Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2010, Article ID 148035, 17 pages
doi:10.1155/2010/148035

Research Article

Blowup Properties for a Semilinear Reaction-Diffusion System with Nonlinear Nonlocal Boundary Conditions

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Received 15 December 2009; Revised 1 April 2010; Accepted 26 April 2010

Academic Editor: Yoshikazu Giga

We investigate the blowup properties of the positive solutions for a semilinear reaction-diffusion system with nonlinear nonlocal boundary condition. We obtain some sufficient conditions for global existence and blowup by utilizing the method of subsolution and supersolution.

1. Introduction

In this paper, we deal with the following semilinear reaction-diffusion system with nonlinear nonlocal boundary conditions and nontrivial nonnegative continuous initial data:

$$
\begin{aligned}
&u_t = \Delta u + v^p, \quad v_t = \Delta v + u^q, \quad x \in \Omega,\ t > 0,\\
&u(x,t) = \int_\Omega \varphi(x,y) u^m(y,t)\,dy, \quad x \in \partial\Omega,\ t > 0,\\
&v(x,t) = \int_\Omega \psi(x,y) v^n(y,t)\,dy, \quad x \in \partial\Omega,\ t > 0,\\
&u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \overline{\Omega},
\end{aligned}
\tag{1.1}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ for $N \geq 1$ with a smooth boundary $\partial\Omega, p, q, m, n > 0$, the weight functions $\varphi(x,y)$ and $\psi(x,y)$ are nonnegative continuous defined in $\partial\Omega \times \overline{\Omega}$, and $\int_\Omega \varphi(x,y)dy, \int_\Omega \psi(x,y)dy > 0$ on $\partial\Omega$. Moreover, for $x \in \partial\Omega$, the initial data $u_0(x), v_0(x)$ satisfy the compatibility conditions $u_0(x) = \int_\Omega \varphi(x,y)u_0^m(y)dy$ and $v_0(x) = \int_\Omega \psi(x,y)v_0^n(y)dy$, respectively.

System (1.1) has been formulated from physical models arising in various fields of applied sciences. For example, it can be interpreted as a heat conduction problem with nonlocal nonlinear sources on the boundary of the material body (see [1, 2]). In this case, $u(x,t)$ and $v(x,t)$ represent the temperatures of the interacting components in the evolution processes.

The local (in time) existence of classical solutions of system (1.1) can be derived easily by standard parabolic theory. We say that the solution $(u(x,t), v(x,t))$ of problem (1.1) blows up in finite time if there exists a positive constant $T < \infty$ such that

$$\lim_{t\to T^-}\left(|u(\cdot,t)|_{L^\infty(\Omega)} + |v(\cdot,t)|_{L^\infty(\Omega)}\right) = +\infty. \tag{1.2}$$

In this case, T is called the blowup time. We say that the solution $(u(x,t), v(x,t))$ exists globally if

$$\sup_{t\in(0,+\infty)}\left(|u(\cdot,t)|_{L^\infty(\Omega)} + |v(\cdot,t)|_{L^\infty(\Omega)}\right) < +\infty. \tag{1.3}$$

In the last few years, a lot of efforts have been devoted to the study of properties of solutions to the semilinear parabolic equation $u_t = \Delta u + u^p$ with homogeneous Dirichlet boundary condition (see, e.g., the classical works in [3, 4]) and to the heat equation $u_t = \Delta u$ with Neumann boundary condition $\partial u/\partial \nu = u^p$ (see, e.g., [5]).

Blowup properties for the problem of systems

$$\begin{gathered}
u_t = \Delta u + v^p, \quad v_t = \Delta v + u^q, \quad x \in \Omega,\ t > 0,\\
u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega,\ t > 0,\\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \overline{\Omega},\\
u_t = \Delta u + v^p, \quad v_t = \Delta v + u^q, \quad x \in \Omega,\ t > 0,\\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0,\\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \overline{\Omega},\\
u_t = \Delta u, \quad v_t = \Delta v, \quad x \in \Omega,\ t > 0,\\
\frac{\partial u}{\partial \nu} = v^p, \quad \frac{\partial v}{\partial \nu} = u^q, \quad x \in \partial\Omega,\ t > 0,\\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \overline{\Omega}
\end{gathered} \tag{1.4}$$

have been studied very extensively over past years by many researchers. Here $p, q > 0$, ν denotes the unit outer normal vector on $\partial\Omega$. They were concerned with the existence, uniqueness, and regularity of solutions. Furthermore, they investigated the global and nonglobal existence, the blowup set, and the blowup rate for the above systems (see, e.g., [1, 2, 6–9] and the references cited therein). For blowup results for other parabolic systems, we refer the readers to [10–13] and the references cited therein.

Moreover, in recent years, many authors (see studies such as those in [14, 15] and the references cited therein) considered semilinear reaction-diffusion systems with nonlocal Dirichlet boundary conditions of the form

$$\begin{gathered} u_t = \Delta u + f(u,v), \quad v_t = \Delta v + g(u,v), \quad x \in \Omega,\ t > 0, \\ u = \int_\Omega \varphi(x,y)u(y,t)dy, \quad v = \int_\Omega \psi(x,y)v(y,t)dy, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \overline{\Omega}. \end{gathered} \tag{1.5}$$

They studied how the weight functions $\varphi(x,y)$ and $\psi(x,y)$ in the nonlocal boundary conditions affect the blowup properties of the solutions of (1.5).

However, reaction-diffusion problems coupled with nonlocal nonlinear boundary conditions, to our knowledge, have not been well studied. Recently, Gladkov and Kim [16] considered the following problem for a single semilinear heat equation:

$$\begin{gathered} u_t = \Delta u + c(x,t)u^p, \quad x \in \Omega,\ t > 0, \\ u(x,t) = \int_\Omega f(x,y,t)u^l(y,t)dy, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \end{gathered} \tag{1.6}$$

where $p, l > 0$. They obtained some criteria for the existence of the global solution as well as for blowup of the solution in finite time.

The main purpose of this paper is to understand how the reaction terms, the weight functions and the nonlinear terms in the boundary conditions affect the blowup properties for problem (1.1). We will show that the weight functions $\varphi(x,y)$, $\psi(x,y)$ and the nonlinear terms $u^m(y,t), v^n(y,t)$ in the boundary conditions of (1.1) play substantial roles in determining blowup or not of solutions.

