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REMAINDERS OF POWER SERIES

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ABSTRACT. Suppose $\Sigma_{n=0}^{\infty} a_n z^n$ has radius of convergence R and $\sigma_N(z) = |\Sigma_{n=N}^{\infty} a_n z^n|$. Suppose $|z_1| < |z_2| < R$, and T is either z_2 or a neighborhood of z_2. Put $S = \{N |\ \sigma_N(z_1) > \sigma_N(z) \text{ for } z \in T\}$. Two questions are asked: (a) can S be cofinite? (b) can S be infinite? This paper provides some answers to these questions. The answer to (a) is no, even if $T = z_2$. The answer to (b) is no, for $T = z_2$ if $\lim a_n = a \neq 0$. Examples show (b) is possible if $T = z_2$ and for T a neighborhood of z_2.

KEY WORDS AND PHRASES. Power-Series, Remainders, Radius of Convergence.

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. 30A10.

1. INTRODUCTION.

This paper originated in a question of approximation by power series raised in Query 51 in the American Mathematical Society Notices [1]. (The query originated in considerations of analytically continuing a polynomial series from the interval [-1,1] to the region of convergence of the series.) Suppose $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ has radius of convergence R and $\sigma_N(z) = |\Sigma_{n=N}^{\infty} a_n z^n|$. Suppose $|z_1| < |z_2| < R$ and T is either z_2 or a neighborhood of z_2. Put $S = \{n | \sigma_n(z_1) > \sigma_n(z) \text{ for } z \in T\}$. S is cofinite if its complement is finite. Two questions are asked:

(a) can S be cofinite?

(b) can S be infinite?

One might expect the answer to both questions to be no since one expects the approximation to f by partial sums of its power series to be worse, closer to the circle of convergence.

This paper provides some answers to these questions. Section 2 shows (a) is impossible for any T. Section 3 shows (b) is impossible if $T = z_2$ and $\lim a_n = a \neq 0$. Section 4 shows (b) is possible for $T = z_2$ and Section 5 shows (b) is possible for T a neighborhood of z_2.

Section 5 suggests the conjecture that if T is a neighborhood of z_2, then S must be "thin." The S which appears in Section 5 is lacunary.

These questions can also be raised about other series of orthonormal polynomials with elliptic domains of convergence. (cf. Szegö [5], pp. 309-10).

2. S CANNOT BE COFINITE.

The following theorem was suggested by P. Lax [3].

THEOREM 1. If $\lim |a_n|^{1/n} = 1/R < \infty$, $0 < |z_1|, |z_2| < R$ and $0 < \delta < |z_2|/|z_1|$, then the set $S = \{n \mid |\Sigma_{k=n}^{\infty} a_k z_2^k| < \delta^n |\Sigma_{k=n}^{\infty} a_k z_1^k|\}$ cannot be cofinite.

PROOF. Suppose S contains a nonempty tail set τ; i.e. $n\in\tau$ implies $n+1\in\tau$. Then for $n\in\tau$,

$$\sigma_n(z_1) \geq \sigma_{n+1}(z_1) - |a_n||z_1|^n \geq \delta^{-(n+1)}\ \sigma_{n+1}(z_2) - |a_n||z_1|^n$$

$$\geq \delta^{-(n+1)}[|a_n||z_2|^n - \sigma_n(z_2)] - |a_n||z_1|^n$$

$$\geq |a_n|\ [\delta^{-(n+1)}|z_2|^n - |z_1|^n] - \delta^{-1}\ \sigma_n(z_1)\ .$$

Hence

$$(1+\delta^{-1})\ \sigma_n(z_1) \geq |a_n|\ [\delta^{-(n+1)}|z_2|^n - |z_1|^n]\ .$$

