

**AN INVERSE EIGENVALUE PROBLEM FOR AN ARBITRARY  
 MULTIPLY CONNECTED BOUNDED REGION:  
 AN EXTENSION TO HIGHER DIMENSIONS**

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**ABSTRACT.** The basic problem in this paper is that of determining the geometry of an arbitrary multiply connected bounded region in  $R^3$  together with the mixed boundary conditions, from the complete knowledge of the eigenvalues  $\{\lambda_j\}_{j=1}^{\infty}$  for the negative Laplacian, using the asymptotic expansion of the spectral function  $\Theta(t) = \sum_{j=1}^{\infty} \exp(-t\lambda_j)$  as  $t \rightarrow 0$ .

**KEY WORDS AND PHRASES.** Inverse problem, Laplace's operator, eigenvalue problem and spectral function.

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**1. INTRODUCTION.**

The underlying problem is to deduce the precise shape of a membrane from the complete knowledge of the eigenvalues  $\{\lambda_j\}_{j=1}^{\infty}$  for the negative Laplacian  $-\Delta_3 = -\sum_{i=1}^3 (\frac{\partial}{\partial x^i})^2$  in the  $(x^1, x^2, x^3)$ -space.

Let  $\Omega \subseteq R^3$  be a simply connected bounded domain with a smooth bounding surface  $S$ . Consider the Dirichlet/Neumann problem

$$(\Delta_3 + \lambda)u = 0 \text{ in } \Omega, \tag{1.1}$$

$$u = 0 \text{ or } \frac{\partial u}{\partial n} = 0 \text{ on } S, \tag{1.2}$$

where  $\frac{\partial}{\partial n}$  denotes differentiation along the inward pointing normal to  $S$ . Denote its eigenvalues, counted according to multiplicity, by

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \leq \dots \rightarrow \infty \text{ as } j \rightarrow \infty. \tag{1.3}$$

The problem of determining the geometry of  $\Omega$  has been discussed by Pleijel [4], McKean and Singer [3], Waechter [5], Gottlieb [1], Hsu [2] and Zayed [6-8, 11], using the asymptotic expansion of the spectral function

$$\Theta(t) = \sum_{j=1}^{\infty} \exp(-t\lambda_j) \text{ as } t \rightarrow 0. \tag{1.4}$$

It has been shown that, in the case of Dirichlet boundary conditions (D.b.c)

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} - \frac{|S|}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}} \int_S H \, dS + a_0 + o(t^{1/2}) \text{ as } t \rightarrow 0, \tag{1.5}$$

while, in the case of Neumann boundary conditions (N.b.c.),

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{|S|}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}} \int_S H \, dS + a_0 + o(t^{1/2}) \text{ as } t \rightarrow 0, \tag{1.6}$$

In these formulae,  $V$  and  $|S|$  are respectively the volume and the surface area of  $\Omega$ , while

$H = \frac{1}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$  is the mean curvature of  $S$ , where  $R_1$  and  $R_2$  are the principal radii of curvature. Furthermore, the constant term  $a_0$  in (1.5) and (1.6) has the following forms:

$$a_0 = \begin{cases} \frac{1}{512\pi} \int_S \left( \frac{1}{R_1} - \frac{1}{R_2} \right)^2 ds, & \text{in the case of D.b.c. (see [5]),} \\ \frac{7}{512\pi} \int_S \left( \frac{1}{R_1} - \frac{1}{R_2} \right)^2 ds, & \text{in the case of N.b.c. (see [2]).} \end{cases} \tag{1.7}$$

In terms of the mean curvature  $H$  and Gaussian curvature  $N = \frac{1}{R_1 R_2}$ , (1.7) may be rewritten in the forms:

$$a_0 = \begin{cases} \frac{1}{128\pi} \int_S (H^2 - N) ds, & \text{in the case of D.b.c.,} \\ \frac{7}{128\pi} \int_S (H^2 - N) ds, & \text{in the case of N.b.c.} \end{cases} \tag{1.8}$$

