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Existence of periodic solutions for impulsive evolution equations in ordered Banach spaces

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Abstract

In this paper, we use the perturbation method and the mixed monotone iterative technique to discuss the existence of periodic solutions for impulsive evolution equations in ordered Banach spaces. Under impulsive functions satisfying broader monotone conditions and without assumption that the lower and upper solutions exist, we obtain the existence results of ω-periodic mild solutions. Moreover, an application is given to illustrate our theoretical results.

MSC: 35B10; 47H05

Keywords: semilinear impulsive evolution equation; upper and lower solutions; monotone iterative technique; periodic boundary value problems; C_0-semigroup

1 Introduction

In this paper, by using the perturbation method and the monotone iterative technique, we discuss the periodic solutions for the impulsive evolution equation

$$\begin{cases} u'(t) + Au(t) = f(t,u(t)), & t \geq 0, t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), & k \in \mathbf{N}, \end{cases} \tag{1.1}$$

in an ordered Banach space X, where $A : D(A) \subset X \to X$ is a closed linear operator and $-A$ generates a positive C_0-semigroup $T(t)$ $(t \geq 0)$ in X; $f : [0, +\infty) \times X \to X$ is a continuous function and f is ω-periodic about t. $J = [0, \omega]$, ω is a constant; $0 < t_1 < t_2 < \cdots < t_p < \omega$. $I_k : X \to X$ $(k = 1, 2, \ldots, p)$ are impulsive functions. $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$ denotes the jump of $u(t)$ at $t = t_k$, where $u(t_k^+)$, $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$ $(k \in \mathbf{N})$, respectively.

Obviously, the periodic problem of impulsive evolution equation (1.1) is equal to the periodic boundary value problem of impulsive evolution equation (IPBVP) in J,

$$\begin{cases} u'(t) + Au(t) = f(t,u(t)), & t \in J, t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), & k = 1, 2, \ldots, p, \\ u(0) = u(\omega). \end{cases} \tag{1.2}$$

The study of impulsive differential equations is a new and important branch of differential equation theory for studying evolution processes of real life phenomena not only in natural sciences but also in social sciences such as climate, food supplement, insecticide population, sustainable development that are subjected to sudden changes at certain instants. The theory of impulsive differential equations has been emerging as an important area of investigation in the last few decades; see the monographs of Lakshmikantham *et al.* [1], Benchohra *et al.* [2] and the papers of Chen [3], Li and Liu [4], Yang [5] and Lan [6], where numerous properties of their solutions are studied and detailed bibliographies are given. Consequently, some basic results on impulsive differential equations have been obtained and the applications of the theory of impulsive differential equations to different areas have been considered by many authors, see [4, 7–19] and the references therein.

The monotone iterative method based on lower and upper solutions is an effective and flexible mechanism. This technique is that, for the considered problem, starting from a pair of ordered lower and upper solutions, one constructs two monotone sequences such that they uniformly converge to the extremal solutions between the lower and upper solutions. By using the method of lower and upper solutions and the monotone iterative technique, Du and Lakshmikantham [20], Sun and Zhao [21] studied the existence of solutions to initial value problem of ordinary differential equation without impulse. Later on, Guo and Liu [22], Li and Liu [4] developed the monotone iterative method for impulsive integro-differential equations. Wang and Wang [23] investigated monotone iterative techniques for abstract semilinear evolution equations. Under the condition that the impulsive function is monotone increasing on the order interval, Chen and Mu [24] and Chen and Li [25] discussed the impulsive evolution equations with classical initial conditions.

Luo *et al.* [15] established a monotone iterative method for the antiperiodic boundary value problem of the first-order impulsive ordinary differential equations

$$\begin{cases} u'(t) = f(t, u(t)), & t \in J, t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), & k = 1, 2, \ldots, p, \\ u(0) = -u(\omega), \end{cases} \tag{1.3}$$

where the impulsive functions I_k $(k = 1, 2, \ldots, p)$ are nondecreasing. By applying the lower and upper solution method and the monotone iterative technique, the author obtained the existence of solutions for problem (1.3).

Ahmad and Nieto [26] applied the method of quasilinearization to obtain monotone sequences of approximate solutions converging uniformly and quadratically to the unique solution of the following impulsive anti-periodic problem:

$$\begin{cases} u'(t) = g(t, x(t), x(w(t))), & t \in J = [0, T], t \neq t_k, t_k \in (0, T), \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \ldots, p, \\ x(0) = -x(T), \\ x(t) = x(0), & t \in [-r, 0]. \end{cases}$$

Suppose that impulsive functions I_k $(k = 1, 2, \ldots, p)$ satisfied $-1 \leq I_k'(\cdot) \leq 0$ with $I_k''(\cdot) \geq 0$.

Recently, Chen [3] discussed the existence of solutions to the impulsive periodic boundary value problem in an ordered Banach space X,

$$\begin{cases} u'(t) = f(t, u(t), u(t)), & t \in J, t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k), u(t_k)), & k = 1, 2, \ldots, p, \\ u(0) = u(\omega). \end{cases} \tag{1.4}$$

Assume that problem (1.4) has coupled lower and upper L-quasisolutions v_0 and w_0 with $v_0 \leq w_0$. Suppose that impulsive functions I_k $(k = 1, 2, \ldots, p)$ are satisfied

$$I_k(u_1, v_1) \leq I_k(u_2, v_2), \quad k = 1, 2, \ldots, p,$$

for any $t \in J$ and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $v_0(t) \leq v_2 \leq v_1 \leq w_0(t)$.

Shao and Zhang [27] investigated the periodic solutions for the impulsive evolution equation

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)), & t \geq 0, t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), & k \in \mathbf{N}, \end{cases}$$

where the impulsive functions satisfied the monotone condition $\forall t \in [0, \omega]$, $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$,

$$I_k(x_1) \leq I_k(x_2), \quad k = 1, 2, \ldots, p.$$

In this paper, we consider the existence of ω-periodic mild solutions for the impulsive evolution equation (1.2) by means of the perturbation method and the mixed monotone iterative technique. In the previous results in the related literature, the impulsive functions were considered as nondecreasing functions, which were not easy to satisfy. To our knowledge, there are very few papers to study the periodic boundary value problem of impulsive evolution equation under the impulsive functions satisfying quasimonotonicity. In this paper, we assume that the impulsive functions satisfy quasimonotonicity, which will compensate for the lack in this area. Applying $A = 0$, our results improve and extend the evolution equations without impulse and some relevant results in ordinary differential equations.

2 Preliminaries

Let X be a Banach space, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a C_0-semigroup $T(t)$ $(t \geq 0)$ in X. Then there exist constants $M > 0$ and $\nu \in \mathbf{R}$ such that

$$\begin{aligned} &\|T(t)\| \leq Me^{\nu t}, \quad t \geq 0, \\ &\nu_0 = \inf\{\nu \in \mathbf{R} \mid \exists M > 0, \|T(t)\| \leq Me^{\nu t}, \forall t \geq 0\} \end{aligned} \tag{2.1}$$

and ν_0 can also be expressed by $\nu_0 = \limsup_{t \to +\infty} \frac{\ln \|T(t)\|}{t}$, then ν_0 is called a growth index of the C_0-semigroup $T(t)$ $(t \geq 0)$. If $\nu_0 < 0$, then $T(t)$ $(t \geq 0)$ is called an exponentially stable C_0-semigroup.

