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Research Article

Strong Convergence by a Hybrid Algorithm for Finding a Common Fixed Point of Lipschitz Pseudocontraction and Strict Pseudocontraction in Hilbert Spaces

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We prove a strong convergence theorem by using a hybrid algorithm in order to find a common fixed point of Lipschitz pseudocontraction and κ-strict pseudocontraction in Hilbert spaces. Our results extend the recent ones announced by Yao et al. (2009) and many others.

1. Introduction

Let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $T : C \to C$. Recall that T is said to be a pseudocontraction if

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2 \tag{1.1}$$

is equivalent to

$$\langle x - y, (I - T)x - (I - T)y \rangle \geqslant 0, \tag{1.2}$$

for all $x, y \in C$, and T is said to be a strict pseudocontraction if there exists a constant $0 \leqslant \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \tag{1.3}$$

for all $x, y \in C$. For the second case, we say that T is a κ-strict pseudocontraction. We use $F(T)$ to denote the set of fixed points of T.

The class of strict pseudocontractions extend the class of nonexpansive mapping. (A mapping T is said to be nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$, for all $x, y \in C$) that is, T is nonexpansive if and only if T is a 0-strict pseudocontraction. The pseudocontractive mapping includes the strict pseudocontractive mapping.

Iterative methods for finding fixed points of nonexpansive mappings are an important topic in the theory of nonexpansive mappings and have wide applications in a number of applied areas, such as the convex feasibility problem [1–4], the split feasibility problem [5–7] and image recovery and signal processing [3, 8, 9], and so forth. However, the Picard sequence $\{T^n x\}_{n=0}^{\infty}$ often fails to converge even in the weak topology. Thus, averaged iterations prevail. The Mann iteration [10] is one of the types and is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geqslant 0, \tag{1.4}$$

where $x_0 \in C$ is chosen arbitrarily and $\{\alpha_n\} \subset [0, 1]$. Reich [11] proved that if E is a uniformly convex Banach space with a Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.4) converges weakly to a fixed point of T. However, we note that Mann iterations have only weak convergence even in a Hilbert space (see e.g., [12]). From a practical point of view, strict pseudocontractions have more powerful applications than nonexpansive mappings do in solving inverse problems (see [13]). Therefore, it is important to develop theory of iterative methods for strict pseudocontractions. Indeed, Browder and Petryshyn [14] prove that if the sequence $\{x_n\}$ is generated by the following:

$$x_{n+1} = \alpha x_n + (1 - \alpha) T x_n, \quad n \geqslant 0, \tag{1.5}$$

for any starting point $x_0 \in C$, α is a constant such that $\kappa < \alpha < 1$, $\{x_n\}$ converges weakly to a fixed point of strict pseudocontraction. Marino and Xu [15] extended the result of Browder and Petryshyn [14] to Mann iteration (1.4); they proved $\{x_n\}$ converges weakly to a fixed point of T, provided the control sequence $\{\alpha_n\}$ satisfies the conditions that $\kappa < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty}(\alpha_n - k)(1 - \alpha_n) = \infty$.

The well-known strong convergence theorem for pseudocontractive mapping was proved by Ishikawa [16] in 1974. More precisely, he got the following theorem.

Theorem 1.1 (see [16]). *Let C be a convex compact subset of a Hilbert space H and let $T : C \to C$ be a Lipschitzian pseudocontractive mapping. For any $x_1 \in C$, suppose the sequence $\{x_n\}$ is defined by*

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n) x_n + \beta_n T x_n, \quad n \geqslant 1, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $[0, 1]$ satisfying

(i) $\alpha_n \leqslant \beta_n$, $n \geqslant 1$,

(ii) $\lim_{n \to \infty} \beta_n = 0$,

(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 1.2. (i) Since $0 \leqslant \alpha_n \leqslant \beta_n \leqslant 1$, $n \geqslant 1$ and $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, the iterative sequence (1.6) could not be reduced to a Mann iterative sequence (1.4). Therefore, the iterative sequence (1.6) has some particular cases.

(ii) The iterative sequence (1.6) is usually called the Ishikawa iterative sequence.

(iii) Chidume and Mutangadura [17] gave an example to show that the Mann iterative sequence failed to be convergent to a fixed point of Lipschitzian pseudocontractive mapping.

