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Research Article

Representations for the Generalized Drazin Inverse of the Sum in a Banach Algebra and Its Application for Some Operator Matrices

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We investigate additive properties of the generalized Drazin inverse in a Banach algebra $\mathscr{A}$. We find explicit expressions for the generalized Drazin inverse of the sum $a+b$, under new conditions on $a, b \in \mathscr{A}$. As an application we give some new representations for the generalized Drazin inverse of an operator matrix.

1. Introduction

Let $\mathscr{A}$ be a complex Banach algebra with unite 1. We use $\sigma(a)$ to denote the spectrum of $a \in \mathscr{A}$. The sets of all nilpotent and quasinilpotent elements ($\sigma(a) = \{0\}$) of $\mathscr{A}$ will be denoted by $\mathscr{A}^{\text{nil}}$ and $\mathscr{A}^{\text{qnil}}$, respectively.

The generalized Drazin inverse of $a \in \mathscr{A}$ (introduced by Koliha in [1]) is the element $b \in \mathscr{A}$ which satisfies

$$xax = x, \qquad ax = xa, \qquad a - a^2x \in \mathscr{A}^{\text{qnil}}. \tag{1}$$

If there exists the generalized Drazin inverse, then the generalized Drazin inverse of a is unique and is denoted by a^d. The set of all generalized Drazin invertible elements of $\mathscr{A}$ is denoted by $\mathscr{A}^d$. For interesting properties of the generalized Drazin inverse see [2–6]. For a complete treatment of the generalized Drazin inverse, see [7, Chapter 2].

If $p = p^2 \in \mathscr{A}$ is an idempotent, we denote $\overline{p} = \mathbb{1} - p$. We can represent element $a \in \mathscr{A}$ as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p, \tag{2}$$

where $a_{11} = pap$, $a_{12} = pa\overline{p}$, $a_{21} = \overline{p}ap$, and $a_{22} = \overline{p}a\overline{p}$.

Let $a \in \mathscr{A}^d$ and $a^\pi = \mathbb{1} - aa^d$ be the spectral idempotent of a corresponding to $\{0\}$. It is well known that $a \in \mathscr{A}$ can be represented in the following matrix form ([7, Chapter 2]):

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p, \tag{3}$$

relative to $p = aa^d$, where a_1 is invertible in the algebra $p\mathscr{A}p$, a^d is its inverse in $p\mathscr{A}p$, and a_2 is quasinilpotent in the algebra $\overline{p}\mathscr{A}\overline{p}$. Thus, the generalized Drazin inverse of a can be expressed as

$$a^d = \begin{bmatrix} a_1^d & 0 \\ 0 & 0 \end{bmatrix}_p. \tag{4}$$

Obviously, if $a \in \mathscr{A}^{\text{qnil}}$, then a is generalized Drazin invertible and $a^d = 0$.

In this paper, we first give the formulas of $(a+b)^d$ under the conditions $ab = bab^\pi$ and $ab = a^\pi bab^\pi$, respectively. Then we will apply these formulas to provide some representations for the generalized Drazin inverse of the operator matrix $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ under some conditions.

2. Main Results

First we start the following result which is proved in [8] for matrices, extended in [9] for a bounded linear operator and in [10] for arbitrary elements in a Banach algebra.

Lemma 1 (see [10, Theorem 2.3]). *Let $x, y \in \mathscr{A}$ and $p \in \mathscr{A}$ be an idempotent. Assume that x and y are represented as*

$$x = \begin{bmatrix} a & 0 \\ c & b \end{bmatrix}_p, \qquad y = \begin{bmatrix} b & c \\ 0 & a \end{bmatrix}_p. \tag{5}$$

(i) *If $a \in (p\mathscr{A}p)^d$ and $b \in (\overline{p}\mathscr{A}\overline{p})^d$, then x and y are generalized Drazin invertible, and*

$$x^d = \begin{bmatrix} a^d & 0 \\ u & b^d \end{bmatrix}_p, \qquad y^d = \begin{bmatrix} b^d & u \\ 0 & a^d \end{bmatrix}_p, \tag{6}$$

where

$$u = \sum_{n=0}^{\infty} \left(b^d\right)^{n+2} ca^n a^\pi + \sum_{n=0}^{\infty} b^\pi b^n c \left(a^d\right)^{n+2} - b^d c a^d. \tag{7}$$

(ii) *If $x \in \mathscr{A}^d$ and $a \in (p\mathscr{A}p)^d$, then $b \in (\overline{p}\mathscr{A}\overline{p})^d$ and x^d and y^d are given by (6) and (7).*

Lemma 2 (see [11, Lemma 2.1]). *Let $a, b \in \mathscr{A}^{qnil}$. If $ab = ba$ or $ab = 0$, then $a + b \in \mathscr{A}^{qnil}$.*

The following result is a generalization of [10, Corollary 3.4].