Let $T(t)$ $(t \geq 0)$ be an exponentially stable C_0-semigroup, for $\forall \nu \in (0, |\nu_0|)$, by the definition of ν_0, we have $\exists M \geq 1$,

$$\|T(t)\| \leq Me^{-\nu t}, \quad t \geq 0. \tag{2.2}$$

We define an equivalent norm in X by

$$|x| = \sup_{t \geq 0} \|e^{\nu t} T(t)x\|, \tag{2.3}$$

then $\|x\| \leq |x| \leq M\|x\|$. Respectively, $|T(t)|$ is the norm of the operator $T(t)$ in space $(X, |\cdot|)$. By (2.2), we have

$$\left|T(t)\right| \leq e^{-\nu t} \tag{2.4}$$

and $|T(\omega)| \leq e^{-\nu\omega} < 1$.

Lemma 2.1 [28] *Let $T(t)$ $(t \geq 0)$ be an exponentially stable C_0-semigroup, then the operator $I - T(\omega)$ has a bounded inverse operator $(I - T(\omega))^{-1}$ and satisfies the inequality*

$$\left|\left(I - T(w)\right)^{-1}\right| \leq \frac{1}{1 - e^{-\nu\omega}}.$$

Let X be an ordered Banach space with the norm $\|\cdot\|$ and partial order '$\leq$', whose positive cone $K = \{x \in X \mid x \geq \theta\}$ is normal with normal constant N. $J = [0, \omega]$, ω is a constant. Let $C(J, X)$ denote the Banach space of all continuous X-value functions on interval J with the norm $\|u\|_C = \max_{t\in J} \|u(t)\|$. Then $C(J, X)$ is an ordered Banach space induced by the convex cone $K_C = \{u \in C(J, X) \mid u(t) \geq 0, t \in J\}$, and K_C is also a normal cone.

Let $J' = J \setminus \{t_1, t_2, \ldots, t_p\}$, $J'' = J \setminus \{0, t_1, t_2, \ldots, t_p\}$. Let $J_1 = [t_0, t_1]$, $J_k = (t_{k-1}, t_k]$, $k = 2, 3, \ldots, p + 1$, where $t_0 = 0$, $t_{p+1} = \omega$. Evidently, $PC(J, X) = \{u : J \to X \mid u(t)$ is continuous in J' and left continuous at t_k, and $u(t_k^+)$ exists, $k = 1, 2, \ldots, p\}$. $PC(J, X)$ is a Banach space with the norm $\|\cdot\|_{PC} = \sup_{t\in J} \|u(t)\|$. Evidently, $PC(J, X)$ is also an ordered Banach space with the partial order '$\leq$' induced by the positive cone $K_{PC} = \{u \in PC(J, X) \mid u(t) \geq \theta, t \in J\}$. K_{PC} is normal with the same normal constant N. For $v, w \in PC(J, X)$ with $v \leq w$, we use $[v, w]$ to denote the order interval $\{u \in PC(J, X) \mid v \leq u \leq w\}$ in $PC(J, X)$, and $[v(t), w(t)]$ to denote the order interval $\{u \in X \mid v(t) \leq u(t) \leq w(t), t \in J\}$ in X. We use X_1 to denote the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$.

Definition 2.2 If functions $v_0 \in PC(J, X) \cap C^1(J'', X) \cap C(J', X_1)$ satisfy

$$\begin{cases} v_0'(t) + Av_0(t) \leq f(t, v_0(t)), & t \in J, t \neq t_k, \\ \Delta v_0|_{t=t_k} \leq I_k(v_0(t_k)), & k = 1, 2, \ldots, p, \\ v_0(0) \leq v_0(\omega), \end{cases} \tag{2.5}$$

we call v_0 a lower solution of IPBVP (1.2); if all the inequalities of (2.5) are inverse, we call it an upper solution of IPBVP (1.2).

3 Linear impulsive evolution equation

Let $I_0 = [t_0, T]$. Denote by $C(I_0, X)$ the Banach space of all continuous X-value functions on interval I_0 with the norm $\|u\|_C = \max_{t\in I_0} \|u(t)\|$. It is well known [29] that for any $x_0 \in D(A)$ and $h \in C^1(I_0, X)$, the initial value problem (IVP) of linear evolution equation

$$\begin{cases} u'(t) + Au(t) = h(t), & t \in I_0, \\ u(t_0) = x_0 \end{cases} \tag{3.1}$$

has a unique classical solution $u \in C^1(I_0, X) \cap C(I_0, X_1)$ expressed by

$$u(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t - s)h(s)\, ds, \quad t \in I_0. \tag{3.2}$$

If $x_0 \in X$ and $h \in C(I_0, X)$, the function u given by (3.2) belongs to $C(I_0, X)$. We call it a mild solution of IVP(3.1). For any $h \in PC(J, X)$, we consider the periodic boundary value problem of linear impulsive evolution equation (LIPBVP) in X,

$$\begin{cases} u'(t) + Au(t) = h(t), & t \in J, t \neq t_k, \\ \Delta u|_{t=t_k} + a_k u(t_k) = e_k, & k = 1, 2, \dots, p, \\ u(0) = u(\omega), \end{cases} \tag{3.3}$$

where a_k is constant, $e_k \in X$, $k = 1, 2, \dots, p$.

Theorem 3.1 *Let X be a Banach space, −A generate an exponentially stable C_0-semigroup $T(t)$ $(t \geq 0)$ in X and ν_0 be a growth index of the semigroup $T(t)$. If $\frac{1}{\omega}\sum_{k=1}^{p}\ln(1-a_k) < -\nu_0$, for any $h \in PC(J, X)$, $a_k < 1$ is constant and $e_k \in X$, $k = 1, 2, \dots, p$, LIPBVP (3.3) has a unique mild solution $u \in PC(J, X)$ given by*

$$\begin{aligned} u(t) &= \prod_{k:0<t_k<t}(1-a_k)T(t)B_1(h) + \sum_{k:0<t_k<t}\prod_{i:t_k\leq t_i<t}(1-a_i)\int_{t_{k-1}}^{t_k}T(t-s)h(s)\,ds \\ &\quad + \int_{t_j}^{t}T(t-s)h(s)\,ds + \sum_{k:0<t_k<t_j}\prod_{i:t_k<t_i<t}(1-a_i)T(t-t_k)e_k + T(t-t_j)e_j \\ &= Q_1(h), \end{aligned} \tag{3.4}$$

where $t_j < t$ $(j = 0, 1, 2, \dots, p)$ is the nearest point of t, $j = \max\{k \mid 0 < t_k < t\}$ and

$$\begin{aligned} B_1(h) = {} & \left[I - \prod_{k=1}^{p}(1-a_k)T(\omega)\right]^{-1}\left[\sum_{k=1}^{p}\prod_{i=k}^{p}(1-a_i)\int_{t_{k-1}}^{t_k}T(\omega-s)h(s)\,ds \right. \\ & \left. + \int_{t_p}^{\omega}T(\omega-s)h(s)\,ds + \sum_{k=1}^{p-1}\prod_{i=k+1}^{p}(1-a_i)T(\omega-t_k)e_k + T(\omega-t_p)e_p\right]. \end{aligned}$$

Proof Let $J_1 = [t_0, t_1]$, $J_k = (t_{k-1}, t_k]$, $k = 2, 3, \dots, p+1$, where $t_0 = 0$, $t_{p+1} = \omega$. For any $h \in PC(J, X)$, we first show that the initial value problem of linear impulsive evolution equation

$$\begin{cases} u'(t) + Au(t) = h(t), & t \in J, t \neq t_k, \\ \Delta u|_{t=t_k} + a_k u(t_k) = e_k, & k = 1, 2, \dots, p, \\ u(0) = x_0. \end{cases} \tag{3.5}$$