The object of this paper is to discuss the following more general inverse problem: Let  $\Omega$  be an arbitrary multiply connected bounded region in  $R^3$  which is surrounded internally by simply connected bounded domains  $\Omega_i, i = 1, 2, \dots, m-1$ , and externally by a simply connected bounded domain  $\Omega_m$  with a smooth bounding surface  $S_m$ . Suppose that the eigenvalues (1.3) are given for the eigenvalue equation

$$(\Delta_3 + \lambda)u = 0 \text{ in } \Omega, \tag{1.9}$$

together with one of the following mixed boundary conditions:

$$\frac{\partial u}{\partial n_i} = 0 \text{ on } S_i, \quad i = 1, \dots, k, \quad u = 0 \text{ on } S_i, \quad i = k + 1, \dots, m, \tag{1.10}$$

or

$$u = 0 \text{ on } S_i, \quad i = 1, \dots, k, \quad \frac{\partial u}{\partial n_i} = 0 \text{ on } S_i, \quad i = k + 1, \dots, m, \tag{1.11}$$

where  $\frac{\partial}{\partial n_i}$  denote differentiations along the inward pointing normals to  $S_i, i = 1, \dots, m$ . Determine the geometry of  $\Omega$  from the asymptotic form of the spectral function  $\Theta(t)$  for small positive  $t$ .

Note that problem (1.9)-(1.11) has been investigated recently by Zayed [11] in the special case when  $\Omega$  is an arbitrary doubly connected region (i.e.,  $m = 2$ ).

2. STATEMENT OF RESULTS.

Suppose that the bounding surfaces  $S_i (i = 1, \dots, m)$  of the region  $\Omega$  are given locally by infinitely differentiable functions  $x^n = y^n(\sigma_i), n = 1, 2, 3$ , of the parameters  $\sigma_i^\alpha = \text{constants}$ , are lines of curvature, the first and second fundamental forms of  $S_i (i = 1, \dots, m)$  can be written respectively in the following forms:

$$\prod_{1i}(\sigma_i, \Delta \sigma_i) = a_{11}(\sigma_i)(\Delta \sigma_i^1)^2 + a_{21}(\sigma_i)(\Delta \sigma_i^2)^2,$$

and

$$\prod_{2i}(\sigma_i, \Delta \sigma_i) = b_{1i}(\sigma_i)(\Delta \sigma_i^1)^2 + b_{2i}(\sigma_i)(\Delta \sigma_i^2)^2.$$

In terms of the coefficients  $a_{11}, a_{21}, b_{1i}, b_{2i}$  the principal radii of curvatures for  $S_i (i = 1, \dots, m)$  are given by:

$$R_{1i} = a_{1i}/b_{1i} \text{ and } R_{2i} = a_{2i}/b_{2i}.$$

Consequently, the mean curvatures  $H_i$  and Gaussian curvatures  $N_i$  of the bounding surfaces  $S_i (i = 1, \dots, m)$  are defined by:

$$H_i = \frac{1}{2} \left( \frac{1}{R_{1i}} + \frac{1}{R_{2i}} \right) \text{ and } N_i = \frac{1}{R_{1i} R_{2i}}.$$

Let  $|S_i|, (i = 1, \dots, m)$  be the surface areas of the bounding surfaces  $S_i, (i = 1, \dots, m)$  respectively. Then, the results of problem (1.9)-(1.11) can be summarized in the following cases:

CASE 1. (N.b.c. on  $S_i, i = 1, \dots, k$  and D.b.c. on  $S_i, i = k + 1, \dots, m$ )

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{1}{16\pi t} \left\{ \sum_{i=1}^k |S_i| - \sum_{i=k+1}^m |S_i| \right\} + \frac{1}{12\pi^{3/2} t^{1/2}} \sum_{i=1}^m \int_{S_i} H_i ds_i$$

$$+ \frac{1}{128\pi} \left\{ 7 \sum_{i=1}^k \int_{S_i} (H_i^2 - N_i) ds_i + \sum_{i=k+1}^m \int_{S_i} (H_i^2 - N_i) ds_i \right\}$$

$$\begin{aligned}
 & + (t/\pi^3)^{\frac{1}{2}} \left\{ \frac{13}{1440} \sum_{i=1}^k \int_{S_i} H_i^3 dS_i - \frac{1}{315} \sum_{i=k+1}^m \int_{S_i} H_i^3 dS_i \right\} \\
 & + 0(t) \text{ as } t \rightarrow 0.
 \end{aligned} \tag{2.1}$$

CASE 2. (D.b.c. on  $S_i, i = 1, \dots, k$  and N.b.c. on  $S_i, i = k + 1, \dots, m$ )

In this case, the asymptotic expansion of  $\Theta(t)$  as  $t \rightarrow 0$  follows directly from (2.1) with the interchanges  $S_i, i = 1, \dots, k \rightarrow S_i, i = k + 1, \dots, m$ .

With reference to formulae (1.5), (1.6) and to the articles [1], [2], [7], [11], the asymptotic expansion (2.1) may be interpreted as follows:

(i)  $\Omega$  is an arbitrary multiply connected bounded region in  $R^3$  and we have the mixed boundary conditions (1.10) or (1.11) as indicated in the specifications of the two respective cases.

