Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2014, Article ID 368702, 7 pages
http://dx.doi.org/10.1155/2014/368702

Research Article

A Note on the Rate of Strong Convergence for Weighted Sums of Arrays of Rowwise Negatively Orthant Dependent Random Variables

Qingxia Zhang and Dingcheng Wang

School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China

Correspondence should be addressed to Qingxia Zhang; zqx121981@126.com

Received 24 April 2014; Accepted 3 August 2014; Published 20 August 2014

Academic Editor: Baodong Zheng

Let $\{X_{ni}; i \geq 1, n \geq 1\}$ be an array of rowwise negatively orthant dependent (NOD) random variables. The authors discuss the rate of strong convergence for weighted sums of arrays of rowwise NOD random variables and solve an open problem posed by Huang and Wang (2012).

1. Introduction

Firstly, let us recall the definitions of negatively associated (NA) random variables and NOD random variables as follows.

Definition 1. A finite collection of random variables $\{X_i; 1 \leq i \leq n\}$ is said to be NA if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \ldots, n\}$,

$$\operatorname{Cov}\left(f_1\left(X_i, i \in A_1\right), f_2\left(X_j, j \in A_2\right)\right) \leq 0, \quad (1)$$

whenever f_1 and f_2 are nondecreasing functions such that the covariance exists. An infinite collection of random variables $\{X_i; i \geq 1\}$ is NA if every finite subcollection is NA.

An array of random variables $\{X_{ni}; i \geq 1, n \geq 1\}$ is called rowwise NA random variables if for every $n \geq 1$, $\{X_{ni}; i \geq 1\}$ is a sequence of NA random variables.

Definition 2. A finite collection of random variables $\{X_i; 1 \leq i \leq n\}$ is said to be NOD if

$$\begin{gathered} P\left(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n\right) \leq \prod_{j=1}^{n} P\left(X_j \leq x_j\right), \\ P\left(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n\right) \leq \prod_{j=1}^{n} P\left(X_j > x_j\right), \end{gathered} \quad (2)$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$. An infinite collection of random variables $\{X_i; i \geq 1\}$ is said to be NOD if every finite subcollection is NOD.

An array of random variables $\{X_{ni}; i \geq 1, n \geq 1\}$ is called rowwise NOD random variables if for every $n \geq 1$, $\{X_{ni}; i \geq 1\}$ is a sequence of NOD random variables.

The concepts of NA and NOD random variables were introduced by Joag-Dev and Proschan [1]. Obviously, independent random variables are NOD, and NA implies NOD from the definition of NA and NOD, but NOD does not imply NA. So, NOD is much weaker than NA. Because of the wide applications of NOD random variables, the notion of NOD random variables has been received more and more attention recently. Many applications have been found. We can refer to Volodin [2], Asadian et al. [3], Amini et al. [4, 5], Kuczmaszewska [6], Zarei and Jabbari [7], Wu and Zhu [8], Wu [9], Sung [10], Wang et al. [11], Huang and Wang [12], and so forth. Hence, it is very significant to study limit properties of this wider NOD random variables in probability theory and practical applications.

Let $\{X_n; n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables and let $\{a_{ni}; i \geq 1, n \geq 1\}$ be an array of real constants. As Bai and Cheng [13] remarked, many useful linear statistics, for example, least-squares estimators, nonparametric regression function estimators, and jackknife estimates, are based on weighted sums of i.i.d. random variables. In this respect, the strong convergence for weighted sums $\sum_{i=1}^{n} a_{ni} X_i$ has been studied by many authors (see, e.g., Bai and Cheng [13]; Cuzick [14]; Sung [15]; Tang [16]; etc.).

Cai [17] proved the following complete convergence result for weighted sums of NA random variables.

