{\infty}\frac{1}{|a_i|}\left[Q_i + \sum_{j=i}^{\infty}\big(W_j + |c_j|\big)\right]\\ &\quad \leq \min\big\{A + b(\overline{d} + A) + \underline{d} - \overline{d} - L, L - B - b(\underline{d} + B) - \overline{d} + \underline{d}\big\}.\end{aligned} \tag{3.21}$$

Let the mapping S_L be defined by (3.19). The rest of the proof is similar to that of Theorem 3.3 and is omitted. This completes the proof. □

Remark 3.1 Theorems 3.1-3.4 extend Theorem 1 in [5] under $p \neq \pm 1$. Theorems 3.1-3.4 improve Theorems 2.4-2.7 in [8], respectively. The examples in the fourth section reveal that Theorems 3.1-3.4 extend authentically the corresponding results in [5, 8].

4 Applications

In this section, we assume that $\{\gamma_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta},\{B_n\}_{n\in\mathbb{Z}_\beta})$ with $\gamma_m = \{\gamma_{m,n}\}_{n\in\mathbb{Z}_\beta}$ for each $m \in \mathbb{N}_0$, $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0,1]$ satisfying (3.6) and (3.7).

Now we display four examples as applications of the results presented in Section 3.

Example 4.1 Consider the second order nonlinear neutral delay difference equation

$$\Delta\left[(-1)^{\frac{n(n+1)}{2}} n^3 \Delta\left(x_n + \frac{1}{2}x_{n-\tau} + 6(-1)^n\right)\right] + \Delta\left(n x^2_{n^2+3n} x^3_{n+(-1)^n}\right) + \frac{x^2_{3n^4+2n^2+8n-2}}{n^2(1+x^2_{n^3+3n^2+10n-6})} = \frac{n^2(2n-1)+\sqrt{3n^2-2n+1}}{n^6+n\ln(1+n^2)}, \quad n \geq 1, \tag{4.1}$$

where $n_0 = 1$ and $\tau \in \mathbb{N}$ is fixed. Let $k = 2$, $b = \frac{1}{2}$, $A = 43$, $B = 56$, $\alpha = 0$, $\beta = 1 - \tau$, $\underline{d} = -6$, $\overline{d} = 6$ and

$$\begin{aligned}
&a_n = (-1)^{\frac{n(n+1)}{2}} n^3, \qquad c_n = \frac{n^2(2n-1)+\sqrt{3n^2-2n+1}}{n^6+n\ln(1+n^2)}, \qquad d_n = 6(-1)^n, \\
&f(n,u,v) = nu^2v^3, \qquad g(n,u,v) = \frac{u^2}{n^2(1+v^2)}, \\
&f_1(n) = n^2+3n, \qquad f_2(n) = n+(-1)^n, \\
&g_1(n) = 3n^4+2n^2+8n-2, \qquad g_2(n) = n^3+3n^2+10n-6, \\
&P_n = 10^{12}n, \qquad Q_n = 10^9 n, \\
&R_n = \frac{10^7}{n^2}, \qquad W_n = \frac{3844}{n^2}, \quad (n,u,v) \in \mathbb{N}_{n_0} \times [\underline{d}+A, \overline{d}+B]^2.
\end{aligned}$$

It is easy to show that the conditions (3.1)-(3.3) are satisfied. It follows from Theorem 3.1 that equation (4.1) possesses uncountably bounded nonoscillatory solutions in $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta},\{B_n\}_{n\in\mathbb{Z}_\beta})$, and for any $L \in (A + b(\overline{d}+B), B + b(\underline{d}+A))$, there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + |\beta|$ such that the Mann iterative sequence with error $\{z_m\}_{m\geq 0}$ generated by (3.4) converges to a bounded nonoscillatory solution $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta},\{B_n\}_{n\in\mathbb{Z}_\beta})$ of equation (4.1) and (3.5) holds. Obviously, Theorem 1 in [5] and Theorem 2.5 in [8] are invalid for equation (4.1).

