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MAGNETOHYDRODYNAMIC CROSS-FIELD BOUNDARY LAYER FLOW

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ABSTRACT. The Blasius boundary layer on a flat plate in the presence of a constant ambient magnetic field is examined. A numerical integration of the MHD boundary layer equations from the leading edge is presented showing how the asymptotic solution described by Sears is approached.

KEY WORDS AND PHRASES. *Magnetohydrodynamics, boundary layers, Newton's method, cross field.*

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1. INTRODUCTION.

Amendments to the classical Blasius boundary layer on a semi-infinite flat plate ($y = 0$, $x \geq 0$) created by the flow of a uniform viscous stream of fluid, with kinematic viscosity ν and velocity $\underset{\sim}{U}_o$ parallel to the plate for large values of y, have been examined on numerous occasions by the addition of external body forces. In particular, the effects caused by allowing the fluid to have a non-zero electrical conductivity σ and magnetic permeability μ, together with an ambient applied magnetic field $\underset{\sim}{H}_o$ have been analysed by various authors, usually by assuming that $\underset{\sim}{H}_o$ is constant at large distances from the plate. The case when undisturbed magnetic field is parallel to the plate is now quite well understood, principally due to the work of Greenspan and Carrier (1) and Glauert (2,3). However when the undisturbed field $\underset{\sim}{H}_o$ is normal to the plate the position is far from clear and is the subject of this investigation. Glauert (4), Hasimoto (5), Clauser (6) and others have shown that Alfvén waves are generated at the surface of a solid body in an MHD flow and propagate away from the body in directions which depend on the orientation of both the magnetic field $\underset{\sim}{H}$ and the fluid velocity $\underset{\sim}{U}$. In the present case, these disturbances are first created at the leading edge of the flat plate and propagate in a direction which confines them, initially, within the region dominated by the normal viscous boundary layer. Ultimately these disturbances are expected to emerge from the viscous boundary layer region since the growth rate of the boundary layer thickness is measured in terms of $x^{\frac{1}{2}}$ whereas Alfvén disturbances occupy a region which grows more like x.

A numerical integration of the full boundary layer equations, starting from the leading edge, therefore presents a difficulty in applying a condition on the magnetic field at the outer edge of the boundary layer. It is expected that for values of x less than some finite value, x_c say, the appropriate boundary conditions at the free stream edge of the boundary layer will be the conventional ones, namely that $\underset{\sim}{U} \to U_o \underset{\sim}{i}$ and $\underset{\sim}{H} \to H_o \underset{\sim}{j}$ in this region. However, for $x > x_c$, the Alfvén disturbances emerge from the viscous boundary layer and the appropriate outer boundary condition is then the jump condition of Stewartson (7), which states that $[\underset{\sim}{H} \cdot \underset{\sim}{i}] = \sqrt{\sigma\mu\nu}\, U_o$ must be satisfied across the viscous boundary layer. At first sight it would appear that a full numerical integration should incorporate these conditions in the appropriate regions, changing from one to the other at the location $x = x_c$. However, as will be shown, this does not appear to be the case.

An early investigation of the boundary layer on a flat plate with a normal magnetic field was that of Rossow (8) who assumed that the magnetic Prandtl number, $\varepsilon = \sigma\mu\nu$, was so small that the induced magnetic field could be ignored and obtained a series solution which is valid near the leading edge of the plate, $x = 0$. Lewis (9) obtained an analytical solution for this problem which is valid at large distances x along the plate. Using an explicit numerical procedure he was able to match his solutions for small and large values of x. In the case of finite magnetic Prandtl number, Sears (10) has obtained an asymptotic solution which is valid for large values of x and Hildyard (11) one appropriate near the leading edge. The paper of Sears was essentially in response to the earlier work of Clauser (6) who stated that 'no boundary layer of wake phenomena can possibly exist'. Sears demonstrated that a boundary layer solution can be obtained by an asymptotic analysis for large values of x . Sears imposed the Stewartson jump condition and therefore his solution cannot be matched onto the solution obtained by Hildyard (11) who assumed that all the magnetic field disturbances were contained in the viscous boundary layer, ie. Hildyard imposed $\underset{\sim}{H} \cdot \underset{\sim}{i} = 0$ at the outer edge of the boundary layer.