Theorem 3. *If $a \in \mathscr{A}^{qnil}$, $b \in \mathscr{A}^d$, and $ab = bab^\pi$, then $a + b \in \mathscr{A}^d$ and*

$$(a+b)^d = b^d + \sum_{n=0}^{\infty} \left(b^d\right)^{n+2} a (a+b)^n. \tag{8}$$

Proof. First, suppose that $b \in \mathscr{A}^{\text{qnil}}$. Therefore, $b^\pi = 1$ and from $ab = bab^\pi$ we obtain $ab = ba$. Using Lemma 2, $a + b \in \mathscr{A}^{\text{qnil}}$ and (8) holds.

Now we assume b is not quasinilpotent, using matrix representations of a and b relative to $p = bb^d$. We have

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_p, \qquad b^d = \begin{bmatrix} b_1^d & 0 \\ 0 & 0 \end{bmatrix}_p, \tag{9}$$

where $b_1 \in (p\mathscr{A}p)^{-1}$, $b_2 \in (\overline{p}\mathscr{A}\overline{p})^{\text{qnil}}$.

Let us represent

$$a = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}_p. \tag{10}$$

From $ab = bab^\pi$ and

$$ab = \begin{bmatrix} a_1 b_1 & a_2 b_2 \\ a_3 b_1 & a_4 b_2 \end{bmatrix}_p, \qquad bab^\pi = \begin{bmatrix} 0 & b_1 a_2 \\ 0 & b_2 a_4 \end{bmatrix}_p, \tag{11}$$

we obtain $a_1 b_1 = 0$ and $a_3 b_1 = 0$. Since b_1 is invertible, we have $a_1 = 0$ and $a_3 = 0$.

Hence we have

$$a + b = \begin{bmatrix} b_1 & a_2 \\ 0 & a_4 + b_2 \end{bmatrix}_p. \tag{12}$$

The condition $ab = bab^\pi$ implies that $a_4 b_2 = b_2 a_4$. Hence, using Lemma 2, we get $a_4 + b_2 \in \mathscr{A}^{\text{qnil}}$. By Lemma 1, we obtain that $a + b \in \mathscr{A}^d$ and

$$(a+b)^d = \begin{bmatrix} b_1^d & u \\ 0 & 0 \end{bmatrix}_p, \tag{13}$$

where

$$u = \sum_{n=0}^{\infty} \left(b_1^d\right)^{n+2} a_2 \left(a_4 + b_2\right)^n. \tag{14}$$

Now from (14), using the matrix representation of b^d, a, and $a + b$, we easily obtain formula (8) of the theorem. □

The next result is a generalization of [12, Theorem 2.2] and [10, Example 4.5].

Theorem 4. *Let $a, b \in \mathscr{A}^d$. If $ab = a^\pi bab^\pi$, then $a + b \in \mathscr{A}^d$ and*

$$\begin{aligned} (a+b)^d = {} & b^\pi a^d + b^d a^\pi + \sum_{n=0}^{\infty} \left(b^d\right)^{n+2} a (a+b)^n a^\pi \\ & + b^\pi \sum_{n=0}^{\infty} (a+b)^n b \left(a^d\right)^{n+2} \\ & - \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left(b^d\right)^{k+1} a (a+b)^{n+k} b \left(a^d\right)^{n+2} \\ & - \sum_{n=0}^{\infty} \left(b^d\right)^{n+2} a (a+b)^n b a^d. \end{aligned} \tag{15}$$

Proof. If a is quasinilpotent, we can apply Theorem 3 and we obtain (15) for this particular case. Now we assume that a is neither invertible nor quasinilpotent and consider the following matrix representations of a, a^d, and b relative to the $p = aa^d$:

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p, \qquad a^d = \begin{bmatrix} a_1^d & 0 \\ 0 & 0 \end{bmatrix}_p, \qquad b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_p. \tag{16}$$

The condition $ab = a^\pi bab^\pi$ implies that $a_1 b_1 = 0$ and $a_1 b_2 = 0$. Since a_1 is invertible, we have $b_1 = 0$ and $b_2 = 0$.

Thus, b can be represented as

$$b = \begin{bmatrix} 0 & 0 \\ b_3 & b_4 \end{bmatrix}_p. \tag{17}$$

Therefore, $b_4 \in (\overline{p}\mathscr{A}\overline{p})^d$ and, from Lemma 1, we have

$$b^d = \begin{bmatrix} 0 & 0 \\ \left(b_4^d\right)^2 b_3 & b_4^d \end{bmatrix}_p, \qquad b^\pi = \begin{bmatrix} p & 0 \\ -b_4^d b_3 & b_4^\pi \end{bmatrix}_p. \tag{18}$$

From $ab = a^\pi bab^\pi$ and

$$ab = \begin{bmatrix} 0 & 0 \\ a_2b_3 & a_2b_4 \end{bmatrix}_p,$$
$$a^\pi bab^\pi = \begin{bmatrix} 0 & 0 \\ b_3a_1 - b_4a_2b_4^db_3 & b_4a_2b_4^\pi \end{bmatrix}_p, \tag{19}$$

we obtained $a_2b_4 = b_4a_2b_4^\pi$. From Theorem 3, we get $a_2 + b_4 \in \mathscr{A}^d$ and

$$(a_2 + b_4)^d = b_4^d + \sum_{n=0}^{\infty} \left(b_4^d\right)^{n+2} a_2 (a_2 + b_4)^n. \tag{20}$$