Let $t \in J_1 = [t_0, t_1]$, equation (3.5) is equivalent to the initial value problem of linear evolution equation

$$\begin{cases} u'(t) + Au(t) = h(t), & t \in J_1, \\ u(0) = x_0. \end{cases} \tag{3.6}$$

Then (3.6) has a unique mild solution $u_1 \in C(J_1, X)$ given by

$$u_1(t) = T(t)x_0 + \int_0^t T(t-s)h(s)\,ds.$$

Especially, we have

$$u_1(t_1) = T(t_1)x_0 + \int_0^{t_1} T(t_1 - s)h(s)\,ds \in X. \tag{3.7}$$

Let $t \in J_2 = (t_1, t_2]$, equation (3.5) is equivalent to the initial value problem of linear evolution equation

$$\begin{cases} u'(t) + Au(t) = h(t), & t \in J_2, \\ u(t_1^+) = u_1(t_1) - a_1 u_1(t_1) + e_1. \end{cases} \tag{3.8}$$

Combining with (3.7), then (3.8) has a unique mild solution $u_2 \in C(J_2, X)$ expressed by

$$\begin{aligned} u_2(t) &= T(t - t_1)\big[(1 - a_1)u_1(t_1) + e_1\big] + \int_{t_1}^{t} T(t - s)h(s)\,ds \\ &= T(t - t_1)\left[(1 - a_1)\left(T(t_1)x_0 + \int_0^{t_1} T(t_1 - s)h(s)\,ds\right) + e_1\right] + \int_{t_1}^{t} T(t - s)h(s)\,ds \\ &= (1 - a_1)T(t)x_0 + (1 - a_1)\int_0^{t_1} T(t - s)h(s)\,ds + \int_{t_1}^{t} T(t - s)h(s)\,ds + T(t - t_1)e_1. \end{aligned}$$

Particularly, we have

$$u_2(t_2) = (1 - a_1)T(t_2)x_0 + (1 - a_1)\int_0^{t_1} T(t_2 - s)h(s)\,ds + \int_{t_1}^{t_2} T(t_2 - s)h(s)\,ds + T(t_2 - t_1)e_1.$$

Similarly, let $t \in J_k = (t_{k-1}, t_k]$ ($k = 3, 4, \dots, p + 1$), where $t_{p+1} = \omega$, equation (3.5) is equivalent to the initial value problem of linear evolution equation

$$\begin{cases} u'(t) + Au(t) = h(t), & t \in J_k, \\ u(t_{k-1}^+) = (1 - a_{k-1})u(t_{k-1}) + e_{k-1}, & k = 3, 4, \dots, p + 1. \end{cases} \tag{3.9}$$

Then (3.9) has a unique mild solution $u_k \in C(J_k, X)$ expressed by

$$\begin{aligned} u_k(t) &= T(t - t_{k-1})\big[(1 - a_{k-1})u_{k-1}(t_{k-1}) + e_{k-1}\big] + \int_{t_{k-1}}^{t} T(t - s)h(s)\,ds \\ &= \prod_{k:0<t_k<t} (1 - a_k)T(t)x_0 + \sum_{k:0<t_k<t} \prod_{i:t_k\le t_i<t} (1 - a_i)\int_{t_{k-1}}^{t_k} T(t - s)h(s)\,ds \\ &\quad + \int_{t_j}^{t} T(t - s)h(s)\,ds + \sum_{k:0<t_k<t_j} \prod_{i:t_k<t_i<t} (1 - a_i)T(t - t_k)e_k + T(t - t_j)e_j, \end{aligned}$$

where $t_j < t$ ($j = 0, 1, 2, \dots, p$) is the nearest point of t, $j = \max\{k \mid 0 < t_k < t\}$.

Let

$$u(t) = \begin{cases} u_1(t), & t \in J_1, \\ u_2(t), & t \in J_2, \\ \dots, \\ u_{p+1}(t), & t \in J_{p+1}. \end{cases} \tag{3.10}$$

Inversely, the function $u \in PC(J,X)$ defined by (3.10) is a unique mild solution of the initial value problem of linear evolution equation (3.5).

Next,we show that LIPBVP (3.3) has a unique mild solution $u \in PC(J,X)$ given by (3.4). If a function $u \in PC(J,X)$ defined by (3.10) is a solution of LIPBVP (3.3), then $x_0 = u(\omega)$, namely

$$\begin{aligned} &\left(I - \prod_{k=1}^{p}(1-a_k)T(\omega)\right)x_0 \\ &\quad = \sum_{k=1}^{p}\prod_{i=k}^{p}(1-a_i)\int_{t_{k-1}}^{t_k} T(\omega-s)h(s)\,ds + \int_{t_p}^{\omega} T(\omega-s)h(s)\,ds \\ &\qquad + \sum_{k=1}^{p-1}\prod_{i=k+1}^{p}(1-a_i)T(\omega-t_k)e_k + T(\omega-t_p)e_p. \end{aligned} \tag{3.11}$$

For $\forall \nu \in (0,-\nu_0)$, by Lemma 2.1, since $r(\prod_{k=1}^{p}(1-a_k)T(\omega)) \leq \prod_{k=1}^{p}|1-a_k|e^{-\nu\omega}$, and $\frac{1}{\omega}\sum_{k=1}^{p}\ln(1-a_k) < -\nu_0$, $a_k < 1$, by the arbitrary of ν, we have $r(\prod_{k=1}^{p}(1-a_k)T(\omega)) \leq \prod_{k=1}^{p}(1-a_k)e^{\nu_0\omega} < 1$, and $I - \prod_{k=1}^{p}(1-a_k)T(\omega)$ has a bounded inverse operator. From (3.11), we choose

$$\begin{aligned} x_0 &= \left[I - \prod_{k=1}^{p}(1-a_k)T(\omega)\right]^{-1}\left[\sum_{k=1}^{p}\prod_{i=k}^{p}(1-a_i)\int_{t_{k-1}}^{t_k} T(\omega-s)h(s)\,ds \right. \\ &\quad \left. + \int_{t_p}^{\omega} T(\omega-s)h(s)\,ds + \sum_{k=1}^{p-1}\prod_{i=k+1}^{p}(1-a_i)T(\omega-t_k)e_k + T(\omega-t_p)e_p\right] \\ &\triangleq B_1(h). \end{aligned} \tag{3.12}$$

Combining (3.12) with (3.10), we obtain that the function $u(t) \in PC(J,X)$ given by (3.4) is a unique mild solution of LIBPVP (3.3) on J and the operator $Q_1 : PC(J,X) \to PC(J,X)$ is a continuous operator. So, the conclusion of Theorem 3.1 holds. □

Remark 3.2 In Theorem 3.1, let X be an ordered Banach space, $-A$ generate a positive C_0-semigroup $T(t)$ $(t \geq 0)$ in X and ν_0 be a growth index of the semigroup $T(t)$. If $\frac{1}{\omega}\sum_{k=1}^{p}\ln(1-a_k) < -\nu_0$, for any $h \geq \theta$, $a_k < 1$ is constant and $e_k \geq \theta$, $k = 1,2,\ldots,p$, then the solution operator Q_1 of LIPBVP (3.3) is a positive operator.