Theorem A. *Let $\{X, X_n; n \geq 1\}$ be a sequence of identically distributed NA random variables, and let $\{a_{ni};\ 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying*

$$A_\alpha = \limsup_{n \to \infty} A_{\alpha,n} < \infty, \qquad A_{\alpha,n} = \frac{1}{n} \sum_{i=1}^{n} |a_{ni}|^\alpha, \tag{3}$$

for some $0 < \alpha \leq 2$. Suppose that $EX = 0$ when $1 < \alpha \leq 2$. If

$$E\{\exp(h|X|^\gamma)\} < \infty \quad \text{for some } h > 0,\ \gamma > 0, \tag{4}$$

then, for $b_n = n^{1/\alpha} (\log n)^{1/\gamma}$,

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > \varepsilon b_n \right) < \infty \quad \forall \varepsilon > 0. \tag{5}$$

Wang et al. [11] extended the above result of Cai [17] to arrays of rowwise NOD random variables as follows.

Theorem B. *Let $\{X_{ni}; i \geq 1, n \geq 1\}$ be an array of rowwise NOD random variables which is stochastically dominated by a random variable X and let $\{a_{ni};\ 1 \leq i \leq n, n \geq 1\}$ be an array of real constants. Assume that there exist some δ with $0 < \delta < 1$ and some α with $0 < \alpha < 2$ such that $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n^\delta)$ and assume further that $EX_{ni} = 0$ if $1 < \alpha < 2$. If for some $h > 0$ and $\gamma > 0$ such that (4), then*

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} P\left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} X_{ni} \right| > \varepsilon b_n \right) < \infty \quad \forall \varepsilon > 0, \tag{6}$$

where $p \geq 1/\alpha$ and $b_n = n^{1/\alpha} (\log n)^{1/\gamma}$.

Recently, Huang and Wang [12] partially extended the corresponding theorems of Cai [17] and Wang et al. [11] to NOD random variables under a mild moment condition.

Theorem C. *Let $\{X_n; n \geq 1\}$ be a sequence of NOD random variables which is stochastically dominated by a random variable X and let $\{a_{ni}; i \geq 1, n \geq 1\}$ be a triangular array of real constants such that $a_{ni} = 0$ for $i > n$. Let*

$$A_\beta = \limsup_{n \to \infty} A_{\beta,n} < \infty; \qquad A_{\beta,n} = n^{-1} \sum_{i=1}^{n} |a_{ni}|^\beta, \tag{7}$$

where $\beta = \max(\alpha, \gamma)$ for some $0 < \alpha \leq 2$, $\gamma > 0$, and $\alpha \neq \gamma$. Assume that $EX_n = 0$ for $1 < \alpha \leq 2$ and $E|X|^\beta < \infty$. Then,

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\left| \sum_{i=1}^{n} a_{ni} X_i \right| > \varepsilon b_n \right) < \infty, \tag{8}$$

where $b_n = n^{1/\alpha} (\log n)^{1/\gamma}$.

As Huang and Wang [12] pointed out, Theorem C partially extends only the case of $\alpha > \gamma$ of Theorems A and B. They left an open problem whether the case of $\alpha = \gamma$ of Theorem C holds for NOD random variables.

The main purpose of this paper is to further study strong convergence for weighted sums of NOD random variables and to obtain the rate of strong convergence for weighted sums of arrays of rowwise NOD random variables under a suitable moment condition. We solve the above problem posed by Huang and Wang [12].

We will use the following concept in this paper.

Definition 3. An array of random variables $\{X_{ni}; i \geq 1, n \geq 1\}$ is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_{ni}| > t) \leq CP(|X| > t), \tag{9}$$

for all $t \geq 0, i \geq 1$, and $n \geq 1$.

2. Main Results

Now, we will present the main results of this paper; the detailed proofs will be given in the next section.