In an infinite-dimensional Hilbert spaces, Mann and Ishikawa's iteration algorithms have only weak convergence, in general, even for nonexpansive mapping. In order to obtain a strong convergence theorem for the Mann iteration method (1.4) to nonexpansive mapping, Nakajo and Takahashi [18] modified (1.4) by employing two closed convex sets that are created in order to form the sequence via metric projection so that strong convergence is guaranteed. Later, it is often referred as the hybrid algorithm or the CQ algorithm. After that the hybrid algorithm have been studied extensively by many authors (see e.g., [19–23]). Particularly, Martinez-Yanes and Xu [24] and Plubtieng and Ungchittrakool [20] extended the same results of Nakajo and Takahashi [18] to the Ishikawa iteration process. In 2007, Marino and Xu [15] further generalized the hybrid algorithm from nonexpansive mappings to strict pseudocontractive mappings. In 2008, Zhou [25] established the hybrid algorithm for pseudocontractive mapping in the case of the Ishikawa iteration process.

Recently, Yao et al. [26] introduced the hybrid iterative algorithm which just involved one closed convex set for pseudocontractive mapping in Hilbert spaces as follows.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudocontraction. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C as follows.

$$\begin{gathered} y_n = (1-\alpha_n)x_n + \alpha_n T z_n, \\ C_{n+1} = \left\{ v \in C_n : \left\| \alpha_n (I-T) y_n \right\|^2 \leqslant 2\alpha_n \left\langle x_n - v, (I-T)y_n \right\rangle \right\}, \\ x_{n+1} = P_{C_{n+1}}(x_0). \end{gathered} \tag{1.7}$$

Theorem 1.3 (see [26]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudocontraction such that $F(T) \neq \emptyset$. Assume the sequence $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1/(L+1))$. Then the sequence $\{x_n\}$ generated by* (1.7) *converges strongly to $P_{F(T)}(x_0)$.*

Very recently, Tang et al. [27] generalized the hybrid algorithm (1.7) in the case of the Ishikawa iterative precess as follows:

$$\begin{gathered} y_n = (1-\alpha_n)x_n + \alpha_n T z_n, \\ z_n = \left(1-\beta_n\right)x_n + \beta_n T x_n, \\ C_{n+1} = \Big\{ v \in C_n : \left\| \alpha_n (I-T) y_n \right\|^2 \leqslant 2\alpha_n \left\langle x_n - v, (I-T)y_n \right\rangle \\ +2\alpha_n \beta_n L \| x_n - T x_n \| \left\| y_n - x_n + \alpha_n (I-T) y_n \right\| \Big\}, \\ x_{n+1} = P_{C_{n+1}}(x_0). \end{gathered} \tag{1.8}$$

Under some appropriate conditions of $\{\alpha_n\}$ and $\{\beta_n\}$, they proved that (1.8) converges strongly to $P_{F(T)}(x_0)$.

Motivated and inspired by the above works, in this paper, we generalize (1.7) to the Ishikawa iterative process in the case of finding the common fixed point of Lipschitz pseudocontraction and κ-strict pseudocontraction. More precisely, we provide some applications of the main theorem to find the common zero point of the Lipshitz monotone mapping and γ-inverse strongly monotone mapping in Hilbert spaces.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C x\| \leqslant \|x - y\|, \quad \forall y \in C, \tag{2.1}$$

where P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping. It is also known that H satisfies Opial's condition, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.2}$$

holds for every $y \in H$ with $y \neq x$.

For a given sequence $\{x_n\} \subset C$, let $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denote the weak ω-limit set of $\{x_n\}$.

Now we collect some Lemmas which will be used in the proof of the main result in the next section. We note that Lemmas 2.1 and 2.2 are well known.

Lemma 2.1. *Let H be a real Hilbert space. There holds the following identities:*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$, *for all* $x, y \in H$,

(ii) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2$, *for all* $x, y \in H$ *and* $\lambda \in [0,1]$.

Lemma 2.2. *Let C be a closed convex subset of real Hilbert space H. Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if there holds the relation*

$$\langle x - z, y - z\rangle \leqslant 0, \quad \forall y \in C. \tag{2.3}$$

Proposition 2.3 (see [15, Proposition 2.1]). *Assume C is a closed convex subset of a Hilbert space H; let $T : C \to C$ be a self-mapping of C. If T is a κ-strict pseudocontraction, then T satisfies the Lipschitz condition*

$$\|Tx - Ty\| \leqslant \frac{1+\kappa}{1-\kappa}\|x - y\|, \quad \forall x, y \in C. \tag{2.4}$$

Lemma 2.4 (see [28]). *Let H be a real Hilbert space, let C be a closed convex subset of H, and let $T : C \to C$ be a continuous pseudocontractive mapping, then*

(i) $F(T)$ *is closed convex subset of* C,

(ii) $I - T$ *is demiclosed at zero, that is, if* $\{x_n\}$ *is a sequence in* C *such that* $x_n \rightharpoonup z$ *and* $(I-T)x_n \to 0$, *then* $(I-T)z = 0$.