4 Proof of the main results

Theorem 4.1 *Let X be an ordered Banach space, whose positive cone K is normal, and N_0 be the normal constant. Let $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a compact and positive C_0-semigroup $T(t)$ $(t \geq 0)$ in X. $f \in C(J \times X, X)$ is ω-periodic about t, $I_k \in C(X,X)$, $k = 1,2,\ldots,p$. Assume that IPBVP* (1.2) *has lower and upper solutions v_0 and w_0 with $v_0(t) \leq w_0(t)$ $(t \in J)$. Suppose that the following conditions are satisfied*:

(P_1) *There exists a constant $C \geq 0$ such that*

$$f(t,x_2) - f(t,x_1) \geq -C(x_2 - x_1), \quad t \in J,$$

for $\forall t \in J$, $v_0(t) \leq x_1 \leq x_2 \leq w_0(t)$.

(P_2) $\exists 0 \le N_k < 1$, *for* $\forall t \in J$, $v_0(t) \le x_1 \le x_2 \le w_0(t)$, *impulsive functions* I_k *satisfy*

$$I_k(x_2) - I_k(x_1) \ge -N_k(x_2 - x_1), \quad k = 1, 2, \ldots, p.$$

Then IPBVP (1.2) *has minimal and maximal* ω*-periodic mild solutions* $\underline{u}$ *and* $\overline{u}$ *between* v_0 *and* w_0, *which can be obtained by monotone iterative sequences starting from* v_0 *and* w_0.

Proof Define $D = [v_0, w_0]$. For $\forall h \in D$, we consider the periodic boundary value problem of linear impulsive evolution equation (LIPBVP) in X,

$$\begin{cases} u'(t) + Au(t) + Cu(t) = f(t, h(t)) + Ch(t), & t \in J, t \neq t_k, \\ \Delta u|_{t=t_k} + N_k u(t_k) = I_k(h(t_k)) + N_k h(t_k), & k = 1, 2, \ldots, p, \\ u(0) = u(\omega), \end{cases} \tag{4.1}$$

where $\forall t \in [0, \omega]$, define $t_j < t$ ($j = 0, 1, 2, \ldots, p$) is the nearest point of t, $j = \max\{k \mid 0 < t_k < t\}$, $f_1(t, x) = f(t, x) + Cx$.

Let $C > v_0$, $-(A + CI)$ generate an exponentially stable, compact and positive C_0-semigroup $S(t) = e^{-Ct}T(t)$ ($t \ge 0$) in X, whose growth index is $-C + v_0$. Since $C > v_0$, $0 \le N_k < 1$, so

$$\frac{1}{\omega} \sum_{k=1}^{p} \ln(1 - N_k) < C - v_0. \tag{4.2}$$

From Theorem 3.1, LIPBVP (4.1) has a unique mild solution $u \in PC(J, X)$ given by

$$\begin{aligned} u(t) = {} & \prod_{k:0<t_k<t} (1 - N_k) S(t) B_2(h) + \sum_{k:0<t_k<t} \prod_{i:t_k \le t_i < t} (1 - N_i) \int_{t_{k-1}}^{t_k} S(t - s) f_1\bigl(s, h(s)\bigr)\, ds \\ & + \int_{t_j}^{t} S(t - s) f_1\bigl(s, h(s)\bigr)\, ds + \sum_{k:0<t_k<t_j} \prod_{i:t_k<t_i<t} (1 - N_i) S(t - t_k)\bigl(I_k\bigl(h(t_k)\bigr) + N_k h(t_k)\bigr) \\ & + S(t - t_j)\bigl(I_j\bigl(h(t_j)\bigr) + N_j h(t_j)\bigr) \triangleq Q_2(h), \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} B_2(h) \triangleq {} & \left(I - \prod_{k=1}^{p} (1 - N_k) S(\omega)\right)^{-1} \left[\sum_{k=1}^{p} \prod_{i=k}^{p} (1 - N_i) \int_{t_{k-1}}^{t_k} S(\omega - s) f_1\bigl(s, h(s)\bigr)\, ds \right. \\ & + \int_{t_p}^{\omega} S(\omega - s) f_1\bigl(s, h(s)\bigr)\, ds + \sum_{k=1}^{p-1} \prod_{i=k+1}^{p} (1 - N_i) S(\omega - t_k)\bigl(I_k\bigl(h(t_k)\bigr) + N_k h(t_k)\bigr) \\ & \left. + S(\omega - t_p)\bigl(I_p\bigl(h(t_p)\bigr) + N_p h(t_p)\bigr) \right]. \end{aligned}$$

Since f and I_k are continuous, so $Q_2 : D \to PC(J, X)$ is continuous.

Clearly, the ω-periodic mild solutions of IPBVP (1.2) are equivalent to the fixed points of operator Q_2.

(i) We show $Q_2 : D \to PC(J, X)$ is an increasing operator.

In fact, for $\forall h_1, h_2 \in D$ and $h_1 \le h_2$, from the assumptions (P_1) and (P_2), we have

$$f_1\big(t,h_1(t)\big) = f\big(t,h_1(t)\big) + Ch_1(t) \le f\big(t,h_2(t)\big) + Ch_2(t) = f_1\big(t,h_2(t)\big), \quad t \in J$$

and

$$I_k\big(h_1(t_k)\big) + N_k h_1(t_k) \le I_k\big(h_2(t_k)\big) + N_k h_2(t_k), \quad k = 1,2,\ldots,p.$$

Since $S(t)$ is an exponentially stable and positive C_0-semigroup, combining this with (4.2), then $[I - \prod_{k=1}^{p}(1-N_k)S(\omega)]$ has a bounded inverse operator and it can be expressed that $(I - \prod_{k=1}^{p}(1-N_k)S(\omega))^{-1} = \sum_{n=0}^{\infty}(\prod_{k=1}^{p}(1-N_k))^n S(n\omega)$. Obviously, the operator $(I - \prod_{k=1}^{p}(1-N_k)S(\omega))^{-1}$ is a positive operator. Hence, we have

$$\begin{aligned}
&\sum_{k=1}^{p}\prod_{i=k}^{p}(1-N_i)\int_{t_{k-1}}^{t_k} S(\omega-s)f_1\big(s,h_1(s)\big)\,ds + \int_{t_p}^{\omega} S(\omega-s)f_1\big(s,h_1(s)\big)\,ds \\
&\quad\le \sum_{k=1}^{p}\prod_{i=k}^{p}(1-N_i)\int_{t_{k-1}}^{t_k} S(\omega-s)f_1\big(s,h_2(s)\big)\,ds + \int_{t_p}^{\omega} S(\omega-s)f_1\big(s,h_2(s)\big)\,ds
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{p-1}\prod_{i=k+1}^{p}(1-N_i)S(\omega-t_k)\big(I_k\big(h_1(t_k)\big) + N_k h_1(t_k)\big) + S(\omega-t_p)\big(I_p\big(h_1(t_p)\big) + N_p h_1(t_p)\big) \\
&\quad\le \sum_{k=1}^{p-1}\prod_{i=k+1}^{p}(1-N_i)S(\omega-t_k)\big(I_k\big(h_2(t_k)\big) + N_k h_2(t_k)\big) \\
&\qquad + S(\omega-t_p)\big(I_p\big(h_2(t_p)\big) + N_p h_2(t_p)\big).
\end{aligned}$$

Namely, $B_2(h_1) \le B_2(h_2)$. Thus we obtain the inequality $\prod_{k:0<t_k<t}(1-N_k)S(t)B_2(h_1) \le \prod_{k:0<t_k<t}(1-N_k)S(t)B_2(h_2)$. Combining this with (4.3), we have $Q_2(h_1) \le Q_2(h_2)$.

(ii) We show $v_0 \le Q_2(v_0)$, $Q_2(w_0) \le w_0$.