Before starting the main results, we introduce some useful symbols. Throughout this paper, we let λ be the first eigenvalue of the eigenvalue problem

$$-\Delta\phi(x) = \lambda\phi, \quad x \in \Omega; \qquad \phi(x) = 0, \quad x \in \partial\Omega, \tag{1.7}$$

and $\phi(x)$ the corresponding eigenfunction with $\int_\Omega \phi(x)dx = 1$, $\phi(x) > 0$ in Ω. In addition, for convenience, we denote that $L = \sup_{\overline{\Omega}}\phi(x)$ and

$$M_1 = \sup_{\partial\Omega\times\overline{\Omega}} \varphi(x,y), \quad M_2 = \inf_{\partial\Omega\times\overline{\Omega}} \varphi(x,y), \quad K_1 = \sup_{\partial\Omega\times\overline{\Omega}} \psi(x,y), \quad K_2 = \inf_{\partial\Omega\times\overline{\Omega}} \psi(x,y). \tag{1.8}$$

The main results of this paper are stated as follows.

Theorem 1.1. *Assume that* $0 < pq \leq 1$ *and* $m,n \leq 1$. *Then the solution of problem* (1.1) *exists globally for any positive initial data.*

Theorem 1.2. *Assume that* $pq > 1$ *or* $m,n > 1$. *Then for any* $\varphi(x,y), \psi(x,y) > 0$, *the solution of problem* (1.1) *blows up in finite time for sufficiently large initial data.*

Theorem 1.3. *Assume that* $p > 1$, $q > 1$, $m > 1$, *and* $n > 1$. *Then for any nonnegative continuous* $\varphi(x,y)$ *and* $\psi(x,y)$, *the solution of problem* (1.1) *exists globally for sufficiently small initial data.*

Remark 1.4. When $p = q$, $m = n$, $\varphi(x,y) = \psi(x,y)$, and $u_0(x) = v_0(x)$, system (1.1) is then reduced to a single equation $u_t = \Delta u + u^p$ with nonlocal nonlinear boundary condition $u(x,t) = \int_\Omega \varphi(x,y)u^m(y,t)dy$. In this case, our above results are still true and consistent with those in [16].

The rest of this paper is organized as follows. In Section 2, we establish the comparison principle for problem (1.1). In Sections 3 and 4, we will give the proofs of Theorems 1.1 and 1.2, respectively. Finally, Theorem 1.3 will be proved in Section 5.

2. Preliminaries

In this section, we will give a suitable comparison principle for problem (1.1). Let $\Omega_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$, and $\overline{\Omega}_T = \overline{\Omega} \times [0,T)$. We begin with the precise definitions of a subsolution and supersolution of problem (1.1).

Definition 2.1. A pair of functions $(\underline{u},\underline{v}) \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T) \times C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$ is called a subsolution of problem (1.1) in Ω_T if

$$\begin{aligned}
&\underline{u}_t \leq \Delta\underline{u} + \underline{v}^p, \quad \underline{v}_t \leq \Delta\underline{v} + \underline{u}^q, \quad (x,t) \in \Omega_T,\\
&\underline{u}(x,t) \leq \int_\Omega \varphi(x,y)\underline{u}^m(y,t)dy, \quad (x,t) \in S_T,\\
&\underline{v}(x,t) \leq \int_\Omega \psi(x,y)\underline{v}^n(y,t)dy, \quad (x,t) \in S_T,\\
&\underline{u}(x,0) \leq u_0(x), \quad \underline{v}(x,0) \leq v_0(x), \quad x \in \overline{\Omega}.
\end{aligned} \tag{2.1}$$

Similarly, a pair of functions $(\overline{u},\overline{v}) \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T) \times C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$ is a supersolution of system (1.1) if the reversed inequalities hold in (2.1). We say that (u,v) is a solution of system (1.1) in Ω_T if it is both a subsolution and a supersolution of problem (1.1) in Ω_T.

Let $g_i(x,t), h_i(x,t) \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$, $\chi_i(x,y) \geq 0$ on $\partial\Omega \times \Omega$, $i = 1,2$. We first give some hypotheses as follows, which will be used in the sequel.

(H1) For $x \in \partial\Omega$, $y \in \Omega$, $t > 0$, $\chi_1(x,y)g_1^{m-1}(y,t)$, $\chi_1(x,y)h_1^{m-1}(y,t)$, $\chi_2(x,y)g_2^{n-1}(y,t)$, and $\chi_2(x,y)h_2^{n-1}(y,t)$ are nonnegative. Further, $\int_\Omega m\chi_1(x,y)g_1^{m-1}(y,t)dy \leq 1$,

$\int_\Omega m\chi_1(x,y)h_1^{m-1}(y,t)dy \le 1$, $\int_\Omega n\chi_2(x,y)g_2^{n-1}(y,t)dy \le 1$, and $\int_\Omega n\chi_2(x,y)h_2^{n-1}(y,t)dy \le 1$.

(H2) For $x \in \partial\Omega$, $y \in \Omega$, $t > 0$, there exists $M > 0$ such that $0 \le m\chi_1(x,y)g_1^{m-1}(y,t) \le M$, $0 \le m\chi_1(x,y)h_1^{m-1}(y,t) \le M$, $0 \le n\chi_2(x,y)g_2^{n-1}(y,t) \le M$, and $0 \le n\chi_2(x,y)h_2^{n-1}(y,t) \le M$.

Lemma 2.2. *Let (H1) hold, and $c_{ij}+d_{ij}$ $(i,j=1,2)$ be bounded in Ω_T and let $c_{ij}+d_{ij} \ge 0$ $(i \ne j,\ i,j=1,2)$. Further, assume that $w_i(x,t) \ge s_i(x,t)$ $(i=1,2)$. If $\chi_i(x,y) \ge 0$ on $\partial\Omega \times \Omega$; and $g_i, h_i \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$ $(i=1,2)$ satisfy*