Theorem 4. *Let $\{X_{ni}; i \geq 1, n \geq 1\}$ be an array of rowwise NOD random variables which is stochastically dominated by a random variable X and let $\{a_{ni};\ 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ for some $0 < \alpha \leq 2$. Assume further that $EX_{ni} = 0$ for $1 < \alpha \leq 2$ and $E|X|^\alpha \log(1 + |X|) < \infty$. Then,*

$$\sum_{n=1}^{\infty} n^{-1} P\left(\left| \sum_{i=1}^{n} a_{ni} X_{ni} \right| > \varepsilon b_n \right) < \infty \quad \forall \varepsilon > 0, \tag{10}$$

where $b_n = n^{1/\alpha} (\log n)^{1/\alpha}$.

Similar to the proof of Theorem 4, we can obtain the following result for NOD random variable sequences.

Corollary 5. *Let $\{X_n; n \geq 1\}$ be a sequence of NOD random variables which is stochastically dominated by a random variable X and let $\{a_{ni};\ 1 \leq i \leq n, n \geq 1\}$ be an array of real constants satisfying $\sum_{i=1}^{n} |a_{ni}|^\alpha = O(n)$ for some*

$0 < \alpha \le 2$. *Assume further that* $EX_n = 0$ *for* $1 < \alpha \le 2$ *and* $E|X|^\alpha \log(1+|X|) < \infty$. *Then,*

$$\sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{i=1}^{n} a_{ni} X_i\right| > \varepsilon b_n\right) < \infty \quad \forall \varepsilon > 0, \tag{11}$$

where $b_n = n^{1/\alpha}(\log n)^{1/\alpha}$.

Remark 6. In Theorem 4 and Corollary 5, we consider the case of $\alpha = \gamma$ for $0 < \alpha \le 2$ and obtain some strong convergence results for arrays of rowwise NOD random variables and NOD random variable sequences without assumption of identical distribution. The main result settles the open problem posed by Huang and Wang [12]. In addition, it is still an open problem whether

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni} X_{ni}\right| > \varepsilon b_n\right) < \infty \quad \forall \varepsilon > 0 \tag{12}$$

holds true under the same moment condition of Theorem 4.

3. Proofs

In order to prove our main results, the following lemmas are needed.

Lemma 7 (see Bozorgnia et al. [18]). *Let* $\{X_i; 1 \le i \le n\}$ *be a sequence of NOD random variables, and let* $\{f_i; 1 \le i \le n\}$ *be a sequence of Borel functions all of which are monotone nondecreasing (or all are monotone nonincreasing). Then,* $\{f_i(X_i); 1 \le i \le n\}$ *is a sequence of NOD random variables.*

Lemma 8 (see Asadian et al. [3]). *Let* $M \ge 2$ *and let* $\{X_n; n \ge 1\}$ *be a sequence of NOD random variables with* $EX_n = 0$ *and* $E|X_n|^M < \infty$ *for all* $n \ge 1$. *Then, there exists a positive constant* $C = C(M)$ *depending only on* M *such that, for all* $n \ge 1$,

$$E\left(\left|\sum_{i=1}^{n} X_i\right|^M\right) \le C\left[\sum_{i=1}^{n} E|X_i|^M + \left(\sum_{i=1}^{n} EX_i^2\right)^{M/2}\right]. \tag{13}$$

Lemma 9. *Let* $\{X_n; n \ge 1\}$ *be a sequence of random variables which is stochastically dominated by a random variable* X. *For any* $u > 0$ *and* $t > 0$, *the following two statements hold:*

$$E|X_{ni}|^u I(|X_{ni}| \le t) \le C_1\left(E|X|^u I(|X| \le t) + t^u P(|X| > t)\right), \tag{14}$$

$$E|X_{ni}|^u I(|X_{ni}| > t) \le C_2 E|X|^u I(|X| > t), \tag{15}$$

where C_1 *and* C_2 *are positive constants.*

Lemma 10 (see Sung [15]). *Let* X *be a random variable and let* $\{a_{ni};\ 1 \le i \le n, n \ge 1\}$ *be an array of real constants satisfying* $\sum_{i=1}^n |a_{ni}|^\alpha = O(n)$ *for some* $\alpha > 0$. *Let* $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ *for some* $\gamma > 0$. *Then,*

$$\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P\left(|a_{ni}X| > b_n\right) \le \begin{cases} CE|X|^\alpha, & \text{for } \alpha > \gamma, \\ CE|X|^\alpha \log(1+|X|), & \text{for } \alpha = \gamma, \\ CE|X|^\gamma, & \text{for } \alpha < \gamma. \end{cases} \tag{16}$$