Lemma 2.5 (see [24]). *Let* C *be a closed convex subset of* H. *Let* $\{x_n\}$ *be a sequence in* H, *and let* $u \in H$. *Let* $q = P_C u$. *If* $\{x_n\}$ *is such that* $\omega_w(x_n) \subset C$ *and satisfies the condition*

$$\|x_n - u\| \leqslant \|u - q\|, \quad \forall n, \tag{2.5}$$

then $x_n \to q$.

3. Main Result

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, *let* $T : C \to C$ *be* L_T*-Lipschitz pseudocontraction, and let* $S : C \to C$ *be* κ*-strict pseudocontraction with* $\widetilde{F} := F(S) \cap F(T) \neq \emptyset$. *Let* $x_0 \in H$. *For* $C_1 = C$ *and* $x_1 = P_{C_1}(x_0)$, *define a sequence* $\{x_n\}$ *of* C *as follows:*

$$\begin{aligned}
y_n &= (1-\alpha_n)x_n + \alpha_n T z_n, \\
z_n &= (1-\beta_n)x_n + \beta_n S x_n, \\
C_{n+1} &= \Big\{ v \in C_n : \|\alpha_n (I-T)y_n\|^2 + (1-\kappa)\|(I-S)x_n\|^2 \\
&\leqslant 2\alpha_n \langle x_n - v, (I-T)y_n \rangle + 2\langle x_n - v, (I-S)z_n + (I-S)x_n \rangle \\
&+ 2\alpha_n \beta_n L_T \|x_n - Sx_n\| \|y_n - x_n + \alpha_n (I-T)y_n\| \\
&+ \beta_n \left(\left(\frac{2\beta_n}{1-\kappa} \right)^2 - 1 \right) \|(I-S)x_n\|^2 \Big\}, \\
x_{n+1} &= P_{C_{n+1}}(x_0).
\end{aligned} \tag{3.1}$$

Assume the sequence $\{\alpha_n\}$, $\{\beta_n\}$ *be such that* $0 < a \leqslant \alpha_n \leqslant b < 1/(L_T + 1) < 1$ *and* $0 < \beta_n \leqslant 1$ *for all* $n \in \mathbb{N}$ *with* $\lim_{n\to\infty} \beta_n = 0$. *Then* $\{x_n\}$ *converges strongly to* $P_{\widetilde{F}}(x_0)$.

Proof. By Lemma 2.4(i), we see that $F(S)$ and $F(T)$ are closed and convex, then $\widetilde{F}$ is as well. Hence, $P_{\widetilde{F}}$ is well defined. Next, we will prove by induction that $\widetilde{F} \subset C_n$ for all $n \in \mathbb{N}$. Note that $\widetilde{F} \subset C = C_1$. Assume that $\widetilde{F} \subset C_k$ holds for $k \geqslant 1$. Let $p \in \widetilde{F}$, thus $p \in C_k$, and we observe that

$$\begin{aligned}
\left\|x_k - p - \alpha_k(I-T)y_k\right\|^2 &= \left\|x_k - p\right\|^2 - \left\|\alpha_k(I-T)y_k\right\|^2 \\
&\quad - 2\alpha_k\left\langle (I-T)y_k, x_k - p - \alpha_k(I-T)y_k\right\rangle \\
&= \left\|x_k - p\right\|^2 - \left\|\alpha_k(I-T)y_k\right\|^2 \\
&\quad - 2\alpha_k\left\langle (I-T)y_k - (I-T)p, y_k - p\right\rangle \\
&\quad - 2\alpha_k\left\langle (I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\right\rangle \\
&\leqslant \left\|x_k - p\right\|^2 - \left\|\alpha_k(I-T)y_k\right\|^2 \\
&\quad - 2\alpha_k\left\langle (I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\right\rangle \\
&= \left\|x_k - p\right\|^2 - \left\|\left(x_k - y_k\right) + \left(y_k - x_k + \alpha_k(I-T)y_k\right)\right\|^2 \\
&\quad - 2\alpha_k\left\langle (I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\right\rangle \\
&= \left\|x_k - p\right\|^2 - \left\|x_k - y_k\right\|^2 - \left\|y_k - x_k + \alpha_k(I-T)y_k\right\|^2 \\
&\quad - 2\left\langle x_k - y_k, y_k - x_k + \alpha_k(I-T)y_k\right\rangle \\
&\quad - 2\alpha_k\left\langle (I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\right\rangle \\
&\leqslant \left\|x_k - p\right\|^2 - \left\|x_k - y_k\right\|^2 - \left\|y_k - x_k + \alpha_k(I-T)y_k\right\|^2 \\
&\quad + 2\left|\left\langle x_k - y_k - \alpha_k(I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\right\rangle\right|.
\end{aligned} \tag{3.2}$$

