

ON FURTHER STRENGTHENED HARDY-HILBERT'S INEQUALITY

LÜ ZHONGXUE

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We obtain an inequality for the weight coefficient $\omega(q, n)$ ($q > 1$, $1/p + 1/q = 1$, $n \in \mathbb{N}$) in the form $\omega(q, n) =: \sum_{m=1}^{\infty} (1/(m+n))(n/m)^{1/q} < \pi/\sin(\pi/p) - 1/(2n^{1/p} + (2/a)n^{-1/q})$ where $0 < a < 147/45$, as $n \geq 3$; $0 < a < (1-C)/(2C-1)$, as $n = 1, 2$, and C is an Euler constant. We show a generalization and improvement of Hilbert's inequalities. The results of the paper by Yang and Debnath are improved.

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1. Introduction. The following inequalities are well known as Hardy-Hilbert's inequalities:

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{n=1}^{\infty} a_n^p \right)^{1/p} \left(\sum_{n=1}^{\infty} b_n^q \right)^{1/q}, \quad (1.1)$$

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{a_m}{m+n} \right)^p < \left(\frac{\pi}{\sin(\pi/p)} \right)^p \sum_{n=1}^{\infty} a_n^p. \quad (1.2)$$

In recent years, Gao [1, 2], Xu and Guo [4], Hsu and Wang [3], Yang [6], and Yang and Gao [7] gave some distinct improvements of (1.1). Yang and Debnath [5] gave a strengthened version by the following inequality:

$$\omega(q, n) =: \sum_{m=1}^{\infty} \frac{1}{m+n} \left(\frac{n}{m} \right)^{1/q} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + n^{-1/q}}. \quad (1.3)$$

In this paper, we show a new generalization and improvement of (1.1) by improving (1.3).

First we introduce some lemmas.

LEMMA 1.1 [2]. *Let $f(x) > 0$, $f^{(2r-1)}(x) < 0$, $f^{(2r)}(x) \geq 0$, $x \in [1, \infty)$, $r = 1, 2$, $f^{(r)}(\infty) = 0$, $r = 0, 1, 2, 3, 4$, and $\int_1^{\infty} f(x) dx < \infty$, then*

$$\sum_{m=1}^{\infty} f(m) \leq \int_1^{\infty} f(x) dx + \frac{1}{2}f(1) - \frac{1}{12}f'(1). \quad (1.4)$$

LEMMA 1.2 [5]. *Let $q > 1$, $1/p + 1/q = 1$, $n \in \mathbb{N}$, then*

$$\omega(q, n) < \frac{\pi}{\sin(\pi/p)} - \frac{1}{n^{1/p}} [f_n(p) + g_n(p)], \quad (1.5)$$

where $\omega(q, n)$ is defined by (1.3), and

$$\begin{aligned}
 f_n(p) &:= p + \frac{1}{12p} + \frac{1}{(1+p)n} + \frac{1}{12pn^2} + \frac{1}{3(1+3p)n^3}, \\
 g_n(p) &:= -\frac{1}{12pn} - \frac{1}{2(1+2p)n^2} - \frac{7}{12} - \frac{1}{2n} + \frac{1}{12n^2} - \frac{7}{12n^3}.
 \end{aligned}
 \tag{1.6}$$

LEMMA 1.3 [5]. *Let $p > 1, n \in \mathbb{N}$, then*

$$f_n(p) + g_n(p) > \frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}.
 \tag{1.7}$$

LEMMA 1.4. *Let $q > 1, 1/p + 1/q = 1, n \in \mathbb{N}$, then*

$$\omega(q, n) =: \sum_{m=1}^{\infty} \frac{1}{m+n} \left(\frac{n}{m}\right)^{1/q} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + (2/a)n^{-1/q}},
 \tag{1.8}$$

$$\omega(p, n) =: \sum_{m=1}^{\infty} \frac{1}{m+n} \left(\frac{n}{m}\right)^{1/p} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/q} + (2/a)n^{-1/p}},
 \tag{1.9}$$

where $0 < a < 147/45$ as $n \geq 3$; $0 < a < (1-C)/(2C-1)$ as $n = 1, 2$, and C is an Euler constant.

PROOF. For $n \geq 3$,

$$\left(\frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}\right) \left(x + \frac{1}{ny}\right) = \frac{x}{2} + \frac{1}{n} \left(\frac{1}{2y} - \frac{x}{12} - \frac{1}{12yn} - \frac{x}{2n^2} - \frac{1}{2yn^3}\right),
 \tag{1.10}$$

where $x > 0, y > 0, xy = a$.