Lemma 11 (see Sung [19]). *Let* X *be a random variable and let* $\{a_{ni};\ 1 \le i \le n, n \ge 1\}$ *be an array of real constants satisfying* $a_{ni} = 0$ *or* $|a_{ni}| > 1$ *and* $\sum_{i=1}^n |a_{ni}|^\alpha = O(n)$ *for some* $\alpha > 0$. *Let* $b_n = n^{1/\alpha}(\log n)^{1/\alpha}$. *If* $q > \alpha$, *then*

$$\sum_{n=1}^{\infty} n^{-1} b_n^{-q} \sum_{i=1}^{n} E|a_{ni}X|^q I\left(|a_{ni}X| \le b_n\right) \le CE|X|^\alpha \log(1+|X|). \tag{17}$$

Throughout this paper, let $I(A)$ be the indicator function of the set A. C denotes a positive constant, which may be different in various places and $a_n = O(b_n)$ stands for $a_n \le Cb_n$.

Proof of Theorem 4. Without loss of generality, suppose that $\sum_{i=1}^n |a_{ni}|^\alpha \le Cn$ and $a_{ni} \ge 0$, for all $1 \le i \le n$, $n \ge 1$. For fixed $n \ge 1$, define

$$\begin{aligned} X_i^{(n)} &= -b_n I\left(a_{ni}X_{ni} < -b_n\right) + a_{ni}X_{ni} I\left(|a_{ni}X_{ni}| \le b_n\right) \\ &\quad + b_n I\left(a_{ni}X_{ni} > b_n\right), \quad i \ge 1, \\ T_n^{(n)} &= \sum_{i=1}^{n}\left(X_i^{(n)} - EX_i^{(n)}\right). \end{aligned} \tag{18}$$

Denote

$$\begin{aligned} A &= \bigcap_{i=1}^{n}\left(a_{ni}X_{ni} = X_i^{(n)}\right), \\ B &= \overline{A} = \bigcup_{i=1}^{n}\left(a_{ni}X_{ni} \ne X_i^{(n)}\right) \\ &= \bigcup_{i=1}^{n}\left(|a_{ni}X_{ni}| > b_n\right), \\ E_n &= \left(\left|\sum_{i=1}^{n} a_{ni}X_{ni}\right| > \varepsilon b_n\right). \end{aligned} \tag{19}$$

It is easily seen that, for all $\varepsilon > 0$,

$$\begin{aligned} E_n = E_nA \bigcup E_nB \subset \left(\left|\sum_{i=1}^{n} X_i^{(n)}\right| > \varepsilon b_n\right) \\ \bigcup\left(\bigcup_{i=1}^{n}|a_{ni}X_{ni}| > b_n\right), \end{aligned} \tag{20}$$

which implies that

$$P\left(E_n\right) \leq P\left(\left|\sum_{i=1}^{n} X_i^{(n)}\right| > \varepsilon b_n\right) + P\left(\bigcup_{i=1}^{n} \left|a_{ni}X_{ni}\right| > b_n\right) \leq P\left(\left|T_n^{(n)}\right| > \varepsilon b_n - \left|\sum_{i=1}^{n} EX_i^{(n)}\right|\right) + \sum_{i=1}^{n} P\left(\left|a_{ni}X_{ni}\right| > b_n\right). \quad (21)$$

First, we will prove that

$$b_n^{-1}\left|\sum_{i=1}^{n} EX_i^{(n)}\right| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (22)$$