Consider the last term of (3.2), we obtain

$$\begin{aligned}
&\left|\left\langle x_k - y_k - \alpha_k(I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k\right\rangle\right| \\
&\quad = \alpha_k\left|\left\langle x_k - Tz_k - (I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k\right\rangle\right| \\
&\quad = \alpha_k\left|\left\langle x_k - Tx_k + Tx_k - Tz_k - (I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k\right\rangle\right| \\
&\quad = \alpha_k\left|\left\langle (I-T)x_k - (I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k\right\rangle + \left\langle Tx_k - Tz_k, y_k - x_k + \alpha_k(I-T)y_k\right\rangle\right| \\
&\quad \leqslant \alpha_k\left(L_T + 1\right)\left\|x_k - y_k\right\|\left\|y_k - x_k + \alpha_k(I-T)y_k\right\| \\
&\qquad + \alpha_k L_T\left\|x_k - z_k\right\|\left\|y_k - x_k + \alpha_k(I-T)y_k\right\|
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_k (L_T + 1) \|x_k - y_k\| \|y_k - x_k + \alpha_k (I - T) y_k\| \\
&\quad + \alpha_k \beta_k L_T \|x_k - S x_k\| \|y_k - x_k + \alpha_k (I - T) y_k\| \\
&\leqslant \frac{\alpha_k (L_T + 1)}{2} \left(\|x_k - y_k\|^2 + \|y_k - x_k + \alpha_k (I - T) y_k\|^2 \right) \\
&\quad + \alpha_k \beta_k L_T \|x_k - S x_k\| \|y_k - x_k + \alpha_k (I - T) y_k\|.
\end{aligned}
\tag{3.3}
$$

Substituting (3.3) into (3.2), we obtain

$$
\begin{aligned}
\|x_k - p - \alpha_k (I - T) y_k\|^2 &\leqslant \|x_k - p\|^2 - \|x_k - y_k\|^2 - \|y_k - x_k + \alpha_k (I - T) y_k\|^2 \\
&\quad + \alpha_k (L_T + 1) \left(\|x_k - y_k\|^2 + \|y_k - x_k + \alpha_k (I - T) y_k\|^2 \right) \\
&\quad + 2\alpha_k \beta_k L_T \|x_k - S x_k\| \|y_k - x_k + \alpha_k (I - T) y_k\| \\
&\leqslant \|x_k - p\|^2 + 2\alpha_k \beta_k L_T \|x_k - S x_k\| \|y_k - x_k + \alpha_k (I - T) y_k\|.
\end{aligned}
\tag{3.4}
$$

Notice that

$$
\|x_k - p - \alpha_k (I - T) y_k\|^2 = \|x_k - p\|^2 - 2\alpha_k \langle x_k - p, (I - T) y_k \rangle + \|\alpha_k (I - T) y_k\|^2. \tag{3.5}
$$

Therefore, from (3.4) and (3.5), we get

$$
\|\alpha_k (I - T) y_k\|^2 \leqslant 2\alpha_k \langle x_k - p, (I - T) y_k \rangle + 2\alpha_k \beta_k L_T \|x_k - S x_k\| \|y_k - x_k + \alpha_k (I - T) y_k\|. \tag{3.6}
$$