$$\begin{gathered}
g_{1t} - \left(\sum_{k,l=1}^{n} a_{k,l}^{(1)} \frac{\partial^2 g_1}{\partial x_k \partial x_l} + \sum_{k=1}^{n} b_k^{(1)} \frac{\partial g_1}{\partial x_k}\right) \ge \sum_{i=1}^{2} c_{1i} g_i - \sum_{i=1}^{2} d_{1i} h_i + w_1(x,t), \quad (x,t) \in \Omega_T, \\
g_{2t} - \left(\sum_{k,l=1}^{n} a_{k,l}^{(2)} \frac{\partial^2 g_2}{\partial x_k \partial x_l} + \sum_{k=1}^{n} b_k^{(2)} \frac{\partial g_2}{\partial x_k}\right) \ge \sum_{i=1}^{2} c_{2i} g_i - \sum_{i=1}^{2} d_{2i} h_i + w_2(x,t), \quad (x,t) \in \Omega_T, \\
h_{1t} - \left(\sum_{k,l=1}^{n} a_{k,l}^{(1)} \frac{\partial^2 h_1}{\partial x_k \partial x_l} + \sum_{k=1}^{n} b_k^{(1)} \frac{\partial h_1}{\partial x_k}\right) \le \sum_{i=1}^{2} c_{1i} h_i - \sum_{i=1}^{2} d_{1i} g_i + s_1(x,t), \quad (x,t) \in \Omega_T, \\
h_{2t} - \left(\sum_{k,l=1}^{n} a_{k,l}^{(2)} \frac{\partial^2 h_2}{\partial x_k \partial x_l} + \sum_{k=1}^{n} b_k^{(2)} \frac{\partial h_2}{\partial x_k}\right) \le \sum_{i=1}^{2} c_{2i} h_i - \sum_{i=1}^{2} d_{2i} g_i + s_2(x,t), \quad (x,t) \in \Omega_T, \\
g_1(x,t) \ge \int_\Omega \chi_1(x,y) g_1^m(y,t)dy, \quad g_2(x,t) \ge \int_\Omega \chi_2(x,y) g_2^n(y,t)dy, \quad (x,t) \in S_T, \\
h_1(x,t) \le \int_\Omega \chi_1(x,y) h_1^m(y,t)dy, \quad h_2(x,t) \le \int_\Omega \chi_2(x,y) h_2^n(y,t)dy, \quad (x,t) \in S_T, \\
g_1(x,0) \ge h_1(x,0), \quad g_2(x,0) \ge h_2(x,0), \quad x \in \overline{\Omega}.
\end{gathered} \tag{2.2}$$

Then $(g_1(x,t), g_2(x,t)) \ge (h_1(x,t), h_2(x,t))$ in $\overline{\Omega}_T$.

Proof. For any given $\varepsilon > 0$, define

$$\tilde{g}_i = g_i + \varepsilon e^{\gamma t}, \quad \tilde{h}_i = h_i - \varepsilon e^{\gamma t}, \quad i = 1,2, \tag{2.3}$$

where

$$\gamma > \max\{c_{11} + c_{12} + d_{11} + d_{12}, c_{21} + c_{22} + d_{21} + d_{22}\}. \tag{2.4}$$

Then, a direct computation yields

$$\begin{aligned}
&\tilde{g}_{1t}-\left(\sum_{l,k=1}^{n}a_{l,k}^{(1)}\frac{\partial^2\tilde{g}_1}{\partial x_k\partial x_l}+\sum_{k=1}^{n}b_k^{(1)}\frac{\partial\tilde{g}_1}{\partial x_k}\right)>\sum_{i=1}^{2}c_{1i}\tilde{g}_i-\sum_{i=1}^{2}d_{1i}\tilde{h}_i+w_1(x,t),\quad (x,t)\in\Omega_T,\\
&\tilde{g}_{2t}-\left(\sum_{l,k=1}^{n}a_{l,k}^{(2)}\frac{\partial^2\tilde{g}_2}{\partial x_k\partial x_l}+\sum_{k=1}^{n}b_k^{(2)}\frac{\partial\tilde{g}_2}{\partial x_k}\right)>\sum_{i=1}^{2}c_{2i}\tilde{g}_i-\sum_{i=1}^{2}d_{2i}\tilde{h}_i+w_2(x,t),\quad (x,t)\in\Omega_T,\\
&\tilde{h}_{1t}-\left(\sum_{l,k=1}^{n}a_{l,k}^{(1)}\frac{\partial^2\tilde{h}_1}{\partial x_k\partial x_l}+\sum_{k=1}^{n}b_k^{(1)}\frac{\partial\tilde{h}_1}{\partial x_k}\right)<\sum_{i=1}^{2}c_{1i}\tilde{h}_i-\sum_{i=1}^{2}d_{1i}\tilde{g}_i+s_1(x,t),\quad (x,t)\in\Omega_T,\\
&\tilde{h}_{2t}-\left(\sum_{l,k=1}^{n}a_{l,k}^{(2)}\frac{\partial^2\tilde{h}_2}{\partial x_k\partial x_l}+\sum_{k=1}^{n}b_k^{(2)}\frac{\partial\tilde{h}_2}{\partial x_k}\right)<\sum_{i=1}^{2}c_{2i}\tilde{h}_i-\sum_{i=1}^{2}d_{2i}\tilde{g}_i+s_2(x,t),\quad (x,t)\in\Omega_T.
\end{aligned}\tag{2.5}$$

On the other hand, for $(x,t)\in S_T$, we have

$$\begin{aligned}
\tilde{g}_1(x,t)&\geq \varepsilon e^{\gamma t}+\int_\Omega \chi_1(x,y)g_1^m(y,t)dy\\
&=\int_\Omega \chi_1(x,y)\tilde{g}_1^m(y,t)dy+\varepsilon e^{\gamma t}-\int_\Omega \chi_1(x,y)\left(\tilde{g}_1^m(y,t)-g_1^m(y,t)\right)dy\\
&=\int_\Omega \chi_1(x,y)\tilde{g}_1^m(y,t)dy+\varepsilon e^{\gamma t}-\varepsilon e^{\gamma t}\int_\Omega m\chi_1(x,y)\theta_1^{m-1}(y,t)dy,
\end{aligned}\tag{2.6}$$

where θ_1 is an intermediate value between g_1 and $\tilde{g}_1$. From (H1), it follows that

$$\tilde{g}_1(x,t)>\int_\Omega \chi_1(x,y)\tilde{g}_1^m(y,t)dy\quad \text{for } (x,t)\in S_T.\tag{2.7}$$

Likewise, for any $(x,t)\in S_T$, we have

$$\begin{gathered}
\tilde{g}_2(x,t)>\int_\Omega \chi_2(x,y)\tilde{g}_2^n(y,t)dy,\\
\tilde{h}_1(x,t)<\int_\Omega \chi_1(x,y)\tilde{h}_1^m(y,t)dy,\qquad \tilde{h}_2(x,t)<\int_\Omega \chi_2(x,y)\tilde{h}_2^n(y,t)dy.
\end{gathered}\tag{2.8}$$