Suppose $1/R \neq 0$. Choose $\varepsilon > 0$ so that $(R^{-1} + \varepsilon)|z_1| < 1$ and choose n τ so large that $|a_k|^{1/k} < (1/R + \varepsilon)$ for $k \geq n$. Also choose n so that $|a_n|^{1/n} > 1/R - \varepsilon$. Then

$$\frac{[(R^{-1}+\varepsilon)|z_1|]^n}{1-(R^{-1}+\varepsilon)|z\ \ |} > \Sigma_{k=n}^{\infty}\ |a_k||z_1|^k \geq \sigma_n(z_1)$$

$$\geq \frac{|a_n|}{1+\delta^{-1}}\ [\delta^{-(n+1)}|z_2|^n - |z_1|^n]$$

$$\geq \frac{(R^{-1}-\varepsilon)^n}{1+\delta^{-1}} [\delta^{-(n+1)}|z_2|^n - |z_1|^n]$$

$$= \frac{(R^{-1}-\varepsilon)^n}{1+\delta^{-1}}\left(\frac{|z_2|}{\delta}\right)^n \left[\delta^{-1} - \left(\frac{\delta|z_1|}{z_2}\right)^n\right] .$$

Now in addition to the other conditions on n, choose n large enough so that

$$\left(\frac{\delta|z_1|}{|z_2|}\right)^n < \delta^{-1} .$$

Then, since

$$\frac{(R^{-1}+\varepsilon)|z_1|}{[1-(R^{-1}+\varepsilon)|z_1|]^{1/n}} \geq \frac{R^{-1}-\varepsilon}{(1+\delta^{-1})^{1/n}} \frac{|z_2|}{\delta} \left[\delta^{-1} - \left(\frac{\delta|z_1|}{|z_2|}\right)^n\right]^{1/n} ,$$

one obtains upon letting $\varepsilon \to 0$ and $n \to \infty$:

$$|z_1| \geq \frac{|z_2|}{\delta} ,$$

contradicting $\delta < |z_2|/|z_1|$.

Suppose $R^{-1} = 0$. Then $|a_n|^{1/n}$ converges to zero. If we add zero to the set, $\{|a_n|^{1/n}|n \geq 1\}$ the new set is closed and bounded and thus compact with the largest element $|a_{n_1}|^{1/n_1}$. Deleting $|a_1|$, $|a_2|^{1/2}, \ldots, |a_{n_1}|^{1/n_1}$, there is a largest element $|a_{n_2}|^{1/n_2}$ in the remaining set and so forth. Thus we obtain a sequence n_i, $i = 1,2,\ldots$, with $|a_{n_i}|^{1/n_i} = \varepsilon_i \neq 0$ and $|a_n|^{1/n} \leq \varepsilon_i$ for $n \geq n_i$. Also $\lim_{i\to\infty} \varepsilon_i = 0$. Thus for i large enough that $\varepsilon_i|z_1| < 1$:

$$\frac{[\varepsilon_i|z_1|]^{n_i}}{1-\varepsilon_i|z_1|} \geq \Sigma_{k=n_i}^{\infty} |a_k||z_1|^k \geq \sigma_{n_i}(z_1)$$

$$\geq \frac{|a_{n_i}|}{1+\delta^{-1}} \left[\delta^{-(n_i+1)} |z_2|^{n_i} - |z_1|^{n_i}\right]$$

$$= \frac{|\ n_i|}{1+\delta^{-1}} \frac{|z_2|}{\delta}^{n_i} \left[\delta^{-1} - \left(\frac{\delta|z_1|}{|z_2|}\right)^{n_i}\right].$$

Now choose n_i so that $(\delta|z_1|/|z_2|)^{n_i} < \delta^{-1}$. Then

$$\frac{\varepsilon_i|z_1|}{(1-\varepsilon_i|z_1|)^{1/n_i}} \geq \frac{|a_{n_i}|^{1/n_i}}{(1+\delta^{-1})^{1/n_i}} \frac{|z_2|}{\delta} \left[\delta^{-1} - \left(\frac{\delta|z_1|}{|z_2|}\right)^{n_i}\right]^{1/n_i}$$

or

$$\frac{|z_1|}{(1-\varepsilon_i|z_1|)^{1/n_i}} \geq \frac{|z_2|}{\delta(1+\delta^{-1})^{1/n_i}} \left[\delta^{-1} - \left(\frac{\delta|z_1|}{|z_2|}\right)^{n_i}\right]^{1/n_i} .$$

Letting $\varepsilon_i \to 0$ and $n_i \to \infty$, one obtains

$$|z_1| \geq \frac{|z_2|}{\delta} ,$$

contradicting $\delta < |z_2|/|z_1|$. This completes the proof of Theorem 1.