We first prove

$$\frac{1}{2y} - \frac{x}{12} - \frac{1}{12yn} - \frac{x}{2n^2} - \frac{1}{2yn^3} = \frac{6n^3 - xyn^3 - n^2 - 6xyn - 6}{12yn^3} > 0.
 \tag{1.11}$$

Formula (1.11) is equivalent to $\psi(n) = 6n^3 - xyn^3 - n^2 - 6xyn - 6 > 0$.

Since $\psi'(x) = 18x^2 - 3ax^2 - 2x - 6a, \psi''(x) = 36x - 6ax - 2$. When $0 < a < 147/45, \psi''(x) = 36x - 6ax - 2 > 0, \psi'(3) = 156 - 33a > 0$, then $\psi'(x) > 0$ and $\psi(3) = 147 - 45a > 0$, hence $\psi(n) > 0$ for $n \geq 3$. Therefore $(1/2 - 1/12n - 1/2n^3)(x + 1/ny) > x/2$. Namely, $1/2 - 1/12n - 1/2n^3 > 1/(2 + 2(an)^{-1})$, for $n \geq 3$. By (1.5) and (1.7), we have

$$\begin{aligned}
 \omega(q, n) &< \frac{\pi}{\sin(\pi/p)} - \frac{1}{n^{1/p}} \left(\frac{1}{2} - \frac{1}{12n} - \frac{1}{2n^3}\right) \\
 &< \frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + (2/a)n^{-1/q}}.
 \end{aligned}
 \tag{1.12}$$

Since $\phi(x) = 1/(2x + 2(xn)^{-1})$ is strictly increasing on $(0, \infty)$ and $0 < a < (1 - C)/(2C - 1)$, by $\omega(q, n) < \pi/\sin(\pi/p) - (1 - C)/n^{1/p}$ (see [7]), we have, when $n = 1$,

$$\begin{aligned} \omega(q, 1) &< \frac{\pi}{\sin(\pi/p)} - \frac{1 - C}{1} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2 + 2(2C - 1)/(1 - C)} \\ &\leq \frac{\pi}{\sin(\pi/p)} - \frac{1}{2 + 2/a}; \end{aligned} \tag{1.13}$$

when $n = 2$,

$$\omega(q, 2) < \frac{\pi}{\sin(\pi/p)} - \frac{1 - C}{2^{1/p}} < \frac{\pi}{\sin(\pi/p)} - \frac{1}{2^{1/p} 2 + (2/a)(1/2)}. \tag{1.14}$$

By (1.12), (1.13), and (1.14), (1.8) is valid for any $n \in \mathbb{N}$. Interchanging p, q in (1.8), since $\pi/\sin(\pi/p) = \pi/\sin(\pi/q)$, we have (1.9). The lemma is proved. \square

2. Main results. Now we introduce main results.

THEOREM 2.1. *Let $p > 1, 1/p + 1/q = 1, a_n \geq 0, b_n \geq 0, 0 < \sum_{n=1}^{\infty} a_n^p < \infty$, and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$. Then*

$$\begin{aligned} &\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m + n} \\ &< \left(\left(\frac{\pi}{\sin(\pi/p)} - \frac{b}{2 + 2b} \right) a_1^p + \left(\frac{\pi}{\sin(\pi/p)} - \frac{b}{2^{1/p}(2b + 2)} \right) a_2^p \right. \\ &\quad \left. + \sum_{n=3}^{\infty} \left(\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + (2/a)n^{-1/q}} \right) a_n^p \right)^{1/p} \\ &\quad \times \left(\left(\frac{\pi}{\sin(\pi/p)} - \frac{b}{2 + 2b} \right) b_1^q + \left(\frac{\pi}{\sin(\pi/p)} - \frac{b}{2^{1/q}(2b + 2)} \right) b_2^q \right. \\ &\quad \left. + \sum_{n=3}^{\infty} \left(\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/q} + (2/a)n^{-1/p}} \right) b_n^q \right)^{1/q}, \tag{2.1} \\ &\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{a_n}{m + n} \right)^p \\ &< \left(\frac{\pi}{\sin(\pi/p)} \right)^{p-1} \left(\left(\frac{\pi}{\sin(\pi/p)} - \frac{b}{2 + 2b} \right) a_1^p + \left(\frac{\pi}{\sin(\pi/p)} - \frac{b}{2^{1/p}(2b + 2)} \right) a_2^p \right. \\ &\quad \left. + \sum_{n=3}^{\infty} \left(\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + (2/a)n^{-1/q}} \right) a_n^p \right), \tag{2.2} \end{aligned}$$

where $0 < a < 147/45, 0 < b < (1 - C)/(2C - 1)$.