2. EQUATIONS AND SOLUTIONS

The equations governing the steady MHD flow of an incompressible fluid are

$$(\underset{\sim}{U} \cdot \nabla)\underset{\sim}{U} = -\frac{1}{\rho} \nabla p + \nu\nabla^2 \underset{\sim}{U} + \frac{\mu}{\rho} \underset{\sim}{j} \wedge \underset{\sim}{H} \quad , \tag{1}$$

$$\underset{\sim}{j} = \sigma (\underset{\sim}{E} + \mu \underset{\sim}{U} \wedge \underset{\sim}{H}) = \nabla \wedge \underset{\sim}{H} \quad , \tag{2}$$

$$\nabla \cdot \underset{\sim}{U} = \nabla \cdot \underset{\sim}{H} = 0 \tag{3}$$

$$\nabla \wedge \underset{\sim}{E} = 0 \tag{4}$$

where p, $\underset{\sim}{j}$ and $\underset{\sim}{E}$ are the pressure, current density and electric field strengths respectively. The problem under consideration is that of the flow in the boundary layer on a thin flat plate $y = 0$, $x > 0$ $(x,y,z$ are rectangular Cartesian coordinates) such that the undisturbed flow has $\underset{\sim}{U} = (U_o,0,0)$ and $\underset{\sim}{H} = (0,H_o,0)$. In order to satisfy equations (3) the stream function ψ and magnetic scalar potential A are introduced such that

$$\underset{\sim}{U} = (\frac{\partial\psi}{\partial y} , -\frac{\partial\psi}{\partial x} , 0) \quad \text{and} \quad \underset{\sim}{H} = (\frac{\partial A}{\partial y} , -\frac{\partial A}{\partial x} , 0) \quad , \tag{5}$$

from which it is easily shown that $\underset{\sim}{E} = (0, 0, -\mu U_o H_o)$. Thus the boundary layer equations to be solved become

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \frac{\partial^3\psi}{\partial y^3} + \beta(\frac{\partial A}{\partial y}\frac{\partial^2 A}{\partial x\partial y} - \frac{\partial A}{\partial x}\frac{\partial^2 A}{\partial y^2}) \quad , \tag{6}$$

$$\frac{\partial^2 A}{\partial y^2} = \varepsilon (1 + \frac{\partial\psi}{\partial y} \cdot \frac{\partial A}{\partial x} - \frac{\partial\psi}{\partial x} \cdot \frac{\partial A}{\partial y}) \quad , \tag{7}$$

where all the quantities have been non-dimensionalised with respect to their main stream values and $\beta = \mu H_o^2/\rho U_o^2$.

The problem now reduces to solving equations (6) and (7) subject to the boundary conditions of no-slip on the plate and from symmetry (for a boundary-layer on each side of the plate) $H_x = 0$ on $y = 0$. As $y \to \infty$, $u \to U_o$ and $H_x = \underset{\sim}{H} \cdot \underset{\sim}{i} \to K\sqrt{\frac{\varepsilon}{\beta}}\, H_o$ where K can take the value 0 or 1 depending on whether the outer condition chosen is that of Hildyard $(K = 0)$ or Sears $(K = 1)$.