The following observation about general series was made by a referee. Let $\Sigma_0^{\infty} A_\mu$ be convergent. If $\Sigma_0^{\infty} \mu|b_\mu| < \infty$, then

$$S = \left\{N \,\middle|\, \left|\sum_{\mu \geq N} A_\mu\right| < \left|\sum_{\mu \geq N} A_\mu b_\mu\right|\right\}$$

is not cofinite. For let $R_n = \Sigma_{\mu \geq n} A_\mu$. Then $A_\mu = R_n - R_{n+1}$. If S were cofinite, then for $n \geq n_o$,

$$|A_\mu| \leq |R_n| + |R_{n+1}| \leq 2\ \Sigma_{\mu \geq n} |A_\mu|\ |b_\mu|$$

or

$$\sum_{\mu \geq N} |A_\mu| \leq 2 \sum_{\mu \geq N} \sum_{\mu \geq n} |A_\mu|\ |b_\mu| \leq 2 \sum_{\mu \geq N} \mu |A_\mu|\ |b_\mu| < \infty .$$

If N_o is selected so large that $\mu|b_\mu| < 1/2$, then for $N > N_o$,

$$\sum_{\mu \geq N} |A_\mu| < 2\,\frac{1}{2} \sum_{\mu \geq N} |A_\mu| = \sum_{\mu \geq N} |A_\mu| ,$$

which is a contradiction. If one puts

$$A_\mu = a_\mu\, z_2^\mu\ ,\ b_\mu = \left(\frac{z_1}{z_2}\right)^\mu ,$$

then under the hypothesis of Theorem 1, one obtains the weaker result that the set

$$S = \left\{ n \;\middle|\; \left| \sum_{k=n}^{\infty} a_k {z_2}^k \right| < \left| \sum_{k=n}^{\infty} a_k {z_1}^k \right| \right\}$$

cannot be cofinite.

3. CASE OF $\mathrm{LIM}_{N\to\infty} A_N = A \neq 0$.

In this section it is shown that (b) is impossible for even a single point if $\lim_{n\to\infty} a_n = a \neq 0$. The proof is as follows. For $\varepsilon > 0$, N large enough, and $|z| < R = 1$

$$\sigma_N(z) = \left|\sum_{n=N}^{\infty} a_n z^n\right| = \left|a \sum_{n=N}^{\infty} z^n + \sum_{n=N}^{\infty} (a_n - a)z^n\right|$$

$$\leq |a| \frac{|z|^n}{|1-z|} + \varepsilon \frac{|z|^n}{1-|z|} .$$

Also

$$|a| \frac{|z|^N}{|1-z|} = \left|a\sum_{n=N}^{\infty} z^n\right| = \left|\sum_{n=N}^{\infty} a_n z^n + \sum_{n=N}^{\infty} (a-a_n)z^n\right|$$

$$\leq \sigma_n(z) + \varepsilon \frac{|z|^N}{1-|z|} .$$

Thus

$$|a| \frac{|z|^N}{|1-z|} - \varepsilon \frac{|z|^N}{1-|z|} \leq \sigma_N(z) \leq |a| \frac{|z|^N}{|1-z|} + \varepsilon \frac{|z|^N}{1-|z|} . \tag{1}$$

Suppose $\sigma_N(z_2) < \sigma_N(z_1)$ for infinitely many N. Then (1) gives

$$|a| \frac{|z_2|^N}{|1-z_2|} - \varepsilon \frac{|z_2|^N}{1-|z_2|} \leq \sigma_N(z_2) < \sigma_N(z_1)$$

$$\leq |a| \frac{|z_1|^N}{|1-z_1|} + \varepsilon \frac{|z_1|^N}{1-|z_1|}$$

for infinitely many N. Taking Nth roots, letting $N \to \infty$, and $\varepsilon \to 0$, yields

$$|z_2| \leq |z_1| ,$$

a contradiction of $|z_1| < |z_2|$.