Further, applying Lemma 1 to $a + b$, we get

$$(a + b)^d = \begin{bmatrix} a_1^d & 0 \\ u & (a_2 + b_4)^d \end{bmatrix}, \tag{21}$$

where

$$\begin{aligned} u = {} & \sum_{n=0}^{\infty} \left[(a_2 + b_4)^d\right]^{n+2} b_3 a_1^n a_1^\pi \\ & + \sum_{n=0}^{\infty} (a_2 + b_4)^\pi (a_2 + b_4)^n b_3 \left(a_1^d\right)^{n+2} \\ & - (a_2 + b_4)^d b_3 a_1^d. \end{aligned} \tag{22}$$

Observe that since $a_1 \in (p\mathscr{A}p)^{-1}$, then $a_1^\pi = 0$.

Hence, the expression of u reduces to

$$\begin{aligned} u = {} & \sum_{n=0}^{\infty} (a_2 + b_4)^\pi (a_2 + b_4)^n b_3 \left(a_1^d\right)^{n+2} \\ & - (a_2 + b_4)^d b_3 a_1^d. \end{aligned} \tag{23}$$

From $a_2b_4 = b_4a_2b_4^\pi$ we get $= a_2b_4(b_4^d)^2 = b_4a_2b_4^\pi(b_4^d)^2 = 0$.

Hence, from formula (20) and $a_2b_4^d = 0$, we have

$$\begin{aligned} (a_2 + b_4)^\pi & = \overline{p} - (a_2 + b_4) \\ & \quad \cdot \left(b_4^d + \sum_{n=0}^{\infty} \left(b_4^d\right)^{n+2} a_2 (a_2 + b_4)^n\right) \\ & = \overline{p} - b_4 \left(b_4^d + \sum_{n=0}^{\infty} \left(b_4^d\right)^{n+2} a_2 (a_2 + b_4)^n\right) \\ & = b_4^\pi - \sum_{n=0}^{\infty} \left(b_4^d\right)^{n+1} a_2 (a_2 + b_4)^n. \end{aligned} \tag{24}$$

Then substituting (20) and (24) in (22), we get

$$\begin{aligned} u = {} & \sum_{n=0}^{\infty} b_4^\pi (a_2 + b_4)^n b_3 \left(a_1^d\right)^{n+2} \\ & - \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left(b_4^d\right)^{k+1} a_2 (a_2 + b_4)^{n+k} b_3 \left(a_1^d\right)^{n+2} \\ & - b_4^d b_3 a_1^d - \sum_{n=0}^{\infty} \left(b_4^d\right)^{n+2} a_2 (a_2 + b_4)^n b_3 a_1^d. \end{aligned} \tag{25}$$

Now, replacing u by the above expression and considering matrix representations of a and b, after direct computations, we obtain the formula (15) for $(a + b)^d$. □

3. Applications

In this section, we give some formulas for the generalized Drazin inverse of a 2×2 operator matrix under some conditions.

Finding an explicit representation for the generalized Drazin inverse of an operator matrix $M = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ in terms of A, B, C, D and related generalized Drazin inverse has been studied by several authors [9, 13–15]. Djordjević and Stanimirović [9] generalize the well-known result in [8, 16] concerning the Drazin inverse of block 2×2 upper triangular matrices to the generalized Drazin inverse for a block triangular operator matrix. Further, they consider the case where $BC = 0$, $BD = 0$, and $DC = 0$.

This section is devoted to the generalized Drazin inverse of 2×2 operator matrix:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{26}$$

where $A \in \mathbf{B}(X)$ and $D \in \mathbf{B}(Y)$ are generalized Drazin invertible.

Next we will state some auxiliary lemmas.

Lemma 5 (see [2, 3]). *Let A and D be generalized Drazin invertible and let M be matrix of form (26). If $BC = 0$ and $BD = 0$, then*

$$M^d = \begin{bmatrix} A^d & \left(A^d\right)^2 B \\ X_0 & D^d + X_1 B \end{bmatrix}, \tag{27}$$

where

$$\begin{aligned} X_n = {} & \sum_{i=0}^{\infty} \left(D^d\right)^{i+n+2} CA^i A^\pi \\ & + D^\pi \sum_{i=0}^{\infty} D^i C \left(A^d\right)^{i+n+2} \\ & - \sum_{i=0}^{n} \left(D^d\right)^{i+1} C \left(A^d\right)^{n-i+1}, \quad n \geq 0. \end{aligned} \tag{28}$$

Lemma 6 (see [17, Lemma 3.1]). *If M is matrix of form (26), such that A is generalized Drazin invertible, D is quasinilpotent, and $BD^nC = 0$ for any nonnegative integer n, then M is generalized Drazin invertible and*

$$M^d = \begin{bmatrix} A^d & \Gamma \\ \Delta & \Delta A \Gamma \end{bmatrix}, \tag{29}$$

where

$$\Gamma = \sum_{n=0}^{\infty} \left(A^d\right)^{n+2} BD^n, \qquad \Delta = \sum_{n=0}^{\infty} D^n C \left(A^d\right)^{n+2}. \tag{30}$$