To obtain a solution near $x = 0$, the Blasius type variables

$$\psi = \sqrt{x}\,\phi(\eta,x) , \quad A = -x + \sqrt{x}\,\gamma(\eta,x) , \tag{8}$$

where
$$\eta = y/\sqrt{x} , \tag{9}$$

are introduced. The boundary-layer equations then become

$$x\left(\frac{\partial\phi}{\partial\eta}\frac{\partial^2\phi}{\partial\eta\partial x} - \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial\eta^2}\right) - \frac{\phi}{2}\frac{\partial^2\phi}{\partial\eta^2} = \frac{\partial^3\phi}{\partial\eta^3} + \beta\left[x\left(\frac{\partial\gamma}{\partial\eta}\frac{\partial^2\gamma}{\partial\eta\partial x} - \frac{\partial\gamma}{\partial x}\frac{\partial^2\gamma}{\partial\eta^2}\right) + \left(\sqrt{x} - \frac{\gamma}{2}\right)\frac{\partial^2\gamma}{\partial\eta^2}\right] , \tag{10}$$

$$\frac{\partial^2\gamma}{\partial\eta^2} = \varepsilon\left[\frac{1}{2}\left(\gamma\frac{\partial\phi}{\partial\eta} - \phi\frac{\partial\gamma}{\partial\eta}\right) + \sqrt{x}\left(1 - \frac{\partial\phi}{\partial\eta}\right) + x\left(\frac{\partial\phi}{\partial\eta}\frac{\partial\gamma}{\partial x} - \frac{\partial\phi}{\partial x}\frac{\partial\gamma}{\partial\eta}\right)\right] , \tag{11}$$

subject to the boundary conditions

$$\left.\begin{aligned} &\phi = \frac{\partial\phi}{\partial\eta} = \frac{\partial\gamma}{\partial\eta} = 0 , && \eta = 0 , \quad x > 0 \\ &\frac{\partial\phi}{\partial\eta} = 1 , \quad \frac{\partial\gamma}{\partial\eta} = K\sqrt{\frac{\varepsilon}{\beta}} , && \eta \to \infty , \quad x > 0 \end{aligned}\right\} \tag{12}$$

The parabolic (in x) partial differential equations (10) and (11) are effectively 5th order in η and hence the five boundary conditions in (12) along with the detailed knowledge of the solution at some given value of x , x_1 say, are sufficient to determine the solution for all values of $x \geqslant x_1$. Near $x = 0$, we look for a solution of the form

$$\left.\begin{aligned} \phi(\eta,x) &= \phi_o(\eta) + \sqrt{x}\,\phi_1(\eta) + x\,\phi_2(\eta) + \dots \\ \gamma(\eta,x) &= \gamma_o(\eta) + \sqrt{x}\,\gamma_1(\eta) + x\,\gamma_2(\eta) + \dots \end{aligned}\right\} \tag{13}$$

Substituting expressions (13) into equations (10) and (11) and equating powers of x, results in a set of ordinary differential equations. For given values of ε and β, these equations can be solved numerically whereas if $\varepsilon >> 1$ or $\varepsilon << 1$ then the method of matched asymptotic expansions can be used as described by Glauert (2).

The Sears solution, valid for large values of x, is given by

$$
\begin{aligned}
u &= U_o\left[1 - \exp(-y\sqrt{\varepsilon\beta})\right] \\
H_x &= H_o\sqrt{\frac{\varepsilon}{\beta}}\left[1 - \exp(-y\sqrt{\varepsilon\beta})\right].
\end{aligned}
\tag{14}
$$

The series expansion for small values of x can now be used as the starting point for a numerical integration of the full boundary layer equations (10) and (11). The integration is continued into the region where x can be considered large, in the sense of Sears using a Newton's method in the x-direction as described by Merkin (12). The errors arising from using finite differences were kept small by using a variable step length.