(ii) For the first five terms,  $\Omega$  is an arbitrary multiply connected bounded region in  $R^3$  of volume  $V$ .

In Case 1, the bounding surfaces  $S_i, i = 1, \dots, k$  are of surface areas  $\sum_{i=1}^k |S_i|$ , mean curvatures  $H_i$  and Gaussian curvature  $N_i$  together with Neumann boundary conditions, while the bounding surfaces  $S_i, i = k + 1, \dots, m$  are of surface areas  $\sum_{i=k+1}^m |S_i|$ , mean curvatures  $H_i$  and Gaussian curvature  $N_i$  together with Dirichlet boundary conditions.

We close this section with the following remarks:

REMARK 2.1. On setting  $k = 0$  in (2.1) with the usual definition that  $\sum_{i=1}^0$  is zero, we obtain the result of D.b.c. on  $S_i, i = 1, \dots, m$ .

REMARK 2.2. On setting  $k = m$  in (2.1) with the usual definition that  $\sum_{i=m+1}^m$  is zero, we obtain the result of N.b.c. on  $S_i, i = 1, \dots, m$ .

3. FORMULATION OF THE MATHEMATICAL PROBLEM.

In analogy with the two-dimensional problem (see [9, 10]), it is easy to show that  $\Theta(t)$  associated with problem (1.9)-(1.11) is given by:

$$\Theta(t) = \int \int \int_{\Omega} G(\underline{x}, \underline{x}; t) d\underline{x}, \tag{3.1}$$

where  $G(\underline{x}_1, \underline{x}_2; t)$  is Green's function for the heat equation

$$\left( \Delta_3 - \frac{\partial}{\partial t} \right) u = 0, \tag{3.2}$$

subject to the mixed boundary conditions (1.10) or (1.11) and the initial condition

$$\lim_{t \rightarrow 0} G(\underline{x}_1, \underline{x}_2; t) = \delta(\underline{x}_1 - \underline{x}_2), \tag{3.3}$$

where  $\delta(\underline{x}_1 - \underline{x}_2)$  is the Dirac delta function located at the source point,  $\underline{x}_2$ .

Let us write

$$G(\underline{x}_1, \underline{x}_2; t) = G_0(\underline{x}_1, \underline{x}_2; t) + \chi(\underline{x}_1, \underline{x}_2; t), \tag{3.4}$$

where

$$G_0(\underline{x}_1, \underline{x}_2; t) = (4\pi t)^{-3/2} \exp \left\{ -\frac{|\underline{x}_1 - \underline{x}_2|^2}{4t} \right\}, \tag{3.5}$$

is the "fundamental solution" of the heat equation (3.2) while  $\chi(\underline{x}_1, \underline{x}_2; t)$  is the "regular solution" chosen so that  $G(\underline{x}_1, \underline{x}_2; t)$  satisfies the mixed boundary conditions (1.10) or (1.11).

On setting  $\underline{x}_1 = \underline{x}_2 = \underline{x}$  we find that

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + K(t), \tag{3.6}$$

where

$$K(t) = \int \int \int_{\Omega} \chi(\underline{x}, \underline{x}; t) d\underline{x}. \tag{3.7}$$

In what follows, we shall use Laplace transforms with respect to  $t$ , and use  $s^2$  as the Laplace transform parameter; thus we define

$$\bar{G}(\underline{x}_1, \underline{x}_2; s^2) = \int_0^{+\infty} e^{-s^2 t} G(\underline{x}_1, \underline{x}_2; t) dt. \tag{3.8}$$

An application of the Laplace transform to the heat equation (3.2) shows that  $\bar{G}(\underline{x}_1, \underline{x}_2; s^2)$  satisfies the membrane equation

$$(\Delta_3 - s^2)\bar{G}(\underline{x}_1, \underline{x}_2; s^2) = -\delta(\underline{x}_1 - \underline{x}_2) \text{ in } \Omega, \tag{3.9}$$

together with the mixed boundary conditions (1.10) or (1.11).