Lemma 7. *Let $A \in \mathbb{C}^{n\times n}$. Then $(AA^\pi)^d = 0$, $(A^2A^d)^d = A^d$, $(A^2A^d)^\pi = A^\pi$, and $\operatorname{Ind}(AA^\pi) = \operatorname{Ind}(A)$ and $\operatorname{Ind}(A^2A^d) = 1$.*

Proof. The Jordan canonical form of X permits us to write $A = S(C \oplus N)S^{-1}$, where S and C are nonsingular, and N is nilpotent with index $\operatorname{Ind}(A)$. Thus $A_d = S(C^{-1} \oplus 0)S^{-1}$. Now, it is evident that $A^2A^d = S(C\oplus 0)S^{-1}$ and $AA^\pi = S(0\oplus N)S^{-1}$, which lead to the affirmations of this lemma. □

In [9, Theorem 5.3] authors gave an explicit representation for M^d under conditions $BC = 0$, $DC = 0$, and $BD = 0$. Here we replace the last two conditions by the two weaker conditions $DC = D^\pi CAA^\pi$ and $BD = AA^\pi B$.

Theorem 8. *Let A and D be generalized Drazin invertible and let M be matrix of form (26). If $AA^\pi B = BD$, $DC = D^\pi CAA^\pi$ and $BC = 0$. Then*

$$M^d = \begin{bmatrix} A^d & \left(A^d\right)^2 B + \sum_{n=0}^{\infty} A^n B \left(D^d\right)^{n+2} \\ C\left(A^d\right)^2 & D^d + C\left(A^d\right)^3 B + \sum_{n=1}^{\infty}\sum_{i=1}^{n} D^{i-1}CA^{n-i}B\left(D^d\right)^{n+2} \end{bmatrix}. \tag{31}$$

Proof. We can split matrix M as $M = P + Q$, where

$$P = \begin{bmatrix} AA^\pi & 0 \\ 0 & D \end{bmatrix}, \qquad Q = \begin{bmatrix} A^2A^d & B \\ C & 0 \end{bmatrix},$$
$$P^d = \begin{bmatrix} 0 & 0 \\ 0 & D^d \end{bmatrix}, \qquad P^\pi = \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix}. \tag{32}$$

Since $DC = D^\pi CAA^\pi$ and $AA^\pi B = BD$, we have

$$D^dC = \left(D^d\right)^2 DC = \left(D^d\right)^2 D^\pi CAA^\pi = 0,$$
$$DCA^d = D^\pi CAA^\pi A^d = 0, \tag{33}$$
$$A^dBD = A^dAA^\pi B = 0.$$

From $BC = 0$ and applying Lemma 5 to Q, we obtain

$$\left(Q^d\right)^i = \begin{bmatrix} \left(A^d\right)^i & \left(A^d\right)^{i+1} B \\ X_{i-1} & X_iB \end{bmatrix},$$
$$Q^\pi = \begin{bmatrix} A^\pi & -A^dB \\ -CA^d & I - C\left(A^d\right)^2 B \end{bmatrix}, \tag{34}$$

where X_n is defined in (28). From $D^dC = 0$ and $DCA^d = 0$, we get $X_n = C(A^d)^{n+2}$.

Since $AA^\pi B = BD$ and $DC = D^\pi CAA^\pi$, we obtain $PQ = P^\pi QPQ^\pi$. Applying Theorem 4, we get

$$\begin{aligned}(P+Q)^d &= Q^\pi P^d + Q^dP^\pi \\ &+ \sum_{n=0}^{\infty}\left(Q^d\right)^{n+2} P(P+Q)^n P^\pi \\ &+ Q^\pi \sum_{n=0}^{\infty} (P+Q)^n Q\left(P^d\right)^{n+2} \\ &- \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(Q^d\right)^{k+1} P(P+Q)^{n+k} Q\left(P^d\right)^{n+2} \\ &- \sum_{n=0}^{\infty}\left(Q^d\right)^{n+2} P(P+Q)^n QP^d.\end{aligned} \tag{35}$$

From $A^dBD = 0$, we have

$$Q^\pi P^d = P^d, \qquad Q^dP^\pi = Q^d, \qquad Q^dP = 0. \tag{36}$$

Hence from (35), we obtain

$$(P+Q)^d = P^d + Q^d + Q^\pi \sum_{n=0}^{\infty}(P+Q)^n Q\left(P^d\right)^{n+2}. \tag{37}$$

Since $A^dBD = 0$, we have

$$Q^\pi Q\left(P^d\right)^2 = B\left(D^d\right)^2. \tag{38}$$

The conditions $BC = 0$ and $BD = AA^\pi B$ imply that $BD^nC = 0$. From $BD^nC = 0$ and $A^dBD = 0$, we get

$$Q^\pi \sum_{n=1}^{\infty}(P+Q)^n Q\left(P^d\right)^{n+2} = \begin{bmatrix} 0 & \sum_{n=1}^{\infty} A^n B\left(D^d\right)^{n+2} \\ 0 & \sum_{n=1}^{\infty}\sum_{i=1}^{n} D^{i-1}CA^{n-i}B\left(D^d\right)^{n+2} \end{bmatrix}. \tag{39}$$

From (36), (38), and (39) it follows (31).
The proof is finished. □

Since

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ I_m & 0 \end{bmatrix}\begin{bmatrix} D & C \\ B & A \end{bmatrix}\begin{bmatrix} 0 & I_m \\ I_n & 0 \end{bmatrix}, \tag{40}$$

we can obtain the following result, applying Theorem 8 to $\begin{bmatrix} D & C \\ B & A \end{bmatrix}$.