As indicated earlier, the first attempt was to take $K = 0$ for $x << 1$. In this case for all values of ε and β the function ϕ_o is the Blasius function and $\gamma_o \equiv 0$. Thus $\phi_o''(0) = 0.3321$. Because of the large amount of computing time required for each computation only a few solutions were obtained for different values of ε and β. Since they all exhibited the same general properties only the results for $\varepsilon = \beta = \frac{1}{4}$ are presented here. With $K = 0$ the solution was obtained by a step-by-step integration for increasing values of x in the expectation that the numerical results would indicate when to revert to the Stewartson jump condition $(K = 1)$. Figure 1 shows the variation of the non-dimensional skin friction $\{ = \frac{1}{\sqrt{x}}\frac{\partial^2\phi}{\partial\eta^2}\Big|_{\eta=0}\}$ and the non-dimensional y-component of the magnetic field $(= 1 - \frac{\gamma}{2\sqrt{x}} - \sqrt{x}\,\frac{\partial\gamma}{\partial x})$ at the plate $\eta = 0$ as a function of x. Although this shows that the skin friction appears to be approaching the Sears solution the y component of the magnetic field is far from being unperturbed from its main stream value. Figure 2 shows the variation of the non-dimensional x component of magnetic field (H_x/H_o) as a function of η at several stations of x. This shows that this component of magnetic field is continuing to increase with increasing values of x and that the boundary layer thickness continues to grow as $x^{\frac{1}{2}}$. All variables show no indication that the solution is going awry and consequently there was no means of deciding when to revert to the Stewartson jump condition.

By contrast, pursuing the integration from $x = 0$ and applying the condition $K = 1$ immediately it was found that the boundary layer solution as described by Sears was indeed approached. For $K = 1$, $\varepsilon = \beta = \frac{1}{4}$, functions ϕ_o, ϕ_1, γ_o and γ_1 were computed, giving in particular, $\phi_o''(0) = 0.1922$, $\phi_1''(0) = 0.0772$, $\gamma_o(0) = 2.0271$, $\gamma_1(0) = 1.0701$. The variation of the non-dimensional skin friction and the non-dimensional y-component of the magnetic field at the plate $\eta = 0$ are presented in Figure 3. This shows how, as x is increased, the asymptotic values of these quantities, as predicted by Sears, are being approached. The numerical integration was carried on up to $x = 100$ by which time the numerical results were indistinguishable from the Sears results. A further check on the numerical solution is afforded by using the results given in (15). It was found that the two term series solution and the full numerical solution agree very well for small values of x. The approach to the asymptotic profiles for the x-component of velocity is shown in Figure 4 and similar profiles exist for the x-component of magnetic field.

Thus if the appropriate boundary conditions are chosen for the magnetic field, at the plate and at large distances from the plate, then the solution presented by Sears will be approached. If however, one enforces the boundary conditions that the x-component of the magnetic field is zero at $y = 0$ <u>and</u> as $y \to \infty$, a solution may be obtained which is valid at small values of x but the boundary layer continues to grow as x is increased and the Sears asymptotic solution is no longer approached.

It is clear therefore that the equations (10) and (11) can be integrated step by step for given values of ε and β, given ϕ and γ at some station of x and the five boundary conditions at some stations of η. This has been mathematically achieved here by taking the boundary conditions (12) with $K = 0$ and $K = 1$ and ϕ_o and γ_o as given in expressions (13). Mathematically other 'solutions' could be obtained by taking other magnetic boundary conditions, e.g. a boundary condition on

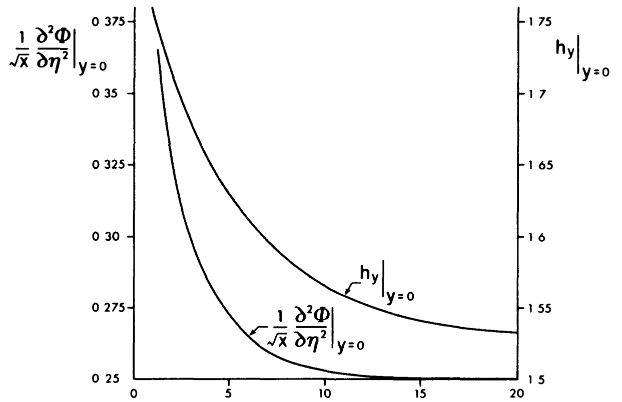

FIGURE 1. THE VARIATION OF THE NON-DIMENSIONAL SKIN FRICTION AND Y COMPONENT OF THE MAGNETIC FIELD AT THE PLATE AS A FUNCTION OF DISTANCE ALONG THE PLATE FOR K = 0.