In addition, it is obvious that

$$\tilde{g}_1(x,0) - \varepsilon \geq \tilde{h}_1(x,0) + \varepsilon, \qquad \tilde{g}_2(x,0) - \varepsilon \geq \tilde{h}_2(x,0) + \varepsilon, \tag{2.9}$$

and hence, we know that

$$\tilde{g}_1(x,0) > \tilde{h}_1(x,0), \qquad \tilde{g}_2(x,0) > \tilde{h}_2(x,0). \tag{2.10}$$

Put

$$h(x,t) = \tilde{g}_1(x,t) - \tilde{g}_2(x,t). \tag{2.11}$$

Next, our task is to show that

$$\left(\eta_1(x,t), \eta_2(x,t)\right) > (0,0). \tag{2.12}$$

Actually, if (2.12) is true; then we can immediately get

$$g_i(x,t) + \varepsilon e^{\gamma t} \geq h_i(x,t) - \varepsilon e^{\gamma t} \quad (i = 1,2),\ \forall (x,t) \in \overline{\Omega}_T, \tag{2.13}$$

which means that $(g_1(x,t), g_2(x,t)) \geq (h_1(x,t), h_2(x,t))$ in $\overline{\Omega}_T$ as desired.

In order to prove (2.12), we set

$$\tilde{\eta}_1 = \eta_1 e^{-\sigma t}, \qquad \tilde{\eta}_2 = \eta_2 e^{-\sigma t} \tag{2.14}$$

with $\sigma > max\{\sup_{\overline{\Omega}_T}(c_{11} + d_{11}), \sup_{\overline{\Omega}_T}(c_{22} + d_{22})\}$. Then from (2.5)–(2.10), we have

$$\begin{gathered}
\tilde{\eta}_{1t} - \left(\sum_{l,k=1}^{n} a_{l,k}^{(1)} \frac{\partial^2 \tilde{\eta}_1}{\partial x_k \partial x_l} + \sum_{k=1}^{n} b_k^{(1)} \frac{\partial \tilde{\eta}_1}{\partial x_k}\right) > (c_{11} + d_{11} - \sigma)\tilde{\eta}_1 + (c_{12} + d_{12})\tilde{\eta}_2, \quad (x,t) \in \Omega_T, \\
\tilde{\eta}_{2t} - \left(\sum_{l,k=1}^{n} a_{l,k}^{(1)} \frac{\partial^2 \tilde{\eta}_2}{\partial x_k \partial x_l} + \sum_{k=1}^{n} b_k^{(1)} \frac{\partial \tilde{\eta}_2}{\partial x_k}\right) > (c_{21} + d_{21})\tilde{\eta}_1 + (c_{22} + d_{22} - \sigma)\tilde{\eta}_2, \quad (x,t) \in \Omega_T, \\
\tilde{\eta}_1(x,t) > \int_\Omega m\chi_1(x,y)\theta_2^{m-1}\tilde{\eta}_1(y,t)\,dy, \quad (x,t) \in S_T, \\
\tilde{\eta}_2(x,t) > \int_\Omega n\chi_2(x,y)\theta_3^{n-1}\tilde{\eta}_2(y,t)\,dy, \quad (x,t) \in S_T, \\
\tilde{\eta}_1(x,0) > 0, \quad \tilde{\eta}_2(x,0) > 0, \quad x \in \overline{\Omega},
\end{gathered} \tag{2.15}$$

where θ_2 is an intermediate value between $\tilde{g}_1$ and $\tilde{h}_1$, θ_3 is an intermediate value between $\tilde{g}_2$ and $\tilde{h}_2$.

Since $\tilde{\eta}_1(x,0), \tilde{\eta}_2(x,0) > 0$, there exists $\delta > 0$ such that $\tilde{\eta}_1, \tilde{\eta}_2 > 0$ for $(x,t) \in \overline{\Omega} \times (0,\delta)$. Suppose a contradiction that

$$\bar{t} = \sup\left\{t \in (0,T) : \tilde{\eta}_1, \tilde{\eta}_2 > 0 \text{ on } \overline{\Omega} \times [0,t]\right\} < T. \tag{2.16}$$

Then $\tilde{\eta}_1, \tilde{\eta}_2 \geq 0$ on $\overline{\Omega}_{\bar{t}}$, and at least one of $\tilde{\eta}_1$, $\tilde{\eta}_2$ vanishes at $(\bar{x},\bar{t})$ for some $\bar{x} \in \overline{\Omega}$. Without loss of generality, suppose that $\tilde{\eta}_1(\bar{x},\bar{t}) = 0 = \inf_{\overline{\Omega}_{\bar{t}}} \tilde{\eta}_1$. If $(\bar{x},\bar{t}) \in \Omega_{\bar{t}}$, by virtue of the first inequality of (2.15), we find that

$$\tilde{\eta}_{1t} - \left(\sum_{k,l=1}^{n} a_{l,k}^{(1)} \frac{\partial^2 \tilde{\eta}_1}{\partial x_k \partial x_l} + \sum_{k=1}^{n} b_k^{(1)} \frac{\partial \tilde{\eta}_1}{\partial x_k}\right) > (a_{11} + b_{11} - \sigma)\tilde{\eta}_1, \quad (x,t) \in \Omega_{\bar{t}}. \tag{2.17}$$

This leads us to conclude that $\tilde{\eta}_1 \equiv 0$ in $\Omega_{\bar{t}}$ by the strong maximum principle, a contradiction. If $(\bar{x},\bar{t}) \in S_{\bar{t}}$, this also results in a contradiction, that is

$$0 = \tilde{\eta}_1\left(\bar{x},\bar{t}\right) > \int_{\Omega} m\varphi(\bar{x},y)\theta_2^{m-1}\tilde{\eta}_1\left(y,\bar{t}\right)dy \geq 0. \tag{2.18}$$

This proves that $\tilde{\eta}_1, \tilde{\eta}_2 > 0$, and in turn $(g_1(x,t), g_2(x,t)) \geq (h_1(x,t), h_2(x,t))$ in $\overline{\Omega}_T$. The proof of Lemma 2.2 is complete. □

Lemma 2.3. *Let the hypotheses of Lemma* 2.2, *with (H1) replaced by (H2), be satisfied. Then*

$$\left(g_1(x,t), g_2(x,t)\right) \geq \left(h_1(x,t), h_2(x,t)\right) \quad \text{in } \overline{\Omega}_T. \tag{2.19}$$

Proof. Choose a positive function f satisfying $f|_{x\in\partial\Omega} = 1$ and $\int_{\Omega} f(y)dy < 1/M$. Set

$$g_i(x,t) = f(x)\rho_i(x,t), \quad h_i(x,t) = f(x)\zeta_i(x,t), \quad i = 1,2. \tag{2.20}$$