Theorem 9. *Let A and D be generalized Drazin invertible and let M be matrix of form (26). If $DD^\pi C = CA$, $AB = A^\pi BDD^\pi$ and $CB = 0$. Then*

$$M^d = \begin{bmatrix} A^d + B\left(D^d\right)^3 C + \sum_{n=1}^{\infty}\sum_{i=1}^{n} A^{n-i}BD^{i-1}C\left(A^d\right)^{n+2} & B\left(D^d\right)^2 \\ \left(D^d\right)^2 C + \sum_{n=0}^{\infty} D^nC\left(A^d\right)^{n+2} & D^d \end{bmatrix}. \tag{41}$$

Theorem 10. *Let A, D, and BC be generalized Drazin invertible and let M be matrix of form (26). If $AB = A^\pi BD$, $DC = D^\pi CA$ and $BC = 0$. Then*

$$M^d = \begin{bmatrix} A^d & \sum_{n=0}^{\infty} A^nB\left(D^d\right)^{n+2} \\ \sum_{n=0}^{\infty} D^nC\left(A^d\right)^{n+2} & D^d + \sum_{n=1}^{\infty}\sum_{i=0}^{n-1} D^iCA^{n-i-1}B\left(D^d\right)^{n+2} \end{bmatrix}. \tag{42}$$

Proof. We can split matrix M as $M = P + Q$, where

$$P = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}, \tag{43}$$

$$P^d = \begin{bmatrix} A^d & 0 \\ 0 & D^d \end{bmatrix}, \qquad P^\pi = \begin{bmatrix} A^\pi & 0 \\ 0 & D^\pi \end{bmatrix}. \tag{44}$$

Since

$$Q^2 = \begin{bmatrix} BC & 0 \\ 0 & CB \end{bmatrix}, \qquad Q^3 = \begin{bmatrix} 0 & BCB \\ CBC & 0 \end{bmatrix}, \tag{45}$$

from $BC = 0$, it is easy to get $Q^3 = 0$. Since Q is nilpotent, we have $Q^d = 0$. Applying Theorem 4 to the particular case, we get

$$(P+Q)^d = P^d + \sum_{n=0}^{\infty} (P+Q)^n Q \left(P^d\right)^{n+2}. \tag{46}$$

The conditions $AB = A^\pi BD$ and $BC = 0$ imply that $BD^nC = 0$, for $n \geq 0$, so we get

$$\sum_{n=0}^{\infty} (P+Q)^n Q \left(P^d\right)^{n+2} = \begin{bmatrix} 0 & \sum_{n=0}^{\infty} A^n B \left(D^d\right)^{n+2} \\ \sum_{n=0}^{\infty} D^n C \left(A^d\right)^{n+2} & \sum_{n=1}^{\infty}\sum_{i=0}^{n-1} D^i C A^{n-i-1} B \left(D^d\right)^{n+2} \end{bmatrix}. \tag{47}$$

From (44) and (47) it follows (42).

The proof is finished. □

Theorem 11. *Let A and D be generalized Drazin invertible and let M be matrix of form (26). If $AA^\pi B = BD^2D^d$, $D^2D^dC = D^\pi CAA^\pi$ and $BD^nC = 0$ for any nonnegative integer n. Then*

$$M^d = \begin{bmatrix} A^d & \Gamma + \sum_{n=0}^{\infty} A^n B \left(D^d\right)^{n+2} \\ \Delta & D^d + \Delta A \Gamma + \sum_{n=1}^{\infty}\sum_{i=0}^{n-1} D^i C A^{n-i-1} B \left(D^d\right)^{n+2} \end{bmatrix}, \tag{48}$$

where Γ and Δ are defined in (30).

Proof. We can split matrix M as $M = P + Q$, where

$$P = \begin{bmatrix} AA^\pi & 0 \\ 0 & D^2D^d \end{bmatrix}, \qquad Q = \begin{bmatrix} A^2A^d & B \\ C & DD^\pi \end{bmatrix},$$
$$P^d = \begin{bmatrix} 0 & 0 \\ 0 & D^d \end{bmatrix}, \qquad P^\pi = \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix}. \tag{49}$$

From $AA^\pi B = BD^2D^d$ and $D^2D^dC = D^\pi CAA^\pi$, we have

$$\begin{aligned} D^dC &= \left(D^d\right)^3 D^2C = \left(D^d\right)^2 D^2D^dC \\ &= \left(D^d\right)^2 D^\pi CAA^\pi = 0, \end{aligned} \tag{50}$$

$$\begin{aligned} A^dBD^d &= A^dBD^2\left(D^d\right)^3 = A^dBD^2D^d\left(D^d\right)^2 \\ &= A^dAA^\pi B\left(D^d\right)^2 = 0, \end{aligned} \tag{51}$$

so we get $D^\pi C = C$.