On the other hand, we found that

$$
\begin{aligned}
\|x_k - p - \beta_k (I - S) z_k\|^2 &= \|x_k - p\|^2 - \|\beta_k (I - S) z_k\|^2 - 2\beta_k \langle (I - S) z_k, x_k - p - \beta_k (I - S) z_k \rangle \\
&= \|x_k - p\|^2 - \|\beta_k (I - S) z_k\|^2 - 2\beta_k \langle (I - S) z_k - (I - S) p, z_k - p \rangle \\
&\quad - 2\beta_k \langle (I - S) z_k, x_k - z_k - \beta_k (I - S) z_k \rangle
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \left\|x_k - p\right\|^2 - \left\|\beta_k (I-S) z_k\right\|^2 - 2\left\langle \beta_k (I-S) z_k, x_k - z_k - \beta_k (I-S) z_k \right\rangle \\
&= \left\|x_k - p\right\|^2 - \left\|\beta_k (I-S) z_k\right\|^2 \\
&\quad + \left(\left\|\beta_k (I-S) z_k\right\|^2 - \|x_k - z_k\|^2 + \left\|x_k - z_k - \beta_k (I-S) z_k\right\|^2 \right) \\
&= \left\|(x_k - z_k) + (z_k - p)\right\|^2 - \|x_k - z_k\|^2 + \left\|\beta_k (I-S) x_k - \beta_k (I-S) z_k\right\|^2 \\
&= \|x_k - z_k\|^2 + 2\left\langle x_k - z_k, z_k - p \right\rangle + \left\|z_k - p\right\|^2 - \|x_k - z_k\|^2 \\
&\quad + \left\|\beta_k (I-S) x_k - \beta_k (I-S) z_k\right\|^2 \\
&= 2\left\langle x_k - z_k, (z_k - x_k) + (x_k - p) \right\rangle + \left\|(1-\beta_k)(x_k - p) + \beta_k (S x_k - p)\right\|^2 \\
&\quad + \left\|\beta_k (I-S) x_k - \beta_k (I-S) z_k\right\|^2 \\
&\leqslant 2\left\langle x_k - p, \beta_k (I-S) x_k \right\rangle + (1-\beta_k)\left\|x_k - p\right\|^2 + \beta_k \left\|S x_k - p\right\|^2 \\
&\quad - \beta_k (1-\beta_k)\|x_k - S x_k\|^2 - 2\beta_k^2 \|(I-S) x_k\|^2 \\
&\quad + \beta_k^2 \left(\frac{1+\kappa}{1-\kappa} + 1 \right)^2 \|x_k - z_k\|^2 \\
&\leqslant 2\left\langle x_k - p, \beta_k (I-S) x_k \right\rangle + (1-\beta_k)\left\|x_k - p\right\|^2 + \beta_k \left\|x_k - p\right\|^2 \\
&\quad + \beta_k \kappa \|(I-S) x_k\|^2 - \beta_k (1-\beta_k)\|(I-S) x_k\|^2 - 2\beta_k^2 \|(I-S) x_k\|^2 \\
&\quad + \beta_k^4 \left(\frac{2}{1-\kappa} \right)^2 \|(I-S) x_k\|^2 \\
&= 2\left\langle x_k - p, \beta_k (I-S) x_k \right\rangle + \left\|x_k - p\right\|^2 - \beta_k (1-\kappa)\|(I-S) x_k\|^2 \\
&\quad - \beta_k^2 \|(I-S) x_k\|^2 + \beta_k^4 \left(\frac{2}{1-\kappa} \right)^2 \|(I-S) x_k\|^2 .
\end{aligned}
\tag{3.7}
$$

Notice that

$$
\left\|x_k - p - \beta_k (I-S) z_k\right\|^2 = \left\|x_k - p\right\|^2 - 2\beta_k \left\langle x_k - p, (I-S) z_k \right\rangle + \beta_k^2 \|(I-S) z_k\|^2 . \tag{3.8}
$$

Combining (3.7) and (3.8) and then it implies that

$$
\beta_k (1-\kappa)\|x_k - S x_k\|^2 \leqslant 2\beta_k \left\langle x_k - p, (I-S) z_k + (I-S) x_k \right\rangle + \beta_k^2 \left(\left(\frac{2\beta_k}{1-\kappa} \right)^2 - 1 \right) \|(I-S) x_k\|^2 . \tag{3.9}
$$

Since $\beta_n > 0$ for all n, so we get

$$(1-\kappa)\|x_k - Sx_k\|^2 \leqslant 2\langle x_k - p, (I-S)z_k + (I-S)x_k\rangle + \beta_k\left(\left(\frac{2\beta_k}{1-\kappa}\right)^2 - 1\right)\|(I-S)x_k\|^2. \tag{3.10}$$