Let

$$\begin{cases} v_0'(t) + Av_0(t) + Cv_0(t) = \overline{h}(t), & t \in J, t \neq t_k, \\ \Delta v_0|_{t=t_k} + N_k v_0(t_k) = \overline{e}_k, & k = 1,2,\ldots,p, \\ v_0(0) = v_0(\omega), \end{cases} \tag{4.4}$$

from the definition of v_0, we have

$$\begin{cases} \overline{h}(t) \le f(t,v_0(t)) + Cv_0(t), & t \in J, t \neq t_k, \\ \overline{e}_k \le I_k(v_0(t_k)) + N_k v_0(t_k), & k = 1,2,\ldots,p. \end{cases} \tag{4.5}$$

By Theorem 3.1, (4.4) and (4.5), we have

$$\begin{aligned}
v_0(t) &= \prod_{k:0<t_k<t}(1-N_k)S(t)B_3(\overline{h}) + \sum_{k:0<t_k<t}\prod_{i:t_k\le t_i<t}(1-N_i)\int_{t_{k-1}}^{t_k} S(t-s)\overline{h}(s)\,ds \\
&\quad + \int_{t_j}^{t} S(t-s)\overline{h}(s)\,ds + \sum_{k:0<t_k<t_j}\prod_{i:t_k<t_i<t}(1-N_i)S(t-t_k)\overline{e}_k + S(t-t_j)\overline{e}_j
\end{aligned}$$

$$\begin{aligned}
&\leq \prod_{k:0<t_k<t}(1-N_k)S(t)B_3(\overline{h}) + \sum_{k:0<t_k<t}\prod_{i:t_k\leq t_i<t}(1-N_j)\int_{t_{k-1}}^{t_k} S(t-s)f_1\bigl(s,v_0(s)\bigr)\,ds \\
&\quad + \int_{t_j}^{t} S(t-s)f_1\bigl(s,v_0(s)\bigr)\,ds \\
&\quad + \sum_{k:0<t_k<t_j}\prod_{i:t_k<t_i<t}(1-N_i)S(t-t_k)\bigl(I_k\bigl(v_0(t_k)\bigr)+N_kv_0(t_k)\bigr) \\
&\quad + S(t-t_j)\bigl(I_j\bigl(v_0(t_j)\bigr)+N_jv_0(t_j)\bigr), \qquad (4.6)
\end{aligned}$$

where

$$\begin{aligned}
B_3(\overline{h}) = {}& \left(I-\prod_{k=1}^{p}(1-N_k)S(\omega)\right)^{-1}\left[\sum_{k=1}^{p}\prod_{i=k}^{p}(1-N_i)\int_{t_{k-1}}^{t_k} S(\omega-s)\overline{h}(s)\,ds\right. \\
&\left. + \int_{t_p}^{\omega} S(\omega-s)\overline{h}(s)\,ds + \sum_{k=1}^{p-1}\prod_{i=k+1}^{p}(1-N_i)S(\omega-t_k)\overline{e}_k + S(\omega-t_p)\overline{e}_p\right].
\end{aligned}$$

Particularly,

$$\begin{aligned}
v_0(\omega) \leq {}& \prod_{k=1}^{p}(1-N_k)S(\omega)B_3(\overline{h}) + \sum_{k=1}^{p}\prod_{i=k}^{p}(1-N_i)\int_{t_{k-1}}^{t_k} S(\omega-s)f_1\bigl(s,v_0(s)\bigr)\,ds \\
& + \int_{t_p}^{\omega} S(\omega-s)f_1\bigl(s,v_0(s)\bigr)\,ds \\
& + \sum_{k=1}^{p-1}\prod_{i=k+1}^{p}(1-N_i)S(\omega-t_k)\bigl(I_k\bigl(v_0(t_k)\bigr)+N_kv_0(t_k)\bigr) \\
& + S(\omega-t_p)\bigl(I_p\bigl(v_0(t_p)\bigr)+N_pv_0(t_p)\bigr). \qquad (4.7)
\end{aligned}$$

By (4.4) and (4.6), then $v_0(0) = B_3(\overline{h})$. Combining $v_0(0) \leq v_0(\omega)$ with (4.7), we have

$$\begin{aligned}
B_3(\overline{h}) \leq {}& \left(I-\prod_{k=1}^{p}(1-N_k)S(\omega)\right)^{-1}\left[\sum_{k=1}^{p}\prod_{i=k}^{p}(1-N_i)\int_{t_{k-1}}^{t_k} S(\omega-s)f_1\bigl(s,v_0(s)\bigr)\,ds\right. \\
& + \int_{t_p}^{\omega} S(\omega-s)f_1\bigl(s,v_0(s)\bigr)\,ds + \sum_{k=1}^{p-1}\prod_{i=k+1}^{p}(1-N_i)S(\omega-t_k)\bigl(I_k\bigl(v_0(t_k)\bigr)+N_kv_0(t_k)\bigr) \\
&\left. + S(\omega-t_p)\bigl(I_p\bigl(v_0(t_p)\bigr)+N_pv_0(t_p)\bigr)\right] \\
= {}& B_2(v_0).
\end{aligned}$$

On the other hand, from (4.3), then

$$\begin{aligned}
Q_2(v_0)(t) = {}& \prod_{k:0<t_k<t}(1-N_k)S(t)B_2(v_0) + \sum_{k:0<t_k<t}\prod_{i:t_k\leq t_i<t}(1-N_i)\int_{t_{k-1}}^{t_k} S(t-s)f_1\bigl(s,v_0(s)\bigr)\,ds \\
& + \int_{t_j}^{t} S(t-s)f_1\bigl(s,v_0(s)\bigr)\,ds
\end{aligned}$$

$$+ \sum_{k:0<t_k<t_j} \prod_{i:t_k<t_i<t} (1-N_i)S(t-t_k)\big(I_k\big(v_0(t_k)\big) + N_k v_0(t_i)\big)$$
$$+ S(t-t_j)\big(I_j\big(v_0(t_j)\big) + N_j v_0(t_j)\big). \tag{4.8}$$

By (4.6) and (4.8), we have

$$Q_2(v_0)(t) - v_0(t) \geq \prod_{k:0<t_k<t} (1-N_k)S(t)\big(B_2(v_0) - B_3(\overline{h})\big) \geq \theta,$$

namely $v_0(t) \leq Q_2(v_0)(t)$. Similarly, it can be shown that $Q_2(w_0)(t) \leq w_0(t)$. Therefore, $Q_2 : [v_0, w_0] \to [v_0, w_0]$ is a continuously increasing operator.

(iii) Next, we will prove that the operator Q_2 has fixed points on $[v_0, w_0]$.

Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ by the iterative scheme

$$v_n = Q_2(v_{n-1}), \qquad w_n = Q_2(w_{n-1}), \quad n = 1, 2, \ldots. \tag{4.9}$$

Then from the monotonicity of operator Q_2 it follows that

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0. \tag{4.10}$$

Next, we prove that $\{v_n\}$ and $\{w_n\}$ are convergent in J. Let $G = \{v_n \mid n \in \mathbf{N}\}$, $G_0 = \{v_{n-1} \mid n \in \mathbf{N}\}$, then $G_0 = \{v_0\} \cup G$ and $G = Q_2(G_0)$. For any $v_{n-1} \in G_0$, let

$$W(v_{n-1})(t) = \sum_{k:0<t_k<t} \prod_{i:t_k\leq t_i<t} (1-N_i) \int_{t_{k-1}}^{t_k} S(t-s)f_1\big(s, v_{n-1}(s)\big)\,ds$$
$$+ \int_{t_j}^{t} S(t-s)f_1\big(s, v_{n-1}(s)\big)\,ds$$
$$+ \sum_{k:0<t_k<t_j} \prod_{i:t_k<t_i<t} (1-N_i)S(t-t_k)\big(I_k\big(v_{n-1}(t_k)\big) + N_k v_{n-1}(t_k)\big)$$
$$+ S(t-t_j)\big(I_j\big(v_{n-1}(t_j)\big) + N_j v_{n-1}(t_j)\big), \tag{4.11}$$

then $Q_2(v_{n-1})(t) = \prod_{k:0<t_k<t}(1-N_k)S(t)B_2(v_{n-1}) + W(v_{n-1})(t)$. First, we will prove that for any $0 < t < \omega$, $Y(t) \stackrel{\text{def}}{=} \{W(v_{n-1})(t) \mid v_{n-1} \in G_0\}$ is relatively compact in X. Let $0 < \epsilon < t$, $\bar{j} = \max\{k \mid 0 < t_k < t-\epsilon\}$ and

$$W_\epsilon(v_{n-1})(t) = \sum_{k:0<t_k<t-\epsilon} \prod_{i:t_k\leq t_i<t-\epsilon} (1-N_i) \int_{t_{k-1}}^{t_k} S(t-s)f_1\big(s, v_{n-1}(s)\big)\,ds$$
$$+ \int_{t_{\bar{j}}}^{t-\epsilon} S(t-s)f_1\big(s, v_{n-1}(s)\big)\,ds$$
$$+ \sum_{k:0<t_k<t_{\bar{j}}} \prod_{i:t_k<t_i<t-\epsilon} (1-N_i)S(t-t_k)\big(I_k\big(v_{n-1}(t_k)\big) + N_k v_{n-1}(t_k)\big)$$
$$+ S(t-t_{\bar{j}})\big(I_{\bar{j}}\big(v_{n-1}(t_{\bar{j}})\big) + N_{\bar{j}} v_{n-1}(t_{\bar{j}})\big)$$
$$= S(\epsilon)\Bigg[\sum_{k:0<t_k<t-\epsilon} \prod_{i:t_k\leq t_i<t-\epsilon} (1-N_i) \int_{t_{k-1}}^{t_k} S(t-\epsilon-s)f_1\big(s, v_{n-1}(s)\big)\,ds$$

$$
\begin{aligned}
&+\int_{t_{\bar{j}}}^{t-\epsilon} S(t-\epsilon-s)f_1\big(s,v_{n-1}(s)\big)\,ds \\
&+\sum_{k:0<t_k<t_{\bar{j}}}\prod_{i:t_k<t_i<t-\epsilon}(1-N_i)S(t-\epsilon-t_k)\big(I_k\big(v_{n-1}(t_k)\big)+N_kv_{n-1}(t_k)\big) \\
&+S(t-\epsilon-t_{\bar{j}})\big(I_{\bar{j}}\big(v_{n-1}(t_{\bar{j}})\big)+N_{\bar{j}}v_{n-1}(t_{\bar{j}})\big)\Bigg]. \qquad (4.12)
\end{aligned}
$$

By assumption (P_1) we know that

$$f\big(t,v_0(t)\big)+Cv_0(t)\le f\big(t,v_{n-1}(t)\big)+Cv_{n-1}(t)\le f\big(t,w_0(t)\big)+Cw_0(t).$$

Since $f(t,v_0(t))$ and $f(t,w_0(t))$ are continuous in the compact set $[0,\omega]$, so their image sets are compact sets in X, namely image sets are bounded. Combining this fact with the normality of cone K in X, we have $\exists M_1>0$, $\forall v_{n-1}\in G_0$,

$$\big\|f_1\big(t,v_{n-1}(t)\big)\big\|\le\big\|f_1\big(t,v_0(t)\big)\big\|+N_0\big\|f_1\big(t,w_0(t)\big)-f_1\big(t,v_0(t)\big)\big\|\le M_1.$$

By assumption (P_2) we know that

$$
\begin{aligned}
I_k\big(v_0(t_k)\big)+N_kv_0(t_k)&\le I_k\big(v_{n-1}(t_k)\big)+N_kv_{n-1}(t_k) \\
&\le I_k\big(w_0(t_k)\big)+N_kw_0(t_k),\quad k=1,2,\ldots,p.
\end{aligned}
$$

By the normality of cone K in X, there exists $M_2>0$ such that

$$
\begin{aligned}
&\big\|I_k\big(v_{n-1}(t_k)\big)+N_kv_{n-1}(t_k)\big\| \\
&\quad\le\big\|I_k\big(v_0(t_k)\big)+N_kv_0(t_k)\big\|+N_0\big\|I_k\big(w_0(t_k)\big)-I_k\big(v_0(t_k)\big)+N_kw_0(t_k)-N_kv_0(t_k)\big\| \\
&\quad\le M_2.
\end{aligned}
$$

Combining (4.12) with the compactness of $S(\epsilon)$, then $Y_\epsilon(t)=\{W_\epsilon(v_{n-1})(t)\mid v_{n-1}\in G_0\}$ is a relatively compact set in X. Let $J_1=[t_0,t_1]$, $J_k=(t_{k-1},t_k]$, $k=2,3,\ldots,p+1$, where $t_0=0$, $t_{p+1}=\omega$. For sufficiently small ϵ and t, $t-\epsilon\in J_k$ $(k=1,2,\ldots,p+1)$, then $j=\bar{j}$ and

$$
\begin{aligned}
&\big\|W(v_{n-1})(t)-W_\epsilon(v_{n-1})(t)\big\| \\
&\quad=\bigg\|\int_{t_j}^{t}S(t-s)f_1\big(s,v_{n-1}(s)\big)\,ds-\int_{t_j}^{t-\epsilon}S(t-s)f_1\big(s,v_{n-1}(s)\big)\,ds\bigg\| \\
&\quad\le\int_{t-\epsilon}^{t}\big\|S(t-s)\big\|\big\|f_1\big(s,v_{n-1}(s)\big)\big\|\,ds\le MM_1\epsilon,
\end{aligned}
$$

hence $Y(t)$ is a totally bounded set in X, thus it is a relatively compact set. Especially, by the compactness of $Y(\omega)$ and the relative compactness of $\{B_2(v_{n-1})\mid v_{n-1}\in G_0\}=(I-\prod_{k=1}^{p}(1-N_k)S(\omega))^{-1}Y(\omega)$ in X, we know that $\{(\prod_{k:0<t_k<t}(1-N_k))S(t)B_2(v_{n-1})\mid v_{n-1}\in G_0\}$ is a relatively compact set.

Noticing

$$\big\{Q_2(t)\mid v_{n-1}\in G_0\big\}=\bigg\{\prod_{k:0<t_k<t}(1-N_k)S(t)B_2(v_{n-1})+W(v_{n-1})(t)\,\bigg|\,v_{n-1}\in G_0\bigg\},$$

and $Q_2(v_{n-1})(0) = B_2(v_{n-1})$, considering

$$\begin{aligned} Q_2(v_{n-1})(\omega) = & \left[\prod_{k=1}^{p}(1-N_k)S(\omega)\left(I-\prod_{k=1}^{p}(1-N_k)S(\omega)\right)^{-1}+I\right] \\ & \times\left[\sum_{k=1}^{p}\prod_{i=k}^{p}(1-N_i)\int_{t_{k-1}}^{t_k}S(\omega-s)f_1\big(s,v_{n-1}(s)\big)\,ds\right. \\ & +\int_{t_p}^{\omega}S(\omega-s)f_1\big(s,v_{n-1}(s)\big)\,ds \\ & +\sum_{k=1}^{p-1}\prod_{i=k+1}^{p}(1-N_i)S(\omega-t_k)\big(I_k\big(v_{n-1}(t_k)\big)+N_kv_{n-1}(t_k)\big) \\ & \left.+S(\omega-t_p)\big(I_p\big(v_{n-1}(t_p)\big)+N_pv_{n-1}(t_p)\big)\right] \\ = & B_2(v_{n-1}), \end{aligned}$$

then $Q_2(v_{n-1})(0) = Q_2(v_{n-1})(\omega) = B_2(v_{n-1})$, namely $\{Q_2(v_{n-1})(0) \mid v_{n-1} \in G_0\} = B_2(G_0)$ is relatively compact.