Then from (2.2), we have

$$\begin{aligned}
&\rho_{1t} - \mathcal{L}_1\rho_1 \geq \sum_{i=1}^{2} \frac{c_{1i}}{f}\rho_i - \sum_{i=1}^{2}\frac{d_{1i}}{f}\zeta_i + \frac{w_1}{f}, \quad (x,t) \in \Omega_T,\\
&\rho_{2t} - \mathcal{L}_2\rho_2 \geq \sum_{i=1}^{2} \frac{c_{2i}}{f}\rho_i - \sum_{i=1}^{2}\frac{d_{2i}}{f}\zeta_i + \frac{w_2}{f}, \quad (x,t) \in \Omega_T,\\
&\zeta_{1t} - \mathcal{L}_1\zeta_1 \leq \sum_{i=1}^{2} \frac{c_{1i}}{f}\zeta_i - \sum_{i=1}^{2}\frac{d_{1i}}{f}\rho_i + \frac{s_1}{f}, \quad (x,t) \in \Omega_T,\\
&\zeta_{2t} - \mathcal{L}_2\zeta_2 \leq \sum_{i=1}^{2} \frac{c_{2i}}{f}\zeta_i - \sum_{i=1}^{2}\frac{d_{2i}}{f}\rho_i + \frac{s_2}{f}, \quad (x,t) \in \Omega_T,\\
&\rho_1(x,t) \geq \int_\Omega \chi_1(x,y) f^m(y)\rho_1^m(y,t)dy, \quad (x,t) \in S_T,\\
&\rho_2(x,t) \geq \int_\Omega \chi_2(x,y) f^n(y)\rho_2^n(y,t)dy, \quad (x,t) \in S_T,\\
&\zeta_1(x,t) \leq \int_\Omega \chi_1(x,y) f^m(y)\zeta_1^m(y,t)dy, \quad (x,t) \in S_T,\\
&\zeta_2(x,t) \leq \int_\Omega \chi_2(x,y) f^n(y)\zeta_2^n(y,t)dy, \quad (x,t) \in S_T,\\
&\rho_1(x,0) \geq \zeta_1(x,0), \quad \rho_2(x,0) \geq \zeta_2(x,0), \quad x \in \overline{\Omega},
\end{aligned} \tag{2.21}$$

where

$$\mathcal{L}_s = \sum_{l,k=1}^{n} a_{l,k}^{(s)} \frac{\partial^2}{\partial x_k \partial x_l} + \sum_{k=1}^{n}\left(\sum_{l=1}^{n} 2a_{k,l}^{(s)}\frac{\partial f}{\partial x_l} + b_k^{(s)} f\right)\frac{1}{f}\frac{\partial}{\partial x_k}, \quad s = 1,2, \tag{2.22}$$

is a uniformly elliptic operator. By (H2), it is easy to see that

$$\begin{aligned}
&\int_\Omega m\chi_1(x,y)\rho_1^{m-1}(y,t) f^m(y)dy \leq M\int_\Omega f(y)dy \leq 1,\\
&\int_\Omega n\chi_2(x,y)\rho_2^{n-1}(y,t) f^n(y)dy \leq M\int_\Omega f(y)dy \leq 1.
\end{aligned} \tag{2.23}$$

Similarly, we have

$$\int_\Omega m\chi_1(x,y)\zeta_1^{m-1}(y,t) f^m(y)dy \leq 1, \qquad \int_\Omega n\chi_2(x,y)\zeta_2^{n-1}(y,t) f^n(y)dy \leq 1. \tag{2.24}$$

Therefore, in view of Lemma 2.2, we have

$$\rho_i(x,t) \geq \zeta_i(x,t), \quad i = 1,2, \tag{2.25}$$

which implies that

$$g_i(x,t) \geq h_i(x,t), \quad i = 1,2. \tag{2.26}$$

The proof of Lemma 2.3 is complete. □

On the basis of the above lemmas, we obtain the following comparison principle for problem (1.1).

Proposition 2.4 (Comparison principle). *Let $(\overline{u},\overline{v})$ and $(\underline{u},\underline{v})$ be a nonnegative supersolution and a nonnegative subsolution of problem* (1.1) *in Ω_T, respectively. Suppose that $(\underline{u},\underline{v}) > (0,0)$ and $(\overline{u},\overline{v}) > (0,0)$ in $\overline{\Omega}_T$ if $\min\{p,q,m,n\} < 1$. If $(\overline{u}(x,0),\overline{v}(x,0)) \geq (\underline{u}(x,0),\underline{v}(x,0))$ for $x \in \overline{\Omega}$, then $(\overline{u},\overline{v}) \geq (\underline{u},\underline{v})$ in $\overline{\Omega}_T$.*

Proof. It is easy to check that $\underline{u}$, $\underline{v}$, $\overline{u}$, $\overline{v}$ and φ, ψ satisfy hypotheses (H2). □

Next, we state the local existence theorem, and its proof is standard; hence we omit it.

Theorem 2.5 (Local existence). *For any nonnegative nontrivial $u_0(x)$,$v_0(x) \in C(\overline{\Omega})$, there exists a constant $T^* = T^*(u_0,v_0) > 0$ such that problem* (1.1) *admits nonnegative solution $(u(x,t),v(x,t)) \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T) \times C^{2,1}(\Omega_T) \cap C(\overline{\Omega}_T)$ for each $T < T^*$. Furthermore, either $T^* = \infty$ or*

$$\lim_{t \to T^*} \sup\left(\|u(x,t)\|_\infty + \|v(x,t)\|_\infty\right) = \infty. \tag{2.27}$$

Remark 2.6. From maximum principle, we know that the solution of system (1.1) is positive when $u_0(x)$ and $v_0(x)$ are positive. Indeed, since $u_t - \Delta u - v^p \geq 0$ and $v_t - \Delta v - u^q \geq 0$, the minimum of (u,v) in $\overline{\Omega}_T$ should be obtained at a parabolic boundary point by maximum principle. Furthermore, $\int_\Omega \varphi(x,y)dy$, $\int_\Omega \psi(x,y)dy > 0$ on $\partial\Omega$ imply that $\varphi(x,t) \not\equiv 0$ and $\psi(x,t) \not\equiv 0$, then we have $(u,v) > (0,0)$ for $(x,t) \in \partial\Omega \times (0,T]$. Thus $(u,v) > (0,0)$ provided that $u_0(x)$ and $v_0(x)$ are positive. In the rest of this paper, we assume that $(u_0(x),v_0(x)) > (0,0)$.