The asymptotic expansion of  $K(t)$  as  $t \rightarrow 0$ , may then be deduced directly from the asymptotic expansion of  $\bar{K}(s^2)$  as  $s \rightarrow \infty$ , where

$$\bar{K}(s^2) = \int \int \int_{\Omega} \bar{\chi}(\underline{x}, \underline{x}; s^2) d\underline{x}. \tag{3.10}$$

4. CONSTRUCTION OF GREEN'S FUNCTION.

It is well known [7] that the membrane equation (3.9) has the fundamental solution

$$\bar{G}_0(\underline{x}_1, \underline{x}_2; s^2) = \frac{\exp(-sr_{\underline{x}_1 \underline{x}_2})}{4\pi r_{\underline{x}_1 \underline{x}_2}}, \tag{4.1}$$

where  $r_{\underline{x}_1, \underline{x}_2} = |\underline{x}_1 - \underline{x}_2|$  is the distance between the points  $\underline{x}_1 = (\underline{x}_1^1, \underline{x}_1^2, \underline{x}_1^3)$  and  $\underline{x}_2 = (\underline{x}_2^1, \underline{x}_2^2, \underline{x}_2^3)$  of the domain  $\Omega$ . The existence of the solution (4.1) enables us to construct integral equations for  $\bar{G}(\underline{x}_1, \underline{x}_2; s^2)$  satisfying the mixed boundary conditions (1.10) or (1.11). Therefore, in Case 1, Green's theorem gives:

$$\begin{aligned} \bar{G}(\underline{x}_1, \underline{x}_2; s^2) &= \frac{\exp(-sr_{\underline{x}_1 \underline{x}_2})}{4\pi r_{\underline{x}_1 \underline{x}_2}} + \frac{1}{2\pi} \sum_{i=1}^k \int_{S_i} \bar{G}(\underline{x}_1, \underline{y}; s^2) \frac{\partial}{\partial n_{i\underline{y}}} \left[ \frac{\exp(-sr_{\underline{y} \underline{x}_2})}{r_{\underline{y} \underline{x}_2}} \right] d\underline{y} \\ &\quad + \frac{1}{2\pi} \sum_{i=k+1}^m \int_{S_i} \frac{\partial}{\partial n_{i\underline{y}}} \left[ \bar{G}(\underline{x}_1, \underline{y}; s^2) \right] \frac{\exp(-sr_{\underline{y} \underline{x}_2})}{r_{\underline{y} \underline{x}_2}} d\underline{y}. \end{aligned} \tag{4.2}$$

On applying the iteration method (see [7], [9], [11]) to the integral equation (4.2), we obtain the Green's function  $\bar{G}(\underline{x}_1, \underline{x}_2; s^2)$  which has the regular part:

$$\begin{aligned} \bar{\chi}(\underline{x}_1, \underline{x}_2; s^2) &= \frac{1}{8\pi^2} \sum_{i=1}^k \int_{S_i} \frac{\exp(-sr_{\underline{x}_1 \underline{y}})}{r_{\underline{x}_1 \underline{y}}} \frac{\partial}{\partial n_{i\underline{y}}} \left[ \frac{\exp(-sr_{\underline{y} \underline{x}_2})}{r_{\underline{y} \underline{x}_2}} \right] d\underline{y} \\ &\quad + \frac{1}{8\pi^2} \sum_{i=k+1}^m \int_{S_i} \frac{\partial}{\partial n_{i\underline{y}}} \left[ \frac{\exp(-sr_{\underline{x}_1 \underline{y}})}{r_{\underline{x}_1 \underline{y}}} \right] \frac{\exp(-sr_{\underline{y} \underline{x}_2})}{r_{\underline{y} \underline{x}_2}} d\underline{y} \\ &\quad + \frac{1}{8\pi^2} \sum_{i=1}^k \int \int_{S_i, S_i} \frac{\exp(-sr_{\underline{x}_1 \underline{y}})}{r_{\underline{x}_1 \underline{y}}} M_i(\underline{y}, \underline{y}') \frac{\partial}{\partial n_{i\underline{y}'}} \left[ \frac{\exp(-sr_{\underline{y}' \underline{x}_2})}{r_{\underline{y}' \underline{x}_2}} \right] d\underline{y} d\underline{y}' \\ &\quad + \frac{1}{8\pi^2} \sum_{i=k+1}^m \int \int_{S_i, S_i} \frac{\partial}{\partial n_{i\underline{y}'}} \left[ \frac{\exp(-sr_{\underline{x}_1 \underline{y}})}{r_{\underline{x}_1 \underline{y}}} \right] M_i^*(\underline{y}, \underline{y}') \frac{\exp(-sr_{\underline{y}' \underline{x}_2})}{r_{\underline{y}' \underline{x}_2}} d\underline{y} d\underline{y}' \\ &\quad + \frac{1}{8\pi^2} \sum_{i=1}^k \int_{S_i} \left\{ \sum_{i=k+1}^m \int_{S_i} \frac{\partial}{\partial n_{i\underline{y}'}} \left[ \frac{\exp(-sr_{\underline{x}_1 \underline{y}})}{r_{\underline{x}_1 \underline{y}}} \right] L_i(\underline{y}, \underline{y}') d\underline{y}' \right\} \times \\ &\quad \times \frac{\partial}{\partial n_{i\underline{y}'}} \left[ \frac{\exp(-sr_{\underline{y}' \underline{x}_2})}{r_{\underline{y}' \underline{x}_2}} \right] d\underline{y}' \\ &\quad + \frac{1}{8\pi^2} \sum_{i=k+1}^m \int_{S_i} \left\{ \sum_{i=1}^k \int_{S_i} \frac{\exp(-sr_{\underline{x}_1 \underline{y}})}{r_{\underline{x}_1 \underline{y}}} L_i^*(\underline{y}, \underline{y}') d\underline{y}' \right\} \frac{\exp(-sr_{\underline{y} \underline{x}_2})}{r_{\underline{y} \underline{x}_2}} d\underline{y}, \end{aligned} \tag{4.3}$$