It follows from (3.6) and (3.10) that we obtain

$$\begin{aligned}
&\left\|\alpha_k(I-T)y_k\right\|^2 + (1-\kappa)\|(I-S)x_k\|^2 \\
&\quad\leqslant 2\alpha_k\langle x_k - v, (I-T)y_k\rangle + 2\langle x_k - v, (I-S)z_k + (I-S)x_k\rangle \\
&\qquad + 2\alpha_k\beta_k L_T\|x_k - Sx_k\|\left\|y_k - x_k + \alpha_k(I-T)y_k\right\| + \beta_k\left(\left(\frac{2\beta_k}{1-\kappa}\right)^2 - 1\right)\|(I-S)x_k\|^2.
\end{aligned} \tag{3.11}$$

Therefore, $p \in C_{k+1}$. By mathematical induction, we have $\widetilde{F} \subset C_n$ for all $n \in \mathbb{N}$. It is easy to check that C_n is closed and convex, and then $\{x_n\}$ is well defined. From $x_n = P_{C_n}(x_0)$, we have $\langle x_0 - x_n, x_n - y\rangle \geqslant 0$ for all $y \in C_n$. Using $\widetilde{F} \subset C_n$, we also have $\langle x_0 - x_n, x_n - u\rangle \geqslant 0$ for all $u \in \widetilde{F}$. So, for $u \in \widetilde{F}$, we have

$$\begin{aligned}
0 &\leqslant \langle x_0 - x_n, x_n - u\rangle = \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle \\
&= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - u\rangle \\
&\leqslant -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - u\|.
\end{aligned} \tag{3.12}$$

Hence, $\|x_0 - x_n\| \leqslant \|x_0 - u\|$, for all $u \in \widetilde{F}$. In particular,

$$\|x_0 - x_n\| \leqslant \left\|x_0 - q\right\|, \quad \text{where } q = P_{\widetilde{F}}(x_0). \tag{3.13}$$

This implies that $\{x_n\}$ is bounded, and then $\{y_n\}$, $\{Ty_n\}$, $\{z_n\}$, $\{Sz_n\}$, and $\{Sx_n\}$ are as well.

From $x_n = P_{C_n}(x_0)$ and $x_{n+1} = P_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geqslant 0. \tag{3.14}$$

Hence

$$\begin{aligned}
0 &\leqslant \langle x_0 - x_n, x_n - x_{n+1}\rangle = \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\
&= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\
&\leqslant -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|,
\end{aligned} \tag{3.15}$$

and; therefore,

$$\|x_0 - x_n\| \leqslant \|x_0 - x_{n+1}\|, \tag{3.16}$$

which implies that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. From Lemma 2.1 and (3.14), we obtain

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\leqslant \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \longrightarrow 0.\end{aligned} \tag{3.17}$$

Since $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\begin{aligned}&\left\|\alpha_n(I-T)y_n\right\|^2 + (1-\kappa)\|(I-S)x_n\|^2 \\ &\quad\leqslant 2\alpha_n\left\langle x_n - x_{n+1}, (I-T)y_n\right\rangle + 2\langle x_n - x_{n+1}, (I-S)z_n + (I-S)x_n\rangle \\ &\qquad + 2\alpha_n\beta_n L_T\|x_n - Sx_n\|\left\|y_n - x_n + \alpha_n(I-T)y_n\right\| \\ &\qquad + \beta_n\left(\left(\frac{2\beta_n}{1-\kappa}\right)^2 - 1\right)\|(I-S)x_n\|^2 \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{3.18}$$

therefore, we obtain

$$\left\|y_n - Ty_n\right\| \longrightarrow 0, \qquad \|x_n - Sx_n\| \longrightarrow 0. \tag{3.19}$$