Remark 2.7. If $pq \geq 1$, $m \geq 1$, and $n \geq 1$, we could obtain the uniqueness of the solution easily by comparison principle.

3. Proof of Theorem 1.1

In this section, by constructing special supersolution, we will give the sufficient condition for the existence of global solution of problem (1.1) under the hypotheses $0 < pq \leq 1$ and $m,n < 1$.

Proof of Theorem 1.1. Since $0 < pq \le 1$, there exist $\alpha, \beta \in (0,1)$ such that

$$\frac{1}{p} \ge \frac{\beta}{\alpha}, \qquad \frac{1}{q} \ge \frac{\alpha}{\beta}. \tag{3.1}$$

Define $k = 1/\alpha + 1/\beta$. Let $\Phi(x,y) \ge \max\{\varphi(x,y), \psi(x,y)\}$ be a continuous function defined for $(x,y) \in \partial\Omega \times \overline{\Omega}$, and

$$a(x) = \left(\int_\Omega \Phi(x,y)\,dy\right)^{(1-\alpha)/\alpha}, \quad b(x) = \left(\int_\Omega \Phi(x,y)\,dy\right)^{(1-\beta)/\beta}, \quad x \in \partial\Omega. \tag{3.2}$$

Suppose that ω is the solution of the following problem:

$$\begin{gathered}\omega_t = \Delta\omega + k\omega, \quad x \in \Omega,\ t > 0,\\ \omega(x,t) = (a(x) + b(x) + 1)\int_\Omega \left(\Phi(x,y) + \frac{1}{|\Omega|}\right)\omega(y,t)\,dy, \quad x \in \partial\Omega,\ t > 0,\\ \omega(x,0) = u_0^{1/\alpha}(x) + v_0^{1/\beta}(x) + 1, \quad x \in \Omega,\end{gathered} \tag{3.3}$$

where $|\Omega|$ denotes the measure of Ω. By Theorem 2.5 in [16], we know that ω is a global solution of (3.3). Moreover, $\omega(x,t) > 1$ in $(x,t) \in \overline{\Omega} \times [0,+\infty)$ by the maximum principle.

Set $(\overline{u}, \overline{v}) = (\omega^\alpha, \omega^\beta)$. A simple computation shows that

$$\overline{u}_t = \alpha\omega^{\alpha-1}\omega_t = \alpha\omega^{\alpha-1}\Delta\omega + \left(1 + \frac{\alpha}{\beta}\right)\omega^\alpha, \qquad \Delta\overline{u} = \alpha\omega^{\alpha-1}\Delta\omega + \alpha(\alpha-1)\omega^{\alpha-2}|\nabla\omega|^2, \tag{3.4}$$

and thus

$$\overline{u}_t - \Delta\overline{u} = \left(1 + \frac{\alpha}{\beta}\right)\omega^\alpha - \alpha(\alpha-1)\omega^{\alpha-2}|\nabla\omega|^2 \ge \left(1 + \frac{\alpha}{\beta}\right)\omega^\alpha \ge \left(\omega^\beta\right)^{\alpha/\beta} \ge \overline{v}^p. \tag{3.5}$$

here, we used $p \le \alpha/\beta$ and $\omega > 1$.

When $(x,t) \in \partial\Omega \times (0,+\infty)$, according to Hölder's inequality, we have that

$$\begin{aligned}
\overline{u}(x,t) &\geq a^{\alpha}(x)\left\{\int_{\Omega}\Phi(x,y)\omega(y,t)dy\right\}^{\alpha} \\
&= \left\{\int_{\Omega}\Phi(x,y)dy\right\}^{1-\alpha}\left\{\int_{\Omega}\Phi(x,y)\omega(y,t)dy\right\}^{\alpha} \\
&\geq \left\{\int_{\Omega}\varphi(x,y)dy\right\}^{1-\alpha}\left\{\int_{\Omega}\varphi(x,y)\omega(y,t)dy\right\}^{\alpha} \\
&\geq \int_{\Omega}\varphi(x,y)\omega^{\alpha}(y,t)dy = \int_{\Omega}\varphi(x,y)\overline{u}(x,t)dy \\
&\geq \int_{\Omega}\varphi(x,y)\overline{u}^{m}(x,t)dy.
\end{aligned} \tag{3.6}$$

Likewise, we also have for $\overline{v}$ that

$$\begin{gathered}
\overline{v}_t - \Delta\overline{v} \geq \overline{u}^{q}, \quad x \in \Omega,\ t > 0, \\
\overline{v}(x,t) \geq \int_{\Omega}\psi(x,y)\overline{v}^{n}(x,t)dy, \quad x \in \partial\Omega,\ t > 0.
\end{gathered} \tag{3.7}$$

On the other hand, since $\alpha < 1$, we have

$$\overline{u}(x,0) = \omega^{\alpha}(x,0) = \left(u_0^{1/\alpha}(x) + v_0^{1/\beta}(x) + 1\right)^{\alpha} \geq u_0(x), \tag{3.8}$$

and similarly,

$$\overline{v}(x,0) = \omega^{\beta}(x,0) = \left(u_0^{1/\alpha}(x) + v_0^{1/\beta}(x) + 1\right)^{\beta} \geq v_0(x). \tag{3.9}$$

Therefore, $(\overline{u},\overline{v})$ is a global supersolution of (1.1); by Proposition 2.4, the solution of (1.1) exists globally. The proof of Theorem 1.1 is complete. □

4. Proof of Theorem 1.2

In this section, we will establish that the solution of system (1.1) blows up in finite time for the case $pq > 1$ or $m,n > 1$. We employ a variant of Kaplan's method (see [17] for more details) to obtain our blowup conclusion.