where

$$M_i(\underline{y}, \underline{y}') = \sum_{\nu=0}^{\infty} K_i^{(\nu)}(\underline{y}, \underline{y}'), \tag{4.4}$$

$$M_i^*(\underline{y}, \underline{y}') = \sum_{\nu=0}^{\infty} {}^*K_i^{(\nu)}(\underline{y}, \underline{y}'), \tag{4.5}$$

$$L_i(\underline{y}, \underline{y}') = \sum_{\nu=0}^{\infty} K_i^{(\nu)}(\underline{y}, \underline{y}'), \tag{4.6}$$

$$L_i^*(\underline{y}, \underline{y}') = \sum_{\nu=0}^{\infty} {}^*K_i^{(\nu)}(\underline{y}, \underline{y}'), \tag{4.7}$$

$$K_i(\underline{y}, \underline{y}') = \frac{1}{2\pi} \frac{\partial}{\partial n_{iy}} \left[ \frac{\exp\left(-sr \frac{\underline{y} \underline{y}'}{r}\right)}{r \frac{\underline{y} \underline{y}'}{r}} \right], \tag{4.8}$$

$${}^*K_i(\underline{y}, \underline{y}') = \frac{1}{2\pi} \frac{\partial}{\partial n_{iy}'} \left[ \frac{\exp\left(-sr \frac{\underline{y} \underline{y}'}{r}\right)}{r \frac{\underline{y} \underline{y}'}{r}} \right], \tag{4.9}$$

$$K_{-i}(\underline{y}, \underline{y}') = \frac{1}{2\pi} \frac{\exp\left(-sr \frac{\underline{y} \underline{y}'}{r}\right)}{r \frac{\underline{y} \underline{y}'}{r}}, \tag{4.10}$$

and

$${}^*K_{-i}(\underline{y}, \underline{y}') = \frac{1}{2\pi} \frac{\partial^2}{\partial n_{iy} \partial n_{iy}'} \left[ \frac{\exp\left(-sr \frac{\underline{y} \underline{y}'}{r}\right)}{r \frac{\underline{y} \underline{y}'}{r}} \right], \tag{4.11}$$

In the same way, we can show that in Case 2, the Green's function  $\bar{G}(\underline{x}_1, \underline{x}_2; s^2)$  has a regular part of the same form (4.3) with the interchanges  $S_{i, i=1, \dots, k} \rightarrow S_{i, i=k+1, \dots, m}$ .

On the basis of (4.3) the function  $\bar{\chi}(\underline{x}_1, \underline{x}_2; s^2)$  will be estimated for  $s \rightarrow \infty$ . The case when  $\underline{x}_1$  and  $\underline{x}_2$  lie in the neighborhood of the bounding surfaces  $S_i, i=1, \dots, m$  of  $\Omega$  is particularly interesting. For this case, we need to use the following coordinates.