We note that

$$\begin{aligned}\|x_n - Tx_n\| &\leqslant \left\|x_n - y_n\right\| + \left\|y_n - Ty_n\right\| + \left\|Ty_n - Tx_n\right\| \\ &\leqslant (L_T + 1)\left\|x_n - y_n\right\| + \left\|y_n - Ty_n\right\| \\ &\leqslant \alpha_n(L_T + 1)\|x_n - Tz_n\| + \left\|y_n - Ty_n\right\| \\ &\leqslant \alpha_n(L_T + 1)\|x_n - Tx_n\| + \alpha_n(L_T + 1)\|Tx_n - Tz_n\| + \left\|y_n - Ty_n\right\| \\ &\leqslant \alpha_n(L_T + 1)\|x_n - Tx_n\| + \alpha_n\beta_n L_T(L_T + 1)\|x_n - Sx_n\| + \left\|y_n - Ty_n\right\|,\end{aligned} \tag{3.20}$$

that is,

$$\|x_n - Tx_n\| \leqslant \frac{\alpha_n\beta_n L_T(L_T + 1)}{1 - \alpha_n(L_T + 1)}\|x_n - Sx_n\| + \frac{1}{1 - \alpha_n(L_T + 1)}\left\|y_n - Ty_n\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.21}$$

By Lemma 2.4(ii), $I - T$ and $I - S$ are demiclosed at zero. Together with the fact that $\{x_n\}$ is bounded, which guarantees that every weak limit point of $\{x_n\}$ is a fixed point of T

and S, that is $\omega_w(x_n) \subset F(T) \cap F(S) = \widetilde{F}$, therefore, by inequality (3.13) and Lemma 2.5, we know that $\{x_n\}$ converges strongly to $q = P_{\widetilde{F}}(x_0)$. This completes the proof. □

If $S = I$, then we obtain the following corollary.

Corollary 3.2 (Yao et al. [26, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be L-Lipschitz pseudocontraction such that $F(T) \neq \emptyset$. Assume the sequence $\{\alpha_n\}$ be such that $0 < a \leqslant \alpha_n \leqslant b < 1/(L+1) < 1$ for all n. Then the sequence $\{x_n\}$ generated by* (1.7) *converges strongly to $P_{F(T)}(x_0)$.*

If T and S are nonexpansive, then we also have the following corollary.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $S, T : C \to C$ be nonexpansive mappings. Suppose that $\widetilde{F} := F(S) \cap F(T) \neq \emptyset$. Assume the sequence $\{\alpha_n\}$ be such that $0 < a \leqslant \alpha_n \leqslant b < 1/2$ and $0 < \beta_n \leqslant 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} \beta_n = 0$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{aligned}
y_n &= (1-\alpha_n)x_n + \alpha_n T z_n, \\
z_n &= (1-\beta_n)x_n + \beta_n S x_n, \\
C_{n+1} &= \Big\{ v \in C_n : \big\|\alpha_n (I-T)y_n\big\|^2 + \|(I-S)x_n\|^2 \\
&\quad \leqslant 2\alpha_n \langle x_n - v, (I-T)y_n\rangle + 2\langle x_n - v, (I-S)z_n + (I-S)x_n\rangle \\
&\quad +2\alpha_n\beta_n \|x_n - Sx_n\| \big\|y_n - x_n + \alpha_n (I-T)y_n\big\| \\
&\quad +\beta_n\big(4\beta_n^2 - 1\big)\|(I-S)x_n\|^2 \Big\}, \\
x_{n+1} &= P_{C_{n+1}}(x_0).
\end{aligned} \tag{3.22}$$

Then $\{x_n\}$ converges strongly to $P_{\widetilde{F}}(x_0)$.

Recall that a mapping A is said to be monotone if $\langle x - y, Ax - Ay\rangle \geqslant 0$ for all $x, y \in H$ and inverse strongly monotone if there exists a real number $\gamma > 0$ such that $\langle x - y, Ax - Ay\rangle \geqslant \gamma\|Ax - Ay\|^2$ for all $x, y \in H$. For the second case, A is said to be γ-inverse strongly monotone. It follows immediately that if A is γ-inverse strongly monotone, then A is monotone and Lipschitz continuous, that is, $\|Ax - Ay\| \leqslant (1/\gamma)\|x - y\|$. It is well known (see e.g., [29]) that if A is monotone, then the solutions of the equation $Ax = 0$ correspond to the equilibrium points of some evolution systems. Therefore, it is important to focus on finding the zero point of monotone mappings. The pseudocontractive mapping and strictly pseudocontractive mapping are strongly related to the monotone mapping and inverse strongly monotone mapping, respectively. It is well known that

(i) A is monotone $\Leftrightarrow T := (I - A)$ is pseudocontractive,

(ii) A is inverse strongly monotone $\Leftrightarrow T := (I - A)$ is strictly pseudocontractive.