4. FOR $T = \{z_2\}$, (b) IS POSSIBLE.

The following example shows (b) is possible if $T = \{z_2\}$. Let

$$F(z) = (1-2z)(1-z^2)^{-1}$$

$$= 1-2z + z^2 - 2z^3 + z^4 - 2z^5 + \dots .$$

One has:

$$\sigma_{2k}(z) = |z^{2k} - 2z^{2k+1} + z^{2k+2} - 2z^{2k+3} + \dots |$$

$$= |z|^{2k} \; |1 - 2z + z^2 - 2z^3 + \dots |$$

$$= |z|^{2k} \; |1 - 2z| |1 - z^2|^{-1}$$

and thus $\sigma_{2k}(1/2) = 0$. So for any $z_1 \neq 1/2$ and $0 < |z_1| < 1$, $\sigma_{2k}(z_1) > \sigma_{2k}(1/2)$.

Note that for an ε-neighborhood of 1/2: $N = \{z | |z - 1/2| < \varepsilon\}$, $0 < \varepsilon < 1/2$ and for any z_1 with $|z_1| < 1/2 - \varepsilon$, $\sigma_{2k}(z_1)$ converges to zero faster than $\sigma_{2k}(z)$ at any point z in N except 1/2. So we cannot extend the result to a neighborhood of 1/2.

5. CASE OF T A NEIGHBORHOOD OF z_2.

THEOREM 2. For each R, $0 < R \leq \infty$, there exist points z_1 and z_2 with $|z_1| < |z_2| < R$ and a power series $\Sigma_{n=0}^{\infty} a_n z^n$ with radius of convergence R such that for infinitely many values of N, $\sigma_N(z_1)/3 \geq \sigma_N(z)$ for all z in some neighborhood of z_2.

PROOF. Suppose R = 1. Put $n_k = 4^k$ and $P_k(z) = (1/b_k)\, z^{n_{2k-1}} (z - 1/2)^{n_{2k}}$, where $b_k = \max_{0 \leq j \leq n_{2k}} \left\{ \binom{n_{2k}}{j} 2^{-j} \right\}$. The power series $\Sigma_{k=1}^{\infty} P_k(z) = \Sigma_{n=0}^{\infty} a_n z^n$ will be shown to satisfy the Theorem for R = 1 with $z_1 = -1/4$ and $z_2 = 1/2$. Note that

$$n_{2k} + n_{2k-1} < n_{2k+1} \tag{2}$$

and

$$n_{2k-1}\,(\log 4/\log 3 + 1) < n_{2k} \tag{3}$$

for all k. (2) implies that each a_n is either zero or appears exactly once as a coefficient in the expansion of some $P_k(z)$. Let j_k be the integer for which $\max_{0\le j\le n_{2k}} \left\{ \binom{n_{2k}}{j} 2^{-j} \right\}$ is obtained. Then

$$|a_{j+n_{2k-1}}|^{1/(j+n_{2k-1})} = \left(\frac{\binom{n_{2k}}{j} 2^{-j}}{\binom{n_{2k}}{j_k} 2^{-j_k}} \right)^{1/(j+n_{2k-1})} . \quad (0 \le j \le n_{2k})$$

This is less than or equal to one for all j and equal to one for $j = j_k$, which implies the radius of convergence is one.