5. COORDINATES IN THE NEIGHBORHOOD OF  $S_i, i=1, \dots, m$ .

Let  $h_i > 0 (i=1, \dots, m)$  be sufficiently small. Let  $n_i (i=1, \dots, m)$  be the minimum distances from a point  $\underline{x} = (x^1, x^2, x^3)$  of the domain  $\Omega$  to its bounding surfaces  $S_i (i=1, \dots, m)$  respectively. Let  $\underline{n}_i(\sigma_i) (i=1, \dots, m)$  denote the inward drawn unit normals to  $S_i (i=1, \dots, m)$  respectively. We note that the coordinates in the neighborhood of  $S_i (i=k+1, \dots, m)$  are in the same form as in Section 5.1 of [11] with the interchanges  $\sigma_2^1 \leftrightarrow \sigma_i^1, \sigma_2^2 \leftrightarrow \sigma_i^2, n_2 \leftrightarrow n_i, h_2 \leftrightarrow h_i, I_2 \leftrightarrow I_i, \mathfrak{A}(I_2) \leftrightarrow \mathfrak{A}(I_i)$  and  $\delta_2 \leftrightarrow \delta_i, (i=k+1, \dots, m)$ . Thus we have the same formulae (5.1.1)-(5.1.6) of Section 5.1 in [11] with the interchanges  $n_2 \leftrightarrow n_i, \underline{n}_2(\sigma_2) \leftrightarrow \underline{n}_i(\sigma_i), II_1 \leftrightarrow II_{1i}, II_2 \leftrightarrow II_{2i}, H_1 \leftrightarrow H_i$  and  $N_1 \leftrightarrow N_i, (i=k+1, \dots, m)$ .

Similarly, the coordinates in the neighborhood of  $S_i, (i=1, \dots, k)$  are similar to those obtained in Section 5.2 of [11] with the interchanges  $\sigma_1^1 \leftrightarrow \sigma_i^1, \sigma_1^2 \leftrightarrow \sigma_i^2, n_1 \leftrightarrow n_i, h_1 \leftrightarrow h_i, I_1 \leftrightarrow I_i, \mathfrak{A}(I_1) \leftrightarrow \mathfrak{A}(I_i)$  and  $\delta_1 \leftrightarrow \delta_i, (i=1, \dots, k)$ . Thus, we have the same formulae (5.2.1)-(5.2.5) of Section 5.1 in [11] with the interchanges  $n_2 \leftrightarrow n_i, \underline{n}_2(\sigma_2) \leftrightarrow \underline{n}_i(\sigma_i), II_1^* \leftrightarrow II_{1i}, II_2^* \leftrightarrow II_{2i}, H_1^* \leftrightarrow H_i$  and  $N_1^* \leftrightarrow N_i, (i=k+1, \dots, m)$ .

6. SOME LOCAL EXPANSIONS.

It now follows that the local expansions of the functions

$$\frac{\exp\left(-sr \frac{\underline{x} \underline{y}}{r}\right)}{r \frac{\underline{x} \underline{y}}{r}}, \frac{\partial}{\partial n_{iy}} \left[ \frac{\exp\left(-sr \frac{\underline{x} \underline{y}}{r}\right)}{r \frac{\underline{x} \underline{y}}{r}} \right], i=1, \dots, m, \tag{6.1}$$

when the distance between  $\underline{x}$  and  $\underline{y}$  is small are very similar to those obtained in Section 6 of [11]. Consequently, the local behavior of the kernels

$$K_+(\underline{y}', \underline{y}), *K_-(\underline{y}', \underline{y}), \tag{6.2}$$

$$*K_+(\underline{y}', \underline{y}), K_-(\underline{y}', \underline{y}), \tag{6.3}$$

when the distance between  $\underline{y}$  and  $\underline{y}'$  is small, follows directly from the local expansions of the functions (6.1).

**DEFINITION 1.** If  $\xi_1$  and  $\xi_2$  are points in the half-part  $\xi^3 > 0$ , then we define

$$\widehat{\rho}_{12} = \sqrt{(\xi_1^1 - \xi_2^1)^2 + (\xi_1^2 - \xi_2^2)^2 + (\xi_1^3 + \xi_2^3)^2}.$$

An  $e^\lambda(\xi_1, \xi_2; s)$ -function is defined for points  $\xi_1$  and  $\xi_2$  belong to sufficiently small domains  $\mathfrak{M}(I_i)$  except when  $\xi_1 = \xi_2 \in I_i, (i = 1, \dots, m)$  and  $\lambda$  is called the degree of this function. For every positive integer  $\Lambda$ , it has the local expansion (see [11]):

$$e^\lambda(\xi_1, \xi_2; s) = \Sigma^* f(\xi_1^1, \xi_1^2)(\xi_1^3)^{P_1} (\xi_2^3)^{P_2} \left(\frac{\partial}{\partial \xi_1^1}\right)^{\ell_1} \left(\frac{\partial}{\partial \xi_1^2}\right)^{\ell_2} \left(\frac{\partial}{\partial \xi_1^3}\right)^{\ell_3} \frac{\exp(-s\widehat{\rho}_{12})}{\widehat{\rho}_{12}} + R^\Lambda(\xi_1, \xi_2; s),$$