Example 4.2 Consider the second order nonlinear neutral delay difference equation

$$\Delta\left[(n^2+6)\Delta\left(x_n - \frac{1}{5}x_{n-\tau} + 9\sin(n^3+1)\right)\right] + \Delta\left(\frac{\sqrt{n-2}\,x^3_{2n-3}}{n^2+x^2_{2n^2-9}}\right) + \frac{n^3x^4_{7n-12} - (-1)^n x^9_{n^2-6}}{n^7+2n^5+3n^4+2n+1} = \frac{\sqrt{n^2-n+1}}{n^3+3n^2+4n+1}, \quad n \geq 2, \tag{4.2}$$

where $n_0 = 2$ and $\tau \in \mathbb{N}$ is fixed. Let $k = 2$, $b = -\frac{1}{5}$, $A = 11$, $B = 23$, $\alpha = -5$, $\beta = \min\{2-\tau, -5\}$, $\underline{d} = -9$, $\overline{d} = 9$, and

$$\begin{aligned}
&a_n = n^2 + 6, \qquad c_n = \frac{\sqrt{n^2 - n + 1}}{n^3 + 3n^2 + 4n + 1}, \qquad d_n = 9\sin\left(n^3 + 1\right),\\
&f(n,u,v) = \frac{\sqrt{n-2}u^3}{n^2 + v^2}, \qquad g(n,u,v) = \frac{n^3u^4 - (-1)^n v^9}{n^7 + 2n^5 + 3n^4 + 2n + 1},\\
&f_1(n) = 2n - 3, \qquad f_2(n) = 2n^2 - 9, \qquad g_1(n) = 7n - 12, \qquad g_2(n) = n^2 - 6,\\
&P_n = \frac{(5242880 + 3072n^2)\sqrt{n-2}}{n^4}, \qquad Q_n = \frac{32768\sqrt{n-2}}{n^2},\\
&R_n = \frac{131072n^3 + 65536 \times 10^9}{n^7}, \qquad W_n = \frac{1048576n^3 + 262144 \times 10^9}{n^3 + 1},\\
&\quad (n,u,v) \in \mathbb{N}_{n_0} \times [\underline{d} + A, \overline{d} + B]^2.
\end{aligned}$$

It is clear that the conditions (3.1)-(3.3) are fulfilled. It follows from Theorem 3.2 that equation (4.2) possesses uncountably bounded nonoscillatory solutions in $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$, and for any $L \in (A + b(\underline{d} + A), B + b(\overline{d} + B))$, there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + |\beta|$ such that the Mann iterative sequence with error $\{z_m\}_{m\geq 0}$ generated by (3.4) converges to a bounded nonoscillatory solution $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ of equation (4.2) and (3.5) holds. However, Theorem 1 in [5] and Theorem 2.6 in [8] are not applicable for equation (4.2).

Example 4.3 Consider the second order nonlinear neutral delay difference equation

$$\begin{aligned}
&\Delta\left[n^2\left(1-n^3\right)\Delta\left(x_n + 2x_{n-\tau} + \frac{n(2n-1)}{n^2+2}\right)\right] + \Delta\left(n^3 x_{n^2+1}\sin\left(x_{n^2+1}x_{2n-5}\right)\right)\\
&\quad + \frac{x^5_{n^3-2n+1}}{n^2 + x^2_{n^2+2n-1}} = \frac{(-1)^{n-1}2n-5}{3n^4 + 2n^2 + n + 1}, \quad n \geq 2,
\end{aligned} \tag{4.3}$$

where $n_0 = 2$ and $\tau \in \mathbb{N}$ is fixed. Let $k = 2$, $b = 2$, $A = 90$, $B = 100$, $\alpha = -1$, $\beta = \min\{2-\tau, -1\}$, $\underline{d} = \frac{1}{3}$, $\overline{d} = 2$ and

$$\begin{aligned}
&a_n = n^2\left(1-n^3\right), \qquad c_n = \frac{(-1)^{n-1}2n-5}{3n^4 + 2n^2 + n + 1}, \qquad d_n = \frac{n(2n-1)}{n^2+2},\\
&f(n,u,v) = n^3 u \sin(uv), \qquad g(n,u,v) = \frac{u^5}{n^2 + v^2}, \qquad f_1(n) = n^2 + 1, \qquad f_2(n) = 2n - 5,\\
&g_1(n) = n^3 - 2n + 1, \qquad g_2(n) = n^2 + 2n - 1, \qquad P_n = 10^5 n^3, \qquad Q_n = 102n^3,\\
&R_n = \frac{10^9 n^2 + 10^{13}}{n^4}, \qquad W_n = \frac{10^{11}}{n^2}, \quad (n,u,v) \in \mathbb{N}_{n_0} \times [\underline{d} + A, \overline{d} + B]^2.
\end{aligned}$$