Note that DD^π is quasinilpotent, $D^\pi C = C$, and $B(DD^\pi)^nC = BD^nD^\pi C = BD^nC = 0$ for any nonnegative integer n; we can apply Lemma 6 to Q with D replaced by DD^π; we have

$$Q^d = \begin{bmatrix} A^d & \Gamma' \\ \Delta' & \Delta' A \Gamma' \end{bmatrix}, \tag{52}$$

where

$$\Gamma' = \sum_{n=0}^{\infty} \left(A^d\right)^{n+2} BD^nD^\pi, \qquad \Delta' = \sum_{n=0}^{\infty} D^nD^\pi C\left(A^d\right)^{n+2}. \tag{53}$$

Observe that (50) and (51) yield

$$\Gamma = \sum_{n=0}^{\infty} \left(A^d\right)^{n+2} BD^n, \qquad \Delta = \sum_{n=0}^{\infty} D^nC\left(A^d\right)^{n+2}, \tag{54}$$

so we get

$$Q^d = \begin{bmatrix} A^d & \Gamma \\ \Delta & \Delta A \Gamma \end{bmatrix}. \tag{55}$$

The condition $BD^nC = 0$ implies that

$$B\Delta = B\sum_{n=0}^{\infty} D^nC\left(A^d\right)^{n+2} = 0. \tag{56}$$

Hence we have

$$\begin{aligned} QQ^d &= \begin{bmatrix} AA^d + B\Delta & A^2A^d\Gamma + B\Delta A\Gamma \\ CA^d + DD^\pi\Delta & C\Gamma + DD^\pi\Delta A\Gamma \end{bmatrix} \\ &= \begin{bmatrix} AA^d & A\Gamma \\ CA^d + D\Delta & C\Gamma + D\Delta A\Gamma \end{bmatrix}, \\ Q^\pi &= \begin{bmatrix} A^\pi & -A\Gamma \\ -CA^d - D\Delta & I - C\Gamma - D\Delta A\Gamma \end{bmatrix}. \end{aligned} \tag{57}$$

From $AA^\pi B = BD^2D^d$ and $D^2D^dC = D^\pi CAA^\pi$, we obtain $PQ = P^\pi QPQ^\pi$. Applying Theorem 4, we get

$$\begin{aligned}(P+Q)^d &= Q^\pi P^d + Q^d P^\pi \\ &+ \sum_{n=0}^{\infty}\left(Q^d\right)^{n+2} P(P+Q)^n P^\pi \\ &+ Q^\pi \sum_{n=0}^{\infty}(P+Q)^n Q\left(P^d\right)^{n+2} \\ &- \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(Q^d\right)^{k+1} P(P+Q)^{n+k} Q\left(P^d\right)^{n+2} \\ &- \sum_{n=0}^{\infty}\left(Q^d\right)^{n+2} P(P+Q)^n QP^d,\end{aligned} \tag{58}$$

where

$$\begin{aligned}\Gamma D^2D^d &= \sum_{n=0}^{\infty}\left(A^d\right)^{n+2} BD^nD^\pi D^2D^d = 0, \\ \Delta AA^\pi &= \sum_{n=0}^{\infty} D^nD^\pi C\left(A^d\right)^{n+2} AA^\pi = 0,\end{aligned} \tag{59}$$

so we get

$$\begin{aligned}Q^dP &= \begin{bmatrix} A^d & \Gamma \\ \Delta & \Delta A\Gamma \end{bmatrix}\begin{bmatrix} AA^\pi & 0 \\ 0 & D^2D^d \end{bmatrix} \\ &= \begin{bmatrix} A^dAA^\pi & \Gamma D^2D^d \\ \Delta AA^\pi & \Delta A\Gamma D^2D^d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned} \tag{60}$$

Hence from (58) and (60) we obtain

$$\begin{aligned}(P+Q)^d &= Q^\pi P^d + Q^dP^\pi \\ &+ Q^\pi \sum_{n=0}^{\infty}(P+Q)^n Q\left(P^d\right)^{n+2}.\end{aligned} \tag{61}$$

By direct computation we verify that

$$Q^\pi P^d = P^d, \qquad Q^dP^\pi = Q^d. \tag{62}$$

From $BD^nC = 0$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}(P+Q)^n Q\left(P^d\right)^{n+2} \\ &= \begin{bmatrix} 0 & \sum_{n=0}^{\infty} A^nB\left(D^d\right)^{n+2} \\ 0 & \sum_{n=1}^{\infty}\sum_{i=0}^{n-1} D^iCA^{n-i-1}B\left(D^d\right)^{n+2} \end{bmatrix}.\end{aligned} \tag{63}$$