where  $\Sigma^*$  denotes a sum of a finite number of terms in which  $f(\xi^1, \xi^2)$  are infinitely differentiable functions. In this expansion  $P_1, P_2, \ell_1, \ell_2, \ell_3$  are integers, where  $P_1 \geq 0, P_2 \geq 0, \ell_1 \geq 0, \ell_2 \geq 0, \lambda = \min(P_1 + P_2 - q), q = \ell_1 + \ell_2 + \ell_3$  and the minimum is taken over all terms which occur in the summation  $\Sigma^*$ . The remainder  $R^\Lambda(\xi_1, \xi_2; s)$  has continuous derivatives of order  $d \leq \Lambda$  satisfying

$$D^d R^\Lambda(\xi_1, \xi_2; s) = 0 \left[ s^{-\Lambda} \exp(-As\widehat{\rho}_{12}) \right] \text{ as } s \rightarrow \infty,$$

where  $A$  is a positive constant.

Thus, using methods similar to those obtained in Section 7 of [11], we can show that the functions (6.1) are  $e^\lambda$ -functions with degrees  $\lambda = -1, -2$  respectively. Consequently, the functions (6.2) are  $e^\lambda$ -functions with degrees  $\lambda = 0, -1$  while the functions (6.3) are  $e^\lambda$ -functions with degrees  $\lambda = 0, 1$  respectively.

**DEFINITION 2.** If  $\underline{x}_1$  and  $\underline{x}_2$  are points in large domains  $\Omega + S_i$ , then we define

$$\widehat{r}_{12} = \min_{\underline{y}} (r_{\underline{x}_1 \underline{y}} + r_{\underline{x}_2 \underline{y}}) \text{ if } \underline{y} \in S_i, i = 1, \dots, k,$$

and

$$\widehat{R}_{12} = \min_{\underline{y}} (r_{\underline{x}_1 \underline{y}} + r_{\underline{x}_2 \underline{y}}) \text{ if } \underline{y} \in S_i, i = k + 1, \dots, m.$$

An  $E^\lambda(\underline{x}_1, \underline{x}_2; s)$ -function is defined and infinitely differentiable with respect to  $\underline{x}_1$  and  $\underline{x}_2$  when these points belong to large domains  $\Omega + S_i$  except when  $\underline{x}_1 = \underline{x}_2 \in S_i, i = 1, \dots, m$ . Thus, the  $E^\lambda$ -function has a similar local expansion of the  $e^\lambda$ -function (see [7], [11]).

With the help of Section 8 in [11], it is easily seen that formula (4.3) is an  $E^{-2}(\underline{x}_1, \underline{x}_2; s)$ -function and consequently

$$\bar{G}(\underline{x}_1, \underline{x}_2; s^2) = \sum_{i=1}^k 0\left\{ \widehat{r}_{12}^{-2} \exp(-A_i s \widehat{r}_{12}) \right\} + \sum_{i=k+1}^m 0\left\{ \widehat{R}_{12}^{-2} \exp(-A_i s \widehat{R}_{12}) \right\}, \tag{6.4}$$

which is valid for  $s \rightarrow \infty$ , where  $A_i (i = 1, \dots, m)$  are positive constants. Formula (6.4) shows that  $\bar{G}(\underline{x}_1, \underline{x}_2; s^2)$  is exponentially small for  $s \rightarrow \infty$ .

With reference to Sections 7 and 9 in [11], if the  $e^\lambda$ -expansions of the functions (6.1)-(6.3) are introduced into (4.3) and if we use formulae similar to (7.4) and (7.10) of Section 7 in [11], we obtain the following local behavior of  $\bar{\chi}(\underline{x}_1, \underline{x}_2; s^2)$  as  $s \rightarrow \infty$  which is valid when  $\widehat{r}_{12}$  and  $\widehat{R}_{12}$  are small:

$$\bar{\chi}(\underline{x}_1, \underline{x}_2; s^2) = \sum_{i=1}^m \bar{\chi}_i(\underline{x}_1, \underline{x}_2; s^2), \tag{6.5}$$

where, if  $\underline{x}_1$  and  $\underline{x}_2$  belong to sufficiently small domains  $\mathfrak{D}(I_i)$ ,  $i = 1, \dots, m$ , then

$$\bar{\chi}_i(\underline{x}_1, \underline{x}_2; s^2) = -\frac{\exp(-s\hat{\rho}_{12})}{8\pi\hat{\rho}_{12}} + 0\left\{\frac{\exp(-A_i s\hat{\rho}_{12})}{\hat{\rho}_{12}}\right\} \text{ as } s \rightarrow \infty. \tag{6.6}$$