Clearly, the conditions (3.1)-(3.3) hold. It follows from Theorem 3.3 that equation (4.3) possesses uncountably bounded nonoscillatory solutions in $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$, and for any $L \in (B + b(\overline{d} + A) + \overline{d} - \underline{d}, A + b(\underline{d} + B) + \underline{d} - \overline{d})$, there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + |\beta|$ such that the Mann iterative sequence with error $\{z_m\}_{m\geq 0}$ generated by (3.14) converges to a bounded nonoscillatory solution $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ of equation (4.3) and (3.5) holds. But Theorem 1 in [5] and Theorem 2.7 in [8] are not valid for equation (4.3).

Example 4.4 Consider the second order nonlinear neutral delay difference equation

$$\begin{aligned} &\Delta\left[n^2\Delta\left(x_n - 3x_{n-\tau} + 2(-1)^{\frac{n(n+1)(n+2)}{3}}\right)\right] + \Delta\left((-1)^{n^2+3n-1}\sqrt{n-3}\,x_{n^3+n-9}x_{3n-12}\right) \\ &\quad + \frac{x_{n^2+n-6}}{n^2 + x_{2n-5}^2} = \frac{(-1)^{n-1}(2n^2-5) + (-1)^{n+1}(3n+4)\ln(1+n^3)}{n^4+3n^3+2n^2+5n+7}, \quad n \geq 3, \end{aligned} \tag{4.4}$$

where $n_0 = 3$ and $\tau \in \mathbb{N}$ is fixed. Let $k = 2$, $b = -3$, $A = 1$, $B = 5$, $\alpha = -3$, $\beta = \{3-\tau, -3\}$, $\underline{d} = -2$, $\overline{d} = 2$ and

$$\begin{aligned} &a_n = n^2, \qquad c_n = \frac{(-1)^{n-1}(2n^2-5) + (-1)^{n+1}(3n+4)\ln(1+n^3)}{n^4+3n^3+2n^2+5n+7}, \\ &d_n = 2(-1)^{\frac{n(n+1)(n+2)}{3}}, \qquad f(n,u,v) = (-1)^{n^2+3n-1}\sqrt{n-3}\,uv, \\ &g(n,u,v) = \frac{u}{n^2+v^2}, \qquad f_1(n) = n^3+n-9, \qquad f_2(n) = 3n-12, \qquad g_1(n) = 2n-5, \\ &g_2(n) = n^2+n-6, \qquad P_n = 14\sqrt{n-3}, \qquad Q_n = 49\sqrt{n-3}, \\ &R_n = \frac{14+n^2}{n^4}, \qquad W_n = \frac{7}{n^2}, \quad (n,u,v) \in \mathbb{N}_{n_0} \times [\underline{d}+A, \overline{d}+B]^2. \end{aligned}$$

It is not difficult to verify that the conditions (3.1)-(3.3) are fulfilled. It follows from Theorem 3.4 that equation (4.4) possesses uncountably bounded nonoscillatory solutions in $\Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$, and for any $L \in (B + b(\underline{d}+B) + \overline{d} - \underline{d}, A + b(\overline{d}+A) + \underline{d} - \overline{d})$, there exist $\theta \in (0,1)$ and $T \geq n_0 + \tau + |\beta|$ such that the Mann iterative sequence with error $\{z_m\}_{m\geq 0}$ generated by (3.14) converges to a bounded nonoscillatory solution $x = \{x_n\}_{n\in\mathbb{Z}_\beta} \in \Omega(\{A_n\}_{n\in\mathbb{Z}_\beta}, \{B_n\}_{n\in\mathbb{Z}_\beta})$ of equation (4.4) and (3.5) holds. However, Theorem 1 in [5] and Theorem 2.4 in [8] are unapplicable for equation (4.4).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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Received: 18 August 2015 Accepted: 18 November 2015 Published online: 01 December 2015

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