Therefore, $\{v_n(t)\} = \{Q_2(v_{n-1})(t) \mid v_{n-1} \in G_0, t \in J\}$ is relatively compact in X. Combining this fact with the monotonicity of $\{v_n\}$, we easily prove that $\{v_n(t)\}$ is convergent. Let $\{v_n(t)\} \to \underline{u}(t)$ in $t \in J$.

The same idea can be used to prove that $\{w_n(t)\} \to \overline{u}(t)$ in $t \in J$.

Evidently $\{v_n(t)\}, \{w_n(t)\} \in PC(J,X)$, so $\underline{u}(t)$ and $\overline{u}(t)$ are bounded integrable in J_k ($k = 1,2,\ldots,p$). Since for any $t \in J_k$, $v_n(t) = Q_2(v_{n-1})(t)$, $w_n(t) = Q_2(w_{n-1})(t)$, letting $n \to \infty$, by the Lebesgue dominated convergence theorem, we have $\underline{u}(t) = Q_2(\underline{u})(t)$, $\overline{u}(t) = Q_2(\overline{u})(t)$ and $\underline{u}(t), \overline{u}(t) \in PC(J,X)$. Combining this with monotonicity (4.10), we have $v_0(t) \leq \underline{u}(t) \leq \overline{u}(t) \leq w_0(t)$.

Next, we prove that $\underline{u}(t)$ and $\overline{u}(t)$ are the minimal and maximal fixed points of Q_2 in $[v_0, w_0]$, respectively. In fact, for any $u^* \in [v_0, w_0]$, $Q_2(u^*) = u^*$, we have $v_0 \leq u^* \leq w_0$, and $v_1 = Q_2(v_0) \leq Q_2(u^*) = u^* \leq Q_2(w_0) = w_1$. Continuing such progress, we get $v_n \leq u^* \leq w_n$. Letting $n \to \infty$, we get $\underline{u}(t) \leq u^* \leq \overline{u}(t)$. Therefore, $\underline{u}(t)$ and $\overline{u}(t)$ are the minimal and maximal ω-periodic mild solutions of IPBVP (1.2) between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0, respectively. This completes the proof of Theorem 4.1. □

Remark 4.2 In [15], the impulsive functions are required to be ordered increasing; therefore, Theorem 4.1 in this paper extensively generalizes the main results in [15].

Theorem 4.3 *Let X be an ordered Banach space whose positive cone K is regular, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)$ $(t \geq 0)$ in X. $f \in C(J \times X, X)$ and f is ω-periodic about t, $I_k \in C(X,X)$, $k = 1,2,\ldots,p$. Assume that IPBVP* (1.2) *has coupled lower and upper solutions v_0 and w_0 with $v_0(t) \leq w_0(t)$ $(t \in J)$, and conditions* (P_1) *and* (P_2) *are satisfied, then IPBVP* (1.2) *has minimal and maximal ω-periodic mild solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0.*

Proof From Theorem 4.1 we know that $Q_2 : [v_0, w_0] \to [v_0, w_0]$ is a continuously increasing operator. Similarly, the two sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are defined in $[v_0, w_0]$ by the iterative scheme (4.9). By conditions (P_2), then $\{v_n(t)\}$ and $\{w_n(t)\}$ are ordered-monotonic and ordered-bounded sequences in X.

Using the regularity of the cone K, any ordered-monotonic and ordered-bounded sequence in X is convergent. So, $\{v_n(t)\}$ and $\{w_n(t)\}$ are convergent, namely $\exists v^*(t), w^*(t)$, $v_n(t) \to v^*(t)$, $w_n(t) \to w^*(t)$ in $t \in J'$, where $v^*(t)$, $w^*(t)$ are bounded and strongly measurable. Combining (4.3) with $v_n = Q_2(v_{n-1})$, let $C > v_0$, noticing $S(t) = e^{-Ct}T(t)$ $(t \geq 0)$ is an exponentially stable and positive C_0-semigroup in X, letting $n \to \infty$, by the Lebesgue dominated convergence theorem, we have $v^*(t) = Q_2(v^*)(t) \in PC(J, X)$.

Similarly, we prove that $w^*(t) \in PC(J, X)$ and $w^*(t) = Q_2(w^*)(t)$.

By (4.10), we know $v_0(t) \leq v^*(t) \leq w^*(t) \leq w_0(t)$.

Similar to the proof of Theorem 4.1, we know that $\underline{u}(t)$ and $\overline{u}(t)$ are the minimal and maximal ω-periodic mild solutions of IPBVP (1.2) between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0, respectively. This completes the proof of Theorem 4.3. □

Corollary 4.4 *Let X be an ordered and weakly sequentially complete Banach space, whose positive cone K is normal, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)$ $(t \geq 0)$ in X. $f \in C(J \times X, X)$ and f is ω-periodic about t, $I_k \in C(X, X)$, $k = 1, 2, \dots, p$. Assume that IPBVP* (1.2) *has lower and upper solutions v_0 and w_0 with $v_0(t) \leq w_0(t)$ $(t \in J)$, and conditions* (P_1) *and* (P_2) *are satisfied, then IPBVP* (1.2) *has minimal and maximal ω-periodic mild solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0.*

Proof In an ordered and weakly sequentially complete Banach space, the normal cone K is regular. Then the proof is complete. □

Next, we discuss the existence of the ω-periodic mild solutions of IPBVP (1.2), when the lower and upper solutions of IPBVP (1.2) do not exist.

Theorem 4.5 *Let X be an ordered Banach space, whose positive cone K is normal, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate an exponentially stable, compact and positive C_0-semigroup $T(t)$ $(t \geq 0)$ in X. $f \in C(J \times X, X)$ and f is ω-periodic about t, $I_k \in C(X, X)$, $k = 1, 2, \dots, p$ satisfy* (P_1) *and* (P_2) *and the following conditions*:

(P_3) $\exists 0 < a < -v_0$ *(v_0 is the growth index of $T(t)$), $h \in PC(J, X)$, $h \geq \theta$, such that*

$$-ax - h(t) \leq f(t, -x), \qquad f(t, x) \leq ax + h(t).$$

(P_4) *Let $a_k < 1$, $\frac{1}{\omega} \sum_{k=1}^{p} \ln(1 - a_k) < -a - v_0$, $e_k \geq \theta$, such that*

$$a_k x - e_k \leq I_k(-x), \qquad I_k(x) \leq -a_k x + e_k.$$

Then IPBVP (1.2) *has minimal and maximal ω-periodic mild solutions, which can be obtained by monotone iterative sequences.*