Observe that (51) and $BD^nC = 0$ yield

$$\begin{aligned}&Q^\pi \sum_{n=0}^{\infty}(P+Q)^n Q\left(P^d\right)^{n+2} \\ &= \begin{bmatrix} A^\pi & -A\Gamma \\ -CA^d - D\Delta & I - C\Gamma - D\Delta A\Gamma \end{bmatrix}\begin{bmatrix} 0 & \sum_{n=0}^{\infty} A^nB\left(D^d\right)^{n+2} \\ 0 & \sum_{n=1}^{\infty}\sum_{i=0}^{n-1} D^iCA^{n-i-1}B\left(D^d\right)^{n+2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & A^\pi\sum_{n=0}^{\infty} A^nB\left(D^d\right)^{n+2} - A\Gamma\sum_{n=1}^{\infty}\sum_{i=0}^{n-1} D^iCA^{n-i-1}B\left(D^d\right)^{n+2} \\ 0 & \left(-CA^d - D\Delta\right)\sum_{n=0}^{\infty} A^nB\left(D^d\right)^{n+2} + (I - C\Gamma - D\Delta A\Gamma)\sum_{n=1}^{\infty}\sum_{i=0}^{n-1} D^iCA^{n-i-1}B\left(D^d\right)^{n+2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \sum_{n=0}^{\infty} A^nB\left(D^d\right)^{n+2} \\ 0 & \sum_{n=1}^{\infty}\sum_{i=0}^{n-1} D^iCA^{n-i-1}B\left(D^d\right)^{n+2} \end{bmatrix}.\end{aligned} \tag{64}$$

From (62) and (64) it follows (48).

The proof is finished. □

Theorem 12. *Let A and D be generalized Drazin invertible and let M be matrix of form (26). If $AA^{\pi}BD^{\pi} = BD$, $BC = 0$, $CA^{d} = 0$, and $CBD^{\pi} = 0$. Then*

$$M^{d} = \begin{bmatrix} A^{d} & \left(A^{d}\right)^{2} B \\ X_{0} & D^{d} + \sum_{n=0}^{\infty} X_{n+1} B D^{n} \end{bmatrix}, \tag{65}$$

where

$$X_{n} = \sum_{i=0}^{\infty} \left(D^{d}\right)^{i+n+2} C A^{i}, \quad n \geq 0. \tag{66}$$

Proof. We can split matrix M as $M = P + Q$, where

$$P = \begin{bmatrix} A^{2}A^{d} & B \\ 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} AA^{\pi} & 0 \\ C & D \end{bmatrix},$$
$$P^{d} = \begin{bmatrix} A^{d} & \left(A^{d}\right)^{2} B \\ 0 & 0 \end{bmatrix}, \qquad P^{\pi} = \begin{bmatrix} A^{\pi} & -A^{d}B \\ 0 & I \end{bmatrix}. \tag{67}$$

Applying Lemma 7, we have $(AA^{\pi})^{d} = 0$, so we get

$$\left(Q^{d}\right)^{n} = \begin{bmatrix} 0 & 0 \\ X_{n-1} & \left(D^{d}\right)^{n} \end{bmatrix}, \qquad Q^{\pi} = \begin{bmatrix} I & 0 \\ -DX_{0} & D^{\pi} \end{bmatrix}, \tag{68}$$

where X_{n} is defined in (28).

Since $AA^{\pi}BD^{\pi} = BD$, $BC = 0$, $CBD^{\pi} = 0$, and $CA^{2}A^{d} = 0$, we obtain $PQ = P^{\pi}QPQ^{\pi}$. Applying Theorem 4, we get

$$\begin{aligned} (P+Q)^{d} &= Q^{\pi}P^{d} + Q^{d}P^{\pi} \\ &\quad + \sum_{n=0}^{\infty} \left(Q^{d}\right)^{n+2} P (P+Q)^{n} P^{\pi} \\ &\quad + Q^{\pi} \sum_{n=0}^{\infty} (P+Q)^{n} Q \left(P^{d}\right)^{n+2} \\ &\quad - \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left(Q^{d}\right)^{k+1} P (P+Q)^{n+k} Q \left(P^{d}\right)^{n+2} \\ &\quad - \sum_{n=0}^{\infty} \left(Q^{d}\right)^{n+2} P (P+Q)^{n} Q P^{d}. \end{aligned} \tag{69}$$

From $CA^{d} = 0$, we have $QP^{d} = 0$. Hence from (69) we obtain

$$(P+Q)^{d} = Q^{\pi}P^{d} + Q^{d}P^{\pi} + \sum_{n=0}^{\infty} \left(Q^{d}\right)^{n+2} P (P+Q)^{n} P^{\pi}, \tag{70}$$

where $X_{n}A^{d} = 0$, we get

$$Q^{\pi}P^{d} = P^{d}, \qquad Q^{d}P^{\pi} = Q^{d},$$
$$\left(Q^{d}\right)^{2} PP^{\pi} = \begin{bmatrix} 0 & 0 \\ X_{1} & \left(D^{d}\right)^{2} \end{bmatrix} \begin{bmatrix} A^{2}A^{d} & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^{\pi} & -A^{d}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & X_{1}B \end{bmatrix}. \tag{71}$$

The conditions $AA^{\pi}BD^{\pi} = BD$ and $BC = 0$ imply that $BD^{i}C = 0$. So we get

$$\sum_{n=0}^{\infty} \left(Q^{d}\right)^{n+2} P (P+Q)^{n} P^{\pi} = \begin{bmatrix} 0 & 0 \\ 0 & \sum_{n=1}^{\infty} X_{n+1} B D^{n} \end{bmatrix}, \quad n \geq 1. \tag{72}$$

From (71) and (72) it follows (65).
The proof is finished. □

Using (40) and Theorem 12, we have the following result.