Indeed, for (ii), we notice that the following equality always holds in a real Hilbert space:

$$\big\|(I-A)x - (I-A)y\big\|^2 = \big\|x - y\big\|^2 + \big\|Ax - Ay\big\|^2 - 2\big\langle x - y, Ax - Ay\big\rangle, \quad \forall x, y \in H. \tag{3.23}$$

Without loss of generality, we can assume that $\gamma \in (0, 1/2]$, and then it yields

$$
\begin{aligned}
\langle x-y, Ax-Ay\rangle \geqslant \gamma\|Ax-Ay\|^2 & \\
&\Longleftrightarrow -2\langle x-y, Ax-Ay\rangle \leqslant -2\gamma\|Ax-Ay\|^2 \\
&\Longleftrightarrow \|(I-A)x-(I-A)y\|^2 \leqslant \|x-y\|^2 + (1-2\gamma)\|Ax-Ay\|^2 \\
&\qquad\qquad \text{(via (3.23))} \\
&\Longleftrightarrow \|Tx-Ty\|^2 \leqslant \|x-y\|^2 + \kappa\|(I-T)x-(I-T)y\|^2 \\
&\qquad\qquad (\text{where } T := (I-A),\ \kappa := 1-2\gamma).
\end{aligned}
\tag{3.24}
$$

Due to Theorem 3.1, we have the following corollary which generalize the corresponding results of Yao et al. [26].

Corollary 3.4. *Let $A : H \to H$ be L_A-Lipschitz monotone mapping and let $B : H \to H$ be an $\widehat{\gamma}$-inverse strongly monotone which $A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Assume the sequence $\{\alpha_n\}$ be such that $0 < a \leqslant \alpha_n \leqslant b < 1/(L_A + 2)$, $0 < \beta_n \leqslant 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty}\beta_n = 0$ and $\gamma \in (0, 1/2]$ such that $\widehat{\gamma} \geqslant \gamma$. Let $x_0 \in H$. For $C_1 = H$ and $x_1 = P_{C_1}(x_0) = x_0$, define a sequence $\{x_n\}$ as follows:*

$$
\begin{aligned}
y_n &= x_n - \alpha_n(x_n - z_n) - \alpha_n A z_n, \\
z_n &= x_n - \beta_n B x_n, \\
C_{n+1} &= \Big\{ v \in C_n : \|\alpha_n A y_n\|^2 + 2\gamma\|Bx_n\|^2 \\
&\quad \leqslant 2\alpha_n\langle x_n - v, Ay_n\rangle + 2\langle x_n - v, Bz_n + Bx_n\rangle \\
&\quad + 2\alpha_n\beta_n(L_A+1)\|Bx_n\|\|y_n - x_n + \alpha_n A y_n\| \\
&\quad + \beta_n\left(\left(\frac{\beta_n}{\gamma}\right)^2 - 1\right)\|Bx_n\|^2 \Big\}, \\
x_{n+1} &= P_{C_{n+1}}(x_0).
\end{aligned}
\tag{3.25}
$$

Then $\{x_n\}$ converges strongly to $P_{A^{-1}(0)\cap B^{-1}(0)}(x_0)$.

Proof. Let $T := (I - A)$ and let $S := (I - B)$. Then T and S are pseudocontractive and $(1 - 2\gamma)$-pseudocontractive, respectively. Moreover, T is also (L_A+1)-Lipschitz, and if we set $\kappa := 1-2\gamma$, S is also $((1-\gamma)/\gamma)$-Lipschitz, and then $(2/(1-\kappa))^2 = 1/\gamma^2$. Hence, it follows from Theorem 3.1 that we have the desired result. □

If $B = 0$ (zero mapping), then $z_n = x_n$ and $B^{-1}(0) = H$. So, we obtain the following corollary.

Corollary 3.5 (Yao et al. [26, Corollary 3.2]). *Let $A : H \to H$ be a L_A-Lipschitz monotone mapping for which $A^{-1}(0) \neq \emptyset$. Assume that the sequence $\{\alpha_n\}$ be as in Corollary 3.4. Then the sequence $\{x_n\}$ generated by*

$$y_n = x_n - \alpha_n A z_n,$$
$$C_{n+1} = \left\{ v \in C_n : \left\| \alpha_n A y_n \right\|^2 \leqslant 2\alpha_n \langle x_n - v, A y_n \rangle \right\}, \tag{3.26}$$
$$x_{n+1} = P_{C_{n+1}}(x_0)$$

strongly converges to $P_{A^{-1}(0)}(x_0)$.

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