Proof of Theorem 1.2. Let

$$J_1(t) = \int_{\Omega}u(x,t)\phi(x)dx, \qquad J_2(t) = \int_{\Omega}v(x,t)\phi(x)dx, \tag{4.1}$$

where $\phi(x)$ is defined in (1.7). Taking the derivative of $J_1(t)$ with respect to t, we could obtain

$$\begin{aligned}
J_1'(t) &= \int_\Omega u_t(x,t)\phi(x)dx = \int_\Omega (\Delta u + v^p)\phi\, dx \\
&= -\int_\Omega Du\cdot D\varphi\, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\phi\, dS + \int_\Omega v^p \phi(x)dx \\
&= \int_\Omega u\Delta\phi\, dx - \int_{\partial\Omega} \frac{\partial \phi}{\partial \nu} u\, dS + \int_\Omega v^p\phi\, dx \\
&= -\int_\Omega \lambda u\phi\, dx - \int_{\partial\Omega} \frac{\partial\phi}{\partial\nu}\left(\int_\Omega \varphi(x,y)u^m(y,t)dy\right)dS + \int_\Omega v^p\phi\, dx.
\end{aligned} \tag{4.2}$$

Applying the equality $\int_{\partial\Omega}(\partial\phi/\partial\nu)dS = -\lambda$ to (4.2), we find that

$$J_1'(t) \geq -\lambda\int_\Omega u\phi\, dx + \frac{\lambda M_2}{L}\int_\Omega \phi u^m dx + \int_\Omega v^p\phi\, dx. \tag{4.3}$$

Symmetrically, we deduce that

$$J_2'(t) \geq -\lambda\int_\Omega v\phi\, dx + \frac{\lambda K_2}{L}\int_\Omega \phi v^n dx + \int_\Omega u^q\phi\, dx. \tag{4.4}$$

□

Case 1. For the case $pq > 1$; we first prove the assertion under the stronger assumption $p, q > 1$. Without loss of generality, we assume that $q = \max\{p,q\} > 1$. Then using Jensen's inequality to (4.3) and (4.4), we see that

$$J_1'(t) \geq -\lambda J_1 + {J_2}^p, \qquad J_2'(t) \geq -\lambda J_2 + {J_1}^q. \tag{4.5}$$

Therefore $J = J_1 + J_2$ satisfies

$$J' = J_1' + J_2' \geq -\lambda J + {J_1}^q + {J_2}^p \geq -\lambda J + {J_1}^p + {J_2}^p - J_1. \tag{4.6}$$

In view of the inequality ${J_1}^p + {J_2}^p \geq 2^{1-p}J^p$, we discover that

$$J' \geq -(\lambda+1)J + 2^{1-p}J^p. \tag{4.7}$$

It follows that J blows up in finite time whenever

$$J(0) = \int_\Omega (u_0(x) + v_0(x))\phi(x)dx \tag{4.8}$$

is sufficiently large. Furthermore, the solution of system (1.1) blows up in finite time.

If $p < 1$ or $q < 1$, in order to obtain our conclusion, we consider system (1.1) with zero Dirichlet boundary condition; then in light of Theorem 2 in [1], we obtain our result immediately.

Case 2. Consider now the case that $m, n > 1$. Since $m, n > 1$, Jensen's inequality can be applied to (4.3) and (4.4) like Step 1 to get

$$J_1'(t) \geq -\lambda J_1 + \frac{\lambda M_1}{L} J_1{}^m, \qquad J_2'(t) \geq -\lambda J_2 + \frac{\lambda M_2}{L} J_2{}^n. \tag{4.9}$$

Then the left arguments are the same as those for Case 1, we omit the details. The proof of Theorem 1.2 is complete.

5. Proof of Theorem 1.3

In this section, we will use an idea from Gladkov and Kim [16] to prove Theorem 1.3.

Proof of Theorem 1.3. Let Ω_1 be a bounded domain in $\mathbb{R}^N$ satisfying the property that $\Omega \Subset \Omega_1$ and let λ_1 be the first eigenvalue of $-\Delta$ on Ω_1 with null Dirichlet boundary condition which satisfies the inequality $0 < \lambda_1 < \lambda$.

Since $\varphi(x, y)$ and $\psi(x, y)$ are nonnegative continuous defined in $\partial\Omega \times \overline{\Omega}$; then there exist some constants $0 < A, B < +\infty$ such that

$$\int_\Omega \varphi(x, y)dy \leq A, \qquad \int_\Omega \psi(x, y)dy \leq B. \tag{5.1}$$

Denote $\widetilde{\phi}$ an eigenfunction corresponding the eigenvalue λ_1; then it is obviously that

$$\frac{\sup_{\Omega_1}\widetilde{\phi}}{\inf_{\overline{\Omega}}\widetilde{\phi}} \leq \frac{\sup_{\Omega_1}\widetilde{\phi}}{\inf_{\Omega_1}\widetilde{\phi}} < \delta, \tag{5.2}$$

where $\delta > 1$ is some constant. Choosing any ε which satisfies the inequality

$$0 < \varepsilon \leq \min\left\{ (A\delta^m)^{-1/(m-1)}, (B\delta^n)^{-1/(n-1)} \right\}, \tag{5.3}$$

and taking

$$\sup_{\Omega_1} \widetilde{\phi} = \delta\varepsilon, \tag{5.4}$$

then, from (5.2), it follows easily that

$$\inf_{\partial\Omega} \widetilde{\phi} > \varepsilon. \tag{5.5}$$

Case 1. For $q \geq p > 1$, set

$$f(t) = e^{-\lambda_1 t}\left[1 + \frac{\max\left\{(\delta\varepsilon)^{p-1}, (\delta\varepsilon)^{q-1}\right\}e^{-(p-1)\lambda_1 t}}{\lambda_1}\right]^{-1/(p-1)}. \tag{5.6}$$

It is easy to check that $f(t)$ satisfy the following ordinary differential equation:

$$f'(t) + \lambda_1 f - \max\left\{(\delta\varepsilon)^{p-1}, (\delta\varepsilon)^{q-1}\right\}f^p = 0. \tag{5.7}$$

Observe next that $f(t) < 1$, and so $f^p \geq f^q$ under the condition $q \geq p > 1$.

Let

$$\overline{u}(x,t) = \overline{v}(x,t) = \tilde{\phi}(x)f(t). \tag{5.8}$$

A series of computations yields

$$\begin{aligned} \overline{u}_t - \Delta\overline{u} - \overline{v}^q &= \tilde{\phi}\left(f' + \lambda_1 f - \tilde{\phi}^{q-1}f^q\right) \\ &\geq \tilde{\phi}\left(f' + \lambda_1 f - \max\left\{(\delta\varepsilon)^{p-1}, (\delta\varepsilon)^{q-1}\right\}f^p\right) \\ &\geq 0. \end{aligned} \tag{5.9}$$

And similarly, we have

$$\begin{aligned} \overline{v}_t - \Delta\overline{v} - \overline{u}^p &= \tilde{\phi}\left(f' + \lambda_1 f - \tilde{\phi}^{p-1}f^p\right) \\ &\geq \tilde{\phi}\left(f' + \lambda_1 f - \max\left\{(\delta\varepsilon)^{p-1}, (\delta\varepsilon)^{q-1}\right\}f^p\right) \\ &\geq 0. \end{aligned} \tag{5.10}$$