Proof For $0 < a < -\nu_0$, then $-(A - aI)$ generates an exponentially stable and positive C_0-semigroup $e^{at}T(t)$ $(t \geq 0)$, whose growth index is $a + \nu_0$. For $\frac{1}{\omega}\sum_{k=1}^{p}\ln(1-a_k) < -a-\nu_0$, $h(t) \geq \theta$, $e_k \geq \theta$, by Theorem 3.1, then the periodic boundary value problem of linear impulsive evolution equation (LIPBVP) in X,

$$\begin{cases} u'(t) + Au(t) - au(t) = h(t), & t \in J, t \neq t_k, \\ \Delta u|_{t=t_k} + a_k u(t_k) = e_k, & k = 1,2,\ldots,p, \\ u(0) = u(\omega) \end{cases}$$

has a unique positive solution $u^* \geq \theta$. Let $v_0 = -u^*$, $w_0 = u^*$, by conditions (P_1), (P_2), (P_3) and (P_4), we get

$$\begin{cases} v_0'(t) + Av_0(t) = av_0(t) - h(t) \leq f(t, v_0(t)), & t \in J, t \neq t_k, \\ \Delta v_0|_{t=t_k} = -a_k v_0(t_k) - e_k \leq I_k(v_0(t_k)), & k = 1,2,\ldots,p, \\ u(0) \leq u(\omega) \end{cases}$$

and

$$\begin{cases} w_0'(t) + Aw_0(t) = aw_0(t) + h(t) \geq f(t, w_0(t)), & t \in J, t \neq t_k, \\ \Delta w_0|_{t=t_k} = -a_k w_0(t_k) + e_k \geq I_k(w_0(t_k)), & k = 1,2,\ldots,p, \\ u(0) \geq u(\omega). \end{cases}$$

So, we showed that v_0 and w_0 are a lower solution and an upper solution of IPBVP (1.2). By Theorem 4.1, our conclusion holds. Then the proof is complete. □

Corollary 4.6 *Let X be an ordered Banach space, whose positive cone K is regular, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)$ $(t \geq 0)$ in X. $f \in C(J \times X, X)$ and f is ω-periodic about t, $I_k \in C(X,X)$, $k = 1,2,\ldots,p$. If conditions* (P_1), (P_2), (P_3) *and* (P_4) *are satisfied, then IPBVP* (1.2) *has minimal and maximal ω-periodic mild solutions, which can be obtained by monotone iterative sequences.*

Corollary 4.7 *Let X be an ordered and weakly sequentially complete Banach space, whose positive cone K is normal, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)$ $(t \geq 0)$ in X. $f \in C(J \times X, X)$ and f is ω-periodic about t, $I_k \in C(X,X)$, $k = 1,2,\ldots,p$. If conditions* (P_1), (P_2), (P_3) *and* (P_4) *are satisfied, then IPBVP* (1.2) *has minimal and maximal ω-periodic mild solutions, which can be obtained by monotone iterative sequences.*

5 Example

Example 5.1 In order to apply our results, we consider the following impulsive parabolic partial differential equation:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) + A(x,D)u(x,t) = f(x,t,u(x,t)), & x \in \Omega, t \in J, t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(x,t_k)), & x \in \Omega, k = 1,2,\ldots,m, \\ Bu = 0, & (x,t) \in \partial\Omega \times J, \\ u(x,0) = u(x,\omega), \end{cases} \tag{5.1}$$

where $J = [0,\omega]$, $0 < t_1 < t_2 < \cdots < t_m < \omega$, $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$, $J'' = J \setminus \{0, t_1, t_2, \ldots, t_m\}$, integer $N > 1$, $\Omega \in \mathbf{R}^N$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$,

$$A(x,D) = -\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x)\frac{\partial^2}{\partial x_i\,\partial y_j} + \sum_{i=1}^{N} a_i(x)\frac{\partial}{\partial x_i} + a_0(x)$$

is a strongly elliptic operator of second order, coefficient functions $a_{ij}(x)$, $a_i(x)$ and $a_0(x)$ are Hölder continuous in Ω, $Bu = b_0(x)u + \delta\frac{\partial u}{\partial n}$ is a regular boundary operator on $\partial\Omega$, $f : \overline{\Omega} \times J \times \mathbf{R} \to \mathbf{R}$ is continuous, $I_k : \mathbf{R} \to \mathbf{R}$ are also continuous, $k = 1, 2, \ldots, m$.

Let $X = L^p(\Omega)$ with $p > N + 2$, $K = \{u \in L^p(\Omega) \mid u(x) \geq 0 \text{ a.e. } x \in \Omega\}$, and define the operator A as follows:

$$D(A) = \{u \in W^{2,p}(\Omega) \mid Bu = 0\}, \qquad Au = A(x,D)u.$$

We know that X is a Banach space, K is a regular cone of X, and $-A$ generates a positive and analytic C_0-semigroup $T(t)$ $(t \geq 0)$ in X (see [29]). Define $u(t) = u(\cdot,t)$, $f(t,u(t)) = f(\cdot,t,u(\cdot,t))$, $I_k(u(t_k)) = I_k(u(\cdot,t_k))$, then system (5.1) can be reformulated as IPBVP (1.2) in X. We assume that the following conditions hold:

(i) Let $f(x,t,0,) \geq 0$, $I_k(0) \geq 0$, $u(x,\omega) \geq u(x,0) \geq 0$, $x \in \Omega$.

(ii) There exist $w = w(x,t) \in PC(J,X) \cap C^{2,1}$ and $w(x,t) \geq 0$, $x \in \Omega$, $t \in J_k$ such that

$$\begin{cases} \frac{\partial w}{\partial t} + A(x,D)w \geq f(x,t,w), & x \in \Omega, t \in J, t \neq t_k, \\ \Delta w|_{t=t_k} \geq I_k(w(x,t_k)), & x \in \Omega, k = 1,2,\ldots,m, \\ Bw = 0, & (x,t) \in \partial\Omega \times J, \\ w(x,0) \geq w(x,\omega). \end{cases}$$

(iii) The partial derivative $f'_u(x,t,u)$ is continuous on any bounded domain.

(iv) For any $u_1, u_2 \in [0, w(x,t)]$ with $u_1 \leq u_2$, there exists $0 < N_k < 1$; for any $x \in \Omega$, $k = 1, 2, \ldots, m$, we have

$$I_k(u_2(x,t_k)) - I_k(u_1(x,t_k)) \geq -N_k(u_2(x,t_k) - u_1(x,t_k)).$$

Theorem 5.2 *If assumptions* (i), (ii), (iii) *and* (iv) *are satisfied, then the impulsive parabolic partial differential equation* (5.1) *has minimal and maximal mild solutions between* 0 *and* $w(x,t)$, *which can be obtained by a monotone iterative procedure starting from* 0 *and* $w(x,t)$, *respectively.*

Proof From assumptions (i) and (ii) we know that 0 and $w(x,t)$ are lower and upper solutions of IBPVP (5.1), respectively. (iii) implies that condition (P_1) is satisfied. (iv) implies that condition (P_2) is satisfied. So, by Theorem 4.3, we have the result. Then the proof is complete. □

6 Conclusions

In this paper, we have discussed the existence of ω-periodic mild solutions for the impulsive evolution equation by means of the perturbation method and the mixed monotone iterative technique under the impulsive functions satisfying quasimonotonicity. The main result (Theorem 4.1) is new and the following results appear as its special cases:

(i) If we take $A = 0$ in (1.2), we obtain the results for first-order periodic boundary problem for impulsive ordinary differential equations.
(ii) If $I_k(u(t_k)) = 0$, $k = 1, 2, \ldots, p$, in (1.2), then Theorem 4.1 in this paper is Theorem 3.1 in [30].
(iii) If $N_k = 0$ in condition (P_2), then Theorem 4.1 in this paper is Theorem 1 in [27].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors contributed to each part of this study equally and approved the final version of this manuscript.

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