Actually, for $0 < \alpha \leq 1$, by (14) of Lemma 9, Markov inequality, and $E|X|^\alpha \log(1+|X|) < \infty$, we have that

$$\begin{aligned} b_n^{-1}\left|\sum_{i=1}^{n} EX_i^{(n)}\right| &\leq Cb_n^{-1}\sum_{i=1}^{n}\left|EX_i^{(n)}\right| \\ &\leq Cb_n^{-1}\sum_{i=1}^{n} E\left|a_{ni}X_{ni}\right| I\left(\left|a_{ni}X_{ni}\right| \leq b_n\right) \\ &\quad + C\sum_{i=1}^{n} P\left(\left|a_{ni}X_{ni}\right| > b_n\right) \\ &\leq Cb_n^{-1}\sum_{i=1}^{n}\left(E\left|a_{ni}X\right| I\left(\left|a_{ni}X\right| \leq b_n\right) + b_n P\left(\left|a_{ni}X\right| > b_n\right)\right) \\ &\quad + C\sum_{i=1}^{n} P\left(\left|a_{ni}X\right| > b_n\right) \\ &\leq Cb_n^{-1}\sum_{i=1}^{n}\left(E\left|a_{ni}X\right| I\left(\left|a_{ni}X\right| \leq b_n\right)\right) \\ &\quad + C\sum_{i=1}^{n} P\left(\left|a_{ni}X\right| > b_n\right) \\ &\leq Cb_n^{-\alpha}\sum_{i=1}^{n}\left(E\left|a_{ni}X\right|^\alpha I\left(\left|a_{ni}X\right| \leq b_n\right)\right) \\ &\quad + Cb_n^{-\alpha}\sum_{i=1}^{n} E\left|a_{ni}X\right|^\alpha \\ &\leq Cb_n^{-\alpha}\sum_{i=1}^{n}\left|a_{ni}\right|^\alpha E|X|^\alpha + Cb_n^{-\alpha}\sum_{i=1}^{n}\left|a_{ni}\right|^\alpha E|X|^\alpha \\ &\leq C(\log n)^{-1}E|X|^\alpha \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (23)$$

Next, for $1 < \alpha \leq 2$, by $EX_{ni} = 0$, (15) of Lemmas 9 and 10, Markov inequality, and $E|X|^\alpha \log(1+|X|) < \infty$, we also have that

$$\begin{aligned} b_n^{-1}\left|\sum_{i=1}^{n} EX_i^{(n)}\right| &\leq C\sum_{i=1}^{n} P\left(\left|a_{ni}X_{ni}\right| > b_n\right) \\ &\quad + Cb_n^{-1}\left|\sum_{i=1}^{n} Ea_{ni}X_{ni} I\left(\left|a_{ni}X_{ni}\right| > b_n\right)\right| \\ &\leq C\sum_{i=1}^{n} P\left(\left|a_{ni}X\right| > b_n\right) \\ &\quad + Cb_n^{-1}\sum_{i=1}^{n} E\left|a_{ni}X_{ni}\right| I\left(\left|a_{ni}X_{ni}\right| > b_n\right) \\ &\leq Cb_n^{-\alpha}\sum_{i=1}^{n} E\left|a_{ni}X\right|^\alpha \\ &\quad + Cb_n^{-1}\sum_{i=1}^{n} E\left|a_{ni}X\right| I\left(\left|a_{ni}X\right| > b_n\right) \\ &\leq Cb_n^{-\alpha}\sum_{i=1}^{n} E\left|a_{ni}X\right|^\alpha \\ &\quad + Cb_n^{-\alpha}\sum_{i=1}^{n} E\left|a_{ni}X\right|^\alpha I\left(\left|a_{ni}X\right| > b_n\right) \\ &\leq Cb_n^{-\alpha}\sum_{i=1}^{n} E\left|a_{ni}X\right|^\alpha + Cb_n^{-\alpha}\sum_{i=1}^{n} E\left|a_{ni}X\right|^\alpha \\ &\leq C(\log n)^{-1}E|X|^\alpha \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (24)$$