On the other hand, since $\int_\Omega \varphi(x,y)dy \leq A$, we have on the boundary that

$$\overline{u}(x,t) > \varepsilon f(t) \geq A(\delta\varepsilon)^m f(t) \geq \int_\Omega \varphi(x,y)\left(\tilde{\phi}(y)f(t)\right)^m dy. \tag{5.11}$$

Likewise, we have that

$$\overline{v}(x,t) > \int_\Omega \psi(x,y)\left(\tilde{\phi}(y)f(t)\right)^n dy. \tag{5.12}$$

Thus, by exploiting (5.9)–(5.12) and comparison principle, the solution of (1.1) exists globally provided that

$$\max\{u_0(x), v_0(x)\} \leq \left[1 + \frac{\max\left\{(\delta\varepsilon)^{p-1}, (\delta\varepsilon)^{q-1}\right\}}{\lambda_1}\right]^{-1/(p-1)} \tilde{\phi}(x). \tag{5.13}$$

□

Case 2. For $p > q > 1$, set

$$f(t) = e^{-\lambda_1 t}\left[1 + \frac{\max\left\{(\delta\varepsilon)^{p-1}, (\delta\varepsilon)^{q-1}\right\}e^{-(q-1)\lambda_1 t}}{\lambda_1}\right]^{-1/(q-1)}. \tag{5.14}$$

We can immediately verify that $f(t)$ satisfy the following ordinary differential equation:

$$f'(t) + \lambda_1 f - \max\left\{(\delta\varepsilon)^{p-1}, (\delta\varepsilon)^{q-1}\right\} f^q = 0. \tag{5.15}$$

In addition, it is obvious that $f(t) < 1$. Then we have that $f^q \geq f^p$ under the condition $p > q > 1$.

Let

$$\overline{u}(x,t) = \overline{v}(x,t) = \tilde{\phi}(x) f(t). \tag{5.16}$$

Similar to the arguments for the case $q \geq p > 1$, we can prove that $(\overline{u}(x,t), \overline{v}(x,t))$ is a global supersolution of problem (1.1) provided that

$$\max\{u_0(x), v_0(x)\} \leq \left[1 + \frac{\max\left\{(\delta\varepsilon)^{p-1}, (\delta\varepsilon)^{q-1}\right\}}{\lambda_1}\right]^{-1/(q-1)} \tilde{\phi}(x). \tag{5.17}$$

The proof of Theorem 1.3 is complete.

Acknowledgments

The authors are very grateful to the anonymous referees for their careful reading and useful suggestions, which greatly improved the presentation of the paper. This work is funded by Innovative Talent Training Project, the Third Stage of "211 Project", Chongqing University, project no. S-09110; C. Mu is supported in part by NSF of China (10771226) and in part by Natural Science Foundation Project of CQ CSTC (2007BB0124).

References

[1] M. Escobedo and M. A. Herrero, “A semilinear parabolic system in a bounded domain,” *Annali di Matematica Pura ed Applicata. Serie Quarta*, vol. 165, pp. 315–336, 1993.
[2] J. Zhang, “Boundedness and blow-up behavior for reaction-diffusion systems in a bounded domain,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 35, no. 7, pp. 833–844, 1999.
[3] H. Fujita, “On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$,” *Journal of the Faculty of Science. University of Tokyo. Section IA*, vol. 13, pp. 109–124, 1966.
[4] Y. Giga and R. V. Kohn, “Characterizing blowup using similarity variables,” *Indiana University Mathematics Journal*, vol. 36, no. 1, pp. 1–40, 1987.
[5] V. A. Galaktionov and H. A. Levine, “On critical Fujita exponents for heat equations with nonlinear flux conditions on the boundary,” *Israel Journal of Mathematics*, vol. 94, pp. 125–146, 1996.
[6] S.-C. Fu and J.-S. Guo, “Blow-up for a semilinear reaction-diffusion system coupled in both equations and boundary conditions,” *Journal of Mathematical Analysis and Applications*, vol. 276, no. 1, pp. 458–475, 2002.
[7] K. Deng, “Blow-up rates for parabolic systems,” *Zeitschrift für Angewandte Mathematik und Physik*, vol. 47, no. 1, pp. 132–143, 1996.
[8] M. Escobedo and M. A. Herrero, “Boundedness and blow up for a semilinear reaction-diffusion system,” *Journal of Differential Equations*, vol. 89, no. 1, pp. 176–202, 1991.
[9] B. Hu and H.-M. Yin, “Critical exponents for a system of heat equations coupled in a non-linear boundary condition,” *Mathematical Methods in the Applied Sciences*, vol. 19, no. 14, pp. 1099–1120, 1996.
[10] K. Deng, “Global existence and blow-up for a system of heat equations with non-linear boundary conditions,” *Mathematical Methods in the Applied Sciences*, vol. 18, no. 4, pp. 307–315, 1995.
[11] K. Deng and C.-L. Zhao, “Blow-up for a parabolic system coupled in an equation and a boundary condition,” *Proceedings of the Royal Society of Edinburgh A*, vol. 131, no. 6, pp. 1345–1355, 2001.
[12] K. Deng and H. A. Levine, “The role of critical exponents in blow-up theorems: the sequel,” *Journal of Mathematical Analysis and Applications*, vol. 243, no. 1, pp. 85–126, 2000.
[13] S. Wang, C. Xie, and M. Wang, “Note on critical exponents for a system of heat equations coupled in the boundary conditions,” *Journal of Mathematical Analysis and Applications*, vol. 218, no. 1, pp. 313–324, 1998.
[14] Y. Wang and Z. Xiang, “Blowup analysis for a semilinear parabolic system with nonlocal boundary condition,” *Boundary Value Problems*, vol. 2009, Article ID 516390, 14 pages, 2009.
[15] S. Zheng and L. Kong, “Roles of weight functions in a nonlinear nonlocal parabolic system,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 8, pp. 2406–2416, 2008.
[16] A. Gladkov and K. I. Kim, “Blow-up of solutions for semilinear heat equation with nonlinear nonlocal boundary condition,” *Journal of Mathematical Analysis and Applications*, vol. 338, no. 1, pp. 264–273, 2008.
[17] S. Kaplan, “On the growth of solutions of quasi-linear parabolic equations,” *Communications on Pure and Applied Mathematics*, vol. 16, pp. 305–330, 1963.