Theorem 13. *If $CA = D^{\pi}DCA^{\pi}$, $BD^{d} = 0$, $CB = 0$, and $BCA^{\pi} = 0$. Then*

$$M^{d} = \begin{bmatrix} A^{d} + \sum_{n=0}^{\infty} X_{n+2} C A^{n} & X_{1} \\ \left(D^{d}\right)^{2} C & D^{d} \end{bmatrix}, \tag{73}$$

where

$$X_{n} = \sum_{i=0}^{\infty} \left(A^{d}\right)^{i+n+1} B D^{i}, \quad n \geq 1. \tag{74}$$

Using the case of Theorem 3, we get the following results.

Theorem 14. *Let A and D be generalized Drazin invertible and let M be matrix of form (26). If $DCA^{\pi} = CA$, $ABD = 0$, $BC = 0$, and $CB = 0$. Then*

$$M^{d} = \begin{bmatrix} A^{d} + \sum_{n=0}^{\infty} B \left(D^{d}\right)^{n+3} C A^{n} & \left(A^{d}\right)^{2} B + B \left(D^{d}\right)^{2} \\ \sum_{n=0}^{\infty} \left(D^{d}\right)^{n+2} C A^{n} & D^{d} \end{bmatrix}. \tag{75}$$

Proof. We can split matrix M as $M = P + Q$, where

$$P = \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}. \tag{76}$$

From $ABD = 0$, we have

$$\left(Q^{d}\right)^{i} = \begin{bmatrix} \left(A^{d}\right)^{i} & X_{i} \\ 0 & \left(D^{d}\right)^{i} \end{bmatrix},$$
$$Q^{\pi} = \begin{bmatrix} A^{\pi} & -AX_{1} - BD^{d} \\ 0 & D^{\pi} \end{bmatrix}, \tag{77}$$

where

$$X_{n} = \left(A^{d}\right)^{n+2} B + B \left(D^{d}\right)^{n+2}, \quad n \geq 0. \tag{78}$$

Note that P is quasinilpotent; since $DCA^{\pi} = CA$, $ABD = 0$, $BC = 0$, and $CB = 0$, we obtain $PQ = QPQ^{\pi}$. Applying Theorem 3, we get

$$(P+Q)^{d} = Q^{d} + \sum_{n=0}^{\infty} \left(Q^{d}\right)^{n+2} P (P+Q)^{n}. \tag{79}$$

From $BC = 0$, we have

$$\begin{aligned}\left(Q^d\right)^2 P &= \begin{bmatrix} \left(A^d\right)^2 & X_2 \\ 0 & \left(D^d\right)^2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} \\ &= \begin{bmatrix} X_2 C & 0 \\ \left(D^d\right)^2 C & 0 \end{bmatrix} = \begin{bmatrix} B\left(D^d\right)^3 & 0 \\ \left(D^d\right)^2 C & 0 \end{bmatrix}.\end{aligned} \tag{80}$$

The conditions $DCA^{\pi} = CA$ and $CB = 0$ imply that $CA^i B = 0$. From $ABD = 0$, $CA^i B = 0$, and $BC = 0$, we get

$$\begin{aligned}&\sum_{n=1}^{\infty} \left(Q^d\right)^{n+2} P (P+Q)^n \\ &= \begin{bmatrix} \sum_{n=1}^{\infty} X_{n+2} CA^n & \sum_{n=1}^{\infty} X_{n+2} CA^{n-1} B \\ \sum_{n=1}^{\infty} \left(D^d\right)^{n+2} CA^n & \sum_{n=1}^{\infty} \left(D^d\right)^{n+2} CA^{n-1} B \end{bmatrix} \\ &= \begin{bmatrix} \sum_{n=1}^{\infty} B\left(D^d\right)^{n+3} CA^n & 0 \\ \sum_{n=1}^{\infty} \left(D^d\right)^{n+2} CA^n & 0 \end{bmatrix}.\end{aligned} \tag{81}$$

From (77), (80), and (81) it follows (75). □

Using (40) and Theorem 14, we have the following result.

Theorem 15. *Let A and D be generalized Drazin invertible and let M be matrix of form (26). If $ABD^{\pi} = BD$, $DCA = 0$, $BC = ABCA^d$, and $CB = 0$. Then*

$$M^d = \begin{bmatrix} A^d & \sum_{n=0}^{\infty} \left(A^d\right)^{n+2} BD^n \\ \left(D^d\right)^2 C + C\left(A^d\right)^2 & D^d + \sum_{n=0}^{\infty} C\left(A^d\right)^{n+3} BD^n \end{bmatrix}. \tag{82}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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