From the above statements, we can get (22) immediately. Hence, for n large enough,

$$P\left(E_n\right) \leq P\left(\left|T_n^{(n)}\right| > \frac{\varepsilon b_n}{2}\right). \quad (25)$$

To prove (10), it is sufficient to show that

$$\begin{aligned} I &\triangleq \sum_{n=1}^{\infty} n^{-1} P\left(\left|T_n^{(n)}\right| > \frac{\varepsilon b_n}{2}\right) < \infty, \\ J &\triangleq \sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P\left(\left|a_{ni}X_{ni}\right| > b_n\right) < \infty. \end{aligned} \quad (26)$$

It follows from Lemma 10 and $E|X|^\alpha \log(1+|X|) < \infty$ that

$$\begin{aligned} J &\triangleq \sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P\left(\left|a_{ni}X_{ni}\right| > b_n\right) \\ &\leq C\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P\left(\left|a_{ni}X\right| > b_n\right) \\ &\leq E|X|^\alpha \log\left(1+|X|\right) < \infty. \end{aligned} \quad (27)$$

For fixed $n \geq 1$, it is easily seen that $\{X_i^{(n)} - EX_i^{(n)}, i \geq 1, n \geq 1\}$ is still a sequence of NOD random variables with mean zero by Lemma 7. Hence, it follows from (14) of Lemmas 9 and 8 and Markov inequality (for $M > 2$) that

$$\begin{aligned} I &\triangleq \sum_{n=1}^{\infty} n^{-1} P\left(\left|T_n^{(n)}\right| > \frac{\varepsilon b_n}{2}\right) \\ &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} E\left(\left|T_n^{(n)}\right|^M\right) \\ &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M}\left[\sum_{i=1}^{n} E\left|X_i^{(n)}\right|^M + \left(\sum_{i=1}^{n} E\left|X_i^{(n)}\right|^2\right)^{M/2}\right] \\ &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M}\sum_{i=1}^{n} E\left|X_i^{(n)}\right|^M \\ &\quad + C\sum_{n=1}^{\infty} n^{-1} b_n^{-M}\left(\sum_{i=1}^{n} E\left|X_i^{(n)}\right|^2\right)^{M/2} \\ &\triangleq I_1 + I_2. \end{aligned} \tag{28}$$

It follows from Lemma 10, (14) of Lemma 9, and Markov inequality that

$$\begin{aligned} I_1 &\triangleq C\sum_{n=1}^{\infty} n^{-1} b_n^{-M}\sum_{i=1}^{n} E\left|X_i^{(n)}\right|^M \\ &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M}\left\{\sum_{i=1}^{n} |a_{ni}|^M E|X_{ni}|^M I\left(|a_{ni}X_{ni}| \le b_n\right)\right. \\ &\qquad\left. + \sum_{i=1}^{n} b_n^M P\left(|a_{ni}X_{ni}| > b_n\right)\right\} \\ &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M}\left\{\sum_{i=1}^{n} |a_{ni}|^M E|X|^M I\left(|a_{ni}X| \le b_n\right)\right. \\ &\qquad\left. + 2\sum_{i=1}^{n} b_n^M P\left(|a_{ni}X| > b_n\right)\right\} \\ &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-M}\sum_{i=1}^{n} |a_{ni}|^M E|X|^M I\left(|a_{ni}X| \le b_n\right) \\ &\quad + C\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P\left(|a_{ni}X| > b_n\right) \\ &\triangleq I_{11} + I_{12}. \end{aligned} \tag{29}$$

From Lemma 10 and $E|X|^\alpha \log(1 + |X|) < \infty$, we can obtain that

$$\begin{aligned} I_{12} &\triangleq C\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P\left(|a_{ni}X| > b_n\right) \\ &\le E|X|^\alpha \log\left(1 + |X|\right) < \infty. \end{aligned} \tag{30}$$