In particular, when $a = b = e$, e is a constant,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \left(\sum_{n=1}^{\infty} \left(\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + (2/e)n^{-1/q}} \right) a_n^p \right)^{1/p} \times \left(\sum_{n=1}^{\infty} \left(\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/q} + (2/e)n^{-1/p}} \right) b_n^q \right)^{1/q}, \tag{2.3}$$

$$\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{a_n}{m+n} \right)^p < \left(\frac{\pi}{\sin(\pi/p)} \right)^{p-1} \sum_{n=1}^{\infty} \left(\frac{\pi}{\sin(\pi/p)} - \frac{1}{2n^{1/p} + (2/e)n^{-1/q}} \right) a_n^p.$$

PROOF. By Hölder’s inequality, we have

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m}{(m+n)^{1/p}} \left(\frac{m}{n} \right)^{1/pq} \frac{b_n}{(m+n)^{1/q}} \left(\frac{n}{m} \right)^{1/pq} \\ &\leq \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m^p}{m+n} \left(\frac{m}{n} \right)^{1/q} \right)^{1/p} \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{b_n^q}{m+n} \left(\frac{n}{m} \right)^{1/p} \right)^{1/q} \\ &= \left(\sum_{n=1}^{\infty} \omega(q, n) a_n^p \right)^{1/p} \left(\sum_{n=1}^{\infty} \omega(p, n) b_n^q \right)^{1/q}. \end{aligned} \tag{2.4}$$

By (1.8) and (1.9), (2.1) is valid.

By Hölder’s inequality and (1.9), we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a_m}{m+n} &= \sum_{n=1}^{\infty} \frac{a_n}{(m+n)^{1/p}} \left(\frac{n}{m} \right)^{1/pq} \frac{1}{(m+n)^{1/q}} \left(\frac{m}{n} \right)^{1/pq} \\ &\leq \left(\sum_{n=1}^{\infty} \frac{a_n^p}{m+n} \left(\frac{m}{n} \right)^{1/q} \right)^{1/p} \left(\sum_{n=1}^{\infty} \frac{1}{m+n} \left(\frac{m}{n} \right)^{1/p} \right)^{1/q} \\ &< \left(\frac{\pi}{\sin(\pi/p)} \right)^{1/q} \left(\sum_{n=1}^{\infty} \frac{a_n^p}{m+n} \left(\frac{n}{m} \right)^{(2-\lambda)/q} \right)^{1/p}. \end{aligned} \tag{2.5}$$

Then

$$\begin{aligned} \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{a_n}{m+n} \right)^p &< \left(\frac{\pi}{\sin(\pi/p)} \right)^{p/q} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_n^p}{m+n} \left(\frac{n}{m} \right)^{1/q} \\ &= \left(\frac{\pi}{\sin(\pi/p)} \right)^{p-1} \sum_{n=1}^{\infty} \omega(q, n) a_n^p. \end{aligned} \tag{2.6}$$

By (1.8), (2.2) is valid. The theorem is proved. □

REMARK 2.2. As $a = b = 0$, inequalities (2.1) and (2.2) change to (1.1) and (1.2), respectively, hence inequalities (2.1) and (2.2) are generalization and improvement of (1.1) and (1.2), respectively.

REMARK 2.3. As $a = b = 2$, inequalities (2.1) and (2.2) change to (1.11) and (3.1) in [5], respectively, hence inequalities (2.1) and (2.2) are generalization and improvement of (1.11) and (3.1) in [5], respectively.

REMARK 2.4. We give an open question: how to determine the constant a such that $1/(2n^{1/p} + (2/a)n^{-1/q})$ is best possible.

REFERENCES

- [1] M. Z. Gao, *An improvement of the Hardy-Riesz extension of the Hilbert inequality*, J. Math. Res. Exposition **14** (1994), no. 2, 255-259.
- [2] ———, *A note on the Hardy-Hilbert inequality*, J. Math. Anal. Appl. **204** (1996), no. 1, 346-351.
- [3] L. C. Hsu and Y. J. Wang, *A refinement of Hilbert's double series theorem*, J. Math. Res. Exposition **11** (1991), no. 1, 143-144.
- [4] L. C. Xu and Y. K. Guo, *Note on Hardy-Riesz's extension of Hilbert's inequality*, Chinese Quart. J. Math. **6** (1991), no. 1, 75-77.
- [5] B. Yang and L. Debnath, *On new strengthened Hardy-Hilbert's inequality*, Int. J. Math. Math. Sci. **21** (1998), no. 2, 403-408.
- [6] B. C. Yang, *A refinement of the general Hilbert double series theorem*, J. Math. Study **29** (1996), no. 2, 64-70.
- [7] B. C. Yang and M. Z. Gao, *An optimal constant in the Hardy-Hilbert inequality*, Adv. Math. (China) **26** (1997), no. 2, 159-164.

Lü Zhongxue: School of Science, Nanjing University of Science & Technology, Nanjing 210094, China

Current address: Department of Basic Science of Technology College, Xuzhou Normal University, Xuzhou, Jiangsu 221011, China

E-mail address: lvzx1@pub.xz.jsinfo.net



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