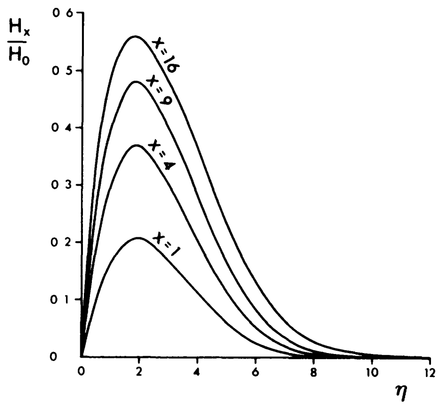

FIGURE 2. THE VARIATION IN THE X COMPONENT OF THE MAGNETIC FIELD AS A FUNCTION OF Y AT VARIOUS STATIONS ALONG THE PLATE FOR K = 0.

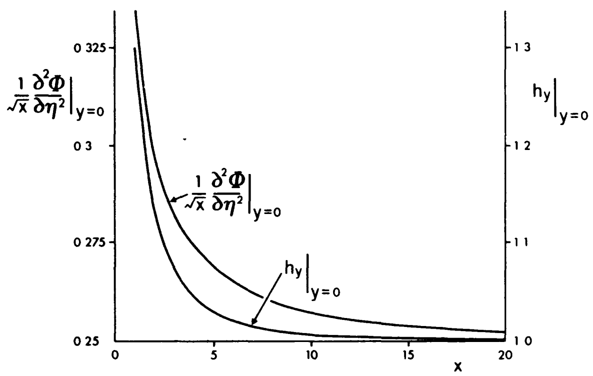

FIGURE 3. THE VARIATION OF THE NON-DIMENSIONAL SKIN FRICTION AND Y COMPONENT OF THE MAGNETIC FIELD AT THE PLATE AS A FUNCTION OF DISTANCE ALONG THE PLATE FOR K = 1.

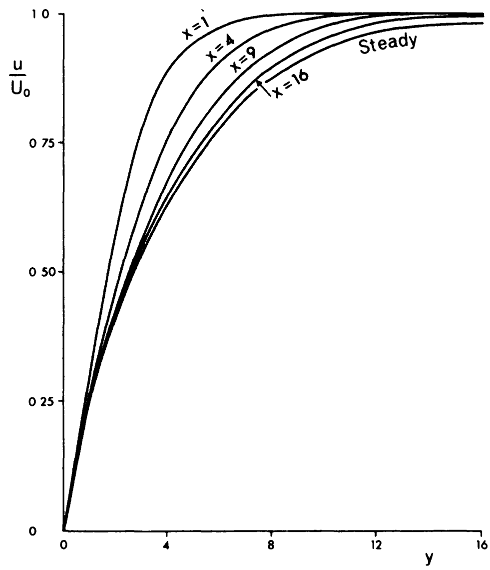

FIGURE 4. THE VARIATION IN THE X COMPONENT OF VELOCITY AS A FUNCTION OF Y AT VARIOUS STATIONS ALONG THE PLATE FOR K = 1.

H_y on the plate rather than H_x could have been assumed. Physically if $H_x = 0$ on $y = 0$ then the solution must satisfy the Stewartson jump condition for $x >> 1$ if a boundary layer solution is to exist and hence the boundary conditions (12) with $K = 1$ must be enforced for $x >> 1$. However, for $x << 1$ the Alfvén disturbance is still confined to within the boundary layer and hence the appropriate boundary conditions are (12) with $K = 0$ for $x << 1$. The results obtained here show that one can integrate the parabolic partial differential equations (12), with the appropriate starting values for ϕ and γ for $x << 1$, step by step without any indication of the fact that the Alfvén disturbance has reached the outer edge of the boundary layer. In general, K must be a function of x which depends on ε and β and hence if K could be determined then a full solution which satisfies the boundary conditions for $x << 1$ and $x >> 1$ should be possible. It is not easy to see how this variation of K with x can be achieved.

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