For all z with $|z - 1/2| < 1/4$:

$$\begin{aligned} |P_{k+1}(z)| &= \frac{1}{b_{k+1}} |z|^{n_{2k+1}} |z - 1/2|^{n_{2k+2}} \\ &< \frac{1}{b_k} |z|^{n_{2k-1}} |z-1/2|^{n_{2k}} |z-1/2|^{n_{2k+2} - n_{2k}} \\ &\le |P_k(z)|\,(1/4)^{n_{2k+2} - n_{2k}} \\ &\le (1/4)\,|P_k(z)| \;. \end{aligned} \tag{4}$$

Next, for $|z - 1/2| < 1/4$,

$$\frac{|P_k(z)|}{|P_k(-1/4)|} = |z|^{n_{2k-1}} |z-1/2|^{n_{2k}} 4^{n_{2k-1}} (4/3)^{n_{2k}}$$

$$< 4^{-n_{2k}} 4^{n_{2k-1}} (4/3)^{n_{2k}} \tag{5}$$

$$= 4^{n_{2k-1}} 3^{-n_{2k}} < 1/4$$

by (3). Hence, for $|z - 1/2) < 1/4$,

$$\sigma_{n_{2k-1}}(z) = \left|\sum_{j=n_{2k-1}}^{\infty} a_j z^j\right| \leq \sum_{j=k}^{\infty} |P_j(z)|$$

$$\leq \left(\sum_{j=k}^{\infty} 4^{k-j}\right) |P_k(z)| \quad \text{by} \quad (4)$$

$$= (4/3)|P_k(z)| < (1/3)|P_k(-1/4)| \quad \text{by} \quad (5)$$

$$\leq (1/3)\left|\sum_{j=k}^{\infty} b_j^{-1} (-1/4)^{n_{2j-1}} (-3/4)^{n_{2j}}\right|$$

$$= (1/3)\, \sigma_{n_{2k-1}}(-1/4) ,$$

since all n_j's are even. This shows that the assertion holds for $z_1 = -1/4$ and $z_2 = 1/2$.

For the case $0 < R < \infty$, use the power series $\Sigma_{n=0}^{\infty} a_n (z/R)^n$. Then the result holds for $z_1 = -R/2$, $z_2 = R/2$, and the neighborhood $|z - R/2| < R/4$.

For the case $R = \infty$, let

$$b_k = (n_{2k-1})^{n_{2k-1}} \binom{n_{2k}}{j_k} 2^{-j_k} .$$

For $0 \leq j \leq n_{2k}$:

$$|a_{j+n_{2k-1}}|^{1/(j+n_{2k-1})} = \left(\frac{\binom{n_{2k}}{j} 2^{-j}}{(n_{2k-1})^{n_{2k-1}} \binom{n_{2k}}{j_k} 2^{-j_k}} \right)^{1/(j+n_{2k-1})}$$

$$\leq (n_{2k-1})^{-n_{2k-1}/(j+n_{2k-1})}$$

$$\leq (n_{2k})^{-n_{2k-1}/(n_{2k}+n_{2k-1})}$$

$$= (n_{2k})^{-1/5} \to 0 \ .$$

as $k \to \infty$ and hence $\overline{\lim} \, |a_n|^{1/n} = 0$. The rest of the proof follows the case $R = 1$.

6. AVERAGE REMAINDER

Suppose $\Sigma \, a_n z^n$ has a radius of convergence R. It follows from results in Pólya and Szegö [4, Part III, problems 307-310] that the geometric mean:

$$G^N(r) = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log \sigma_N(re^{i\theta}) d\theta \right) \quad , \ (r < R)$$

and the pth mean, $p > 0$:

$$I_p^N (r) = \frac{1}{2\pi} \int_0^{2\pi} \sigma_N^p(re^{i\theta}) d \, \theta \ , \ (r < R)$$

are both monotone increasing functions of r for each N and $\log G^N(r)$ and $\log I_p^N(r)$ are convex functions of $\log r$. Thus in the geometric mean sense and pth mean sense, $\sigma_N(z)$ become larger as one approaches the circle of convergence.

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