When  $\hat{r}_{12} \geq \delta_i > 0, i = 1, \dots, k$  and  $\hat{R}_{12} \geq \delta_i > 0, i = k + 1, \dots, m$ , the function  $\bar{\chi}(\underline{x}_1, \underline{x}_2; s^2)$  is of order  $0\{\exp(-sN_0)\}$  as  $s \rightarrow \infty, N_0 > 0$ . Thus, since  $\lim_{\hat{r}_{12} \rightarrow 0} \frac{\hat{r}_{12}}{\hat{\rho}_{12}} = \lim_{\hat{R}_{12} \rightarrow 0} \frac{\hat{R}_{12}}{\hat{\rho}_{12}} = 1$  (see [11]), then the local behavior of the formula (4.3) has the form (6.5), where if  $\underline{x}_1$  and  $\underline{x}_2$  belong to large domains  $\Omega + S_i, i = 1, \dots, k$ , we get

$$\bar{\chi}_i(\underline{x}_1, \underline{x}_2; s^2) = -\frac{\exp(-s\hat{r}_{12})}{8\pi\hat{r}_{12}} + 0\left\{\frac{\exp(-A_i s\hat{r}_{12})}{\hat{r}_{12}}\right\} \text{ as } s \rightarrow \infty, \tag{6.7}$$

while, if  $\underline{x}_1$  and  $\underline{x}_2$  belong to large domains  $\Omega + S_i, i = k + 1, \dots, m$ , we get:

$$\bar{\chi}_i(\underline{x}_1, \underline{x}_2; s^2) = -\frac{\exp(-s\hat{R}_{12})}{8\pi\hat{R}_{12}} + 0\left\{\frac{\exp(-A_i s\hat{R}_{12})}{\hat{R}_{12}}\right\} \text{ as } s \rightarrow \infty. \tag{6.8}$$

### 7. CONSTRUCTION OF RESULTS.

Since for  $\xi^3 \geq h_i > 0, i = 1, \dots, m$  the functions  $\bar{\chi}_i(\underline{x}, \underline{x}; s^2)$  are of orders  $0(e^{-2A_i s h_i})$ , the integral over  $\Omega$  of the function  $\bar{\chi}(\underline{x}, \underline{x}; s^2)$  can be approximated in the following way (see (3.10)):

$$\begin{aligned} \bar{K}(s^2) &= \sum_{i=k+1}^m \int_{S_i} \int_{\xi^3=0}^{h_i} \bar{\chi}_i(\underline{x}, \underline{x}; s^2) \{1 - 2\xi^3 H_i + (\xi^3)^2 N_i\} d\xi^3 dS_i \\ &\quad - \sum_{i=1}^k \int_{S_i} \int_{\xi^3=0}^{h_i} \bar{\chi}_i(\underline{x}, \underline{x}; s^2) \{1 + 2\xi^3 H_i + (\xi^3)^2 N_i\} d\xi^3 dS_i \\ &\quad + \sum_{i=1}^m 0(e^{-2A_i s h_i}) \text{ as } s \rightarrow \infty. \end{aligned} \tag{7.1}$$

If the  $e^\lambda$ -expansions of  $\bar{\chi}_i(\underline{x}, \underline{x}; s^2)$  are introduced into (7.1) and with the help of formula (10.2) of Section 10 in [11], we deduce after inverting Laplace transforms, that

$$K(t) = \frac{a_1}{t} + \frac{a_2}{t^{1/2}} + a_3 + a_4 t^{1/2} + 0(t) \text{ as } t \rightarrow 0, \tag{7.2}$$

where

$$\begin{aligned} a_1 &= \frac{1}{16\pi} \left\{ \sum_{i=1}^k |S_i| - \sum_{i=k+1}^m |S_i| \right\}, a_2 = \frac{1}{12\pi^{3/2}} \sum_{i=1}^m \int_{S_i} H_i dS_i, \\ a_3 &= \frac{1}{128\pi} \left\{ 7 \sum_{i=1}^k \int_{S_i} (H_i^2 - N_i) dS_i + \sum_{i=k+1}^m \int_{S_i} (H_i^2 - N_i) dS_i \right\}, \end{aligned}$$

and

$$a_4 = \frac{1}{\pi^{3/2}} \left\{ \frac{13}{1440} \sum_{i=1}^k \int_{S_i} H_i^3 dS_i - \frac{1}{315} \sum_{i=k+1}^m \int_{S_i} H_i^3 dS_i \right\}.$$

On inserting (7.2) into (3.6) we arrive at our result (2.1).

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