For fixed $n > 1$, we divide $\{a_{ni}, 1 \le i \le n\}$ into three subsets $\{a_{ni} : |a_{ni}| \le 1/(\log n)^m\}$, $\{a_{ni} : 1/(\log n)^m < |a_{ni}| \le 1\}$, and $\{a_{ni} : |a_{ni}| > 1\}$, where $m = (1/(M - \alpha))$. Then,

$$\begin{aligned} I_{11} &\triangleq C\sum_{n=1}^{\infty} n^{-1} b_n^{-M}\sum_{i=1}^{n} |a_{ni}|^M E|X|^M I\left(|a_{ni}X| \le b_n\right) \\ &= C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \\ &\qquad \times \sum_{i:|a_{ni}| \le 1/(\log n)^m} |a_{ni}|^M E|X|^M I\left(|a_{ni}X| \le b_n\right) \\ &\quad + C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \\ &\qquad \times \sum_{i:1/(\log n)^m < |a_{ni}| \le 1} |a_{ni}|^M E|X|^M I\left(|a_{ni}X| \le b_n\right) \\ &\quad + C\sum_{n=1}^{\infty} n^{-1} b_n^{-M}\sum_{i:|a_{ni}| > 1} |a_{ni}|^M E|X|^M I\left(|a_{ni}X| \le b_n\right) \\ &\triangleq I_{11}^{(1)} + I_{11}^{(2)} + I_{11}^{(3)}. \end{aligned} \tag{31}$$

By Lemma 11 and $E|X|^\alpha \log(1 + |X|) < \infty$ again, it follows that

$$\begin{aligned} I_{11}^{(3)} &\triangleq C\sum_{n=1}^{\infty} n^{-1} b_n^{-M}\sum_{i:|a_{ni}| > 1} |a_{ni}|^M E|X|^M I\left(|a_{ni}X| \le b_n\right) \\ &\le E|X|^\alpha \log\left(1 + |X|\right) < \infty \quad \text{for } M > 2 \ge \alpha > 0. \end{aligned} \tag{32}$$

Noting that $\sum_{i:|a_{ni}| \le 1/(\log n)^m} |a_{ni}|^\alpha \le Cn(\log n)^{-m\alpha}$, for $M > \alpha$ and fixed $n > 1$, we have that

$$\begin{aligned} I_{11}^{(1)} &\triangleq C\sum_{n=1}^{\infty} n^{-1} b_n^{-M}\sum_{i:|a_{ni}| \le 1/(\log n)^m} |a_{ni}|^M E|X|^M I\left(|a_{ni}X| \le b_n\right) \\ &\le C\sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha}\sum_{i:|a_{ni}| \le 1/(\log n)^m} |a_{ni}|^\alpha E|X|^\alpha I\left(|a_{ni}X| \le b_n\right) \\ &\le CE|X|^\alpha \sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha}\sum_{i:|a_{ni}| \le 1/(\log n)^m} |a_{ni}|^\alpha \\ &\le CE|X|^\alpha \sum_{n=1}^{\infty} n^{-1} n^{-1} (\log n)^{-1} n (\log n)^{-m\alpha} < \infty. \end{aligned} \tag{33}$$

Noting that $\sum_{i:1/(\log n)^m<|a_{ni}|\le 1}|a_{ni}|^M \le Cn$ and $m = 1/(M-\alpha)$, for $M > 2, 0 < \alpha \le 2$, we have that

$$\begin{aligned}
I_{11}^{(2)} &\triangleq C\sum_{n=1}^{\infty} n^{-1} b_n^{-M} \sum_{i:1/(\log n)^m<|a_{ni}|\le 1} |a_{ni}|^M E|X|^M I(|a_{ni}X| \le b_n) \\
&\le C\sum_{n=1}^{\infty} b_n^{-M} E|X|^M I(|X| \le b_n(\log n)^m) \\
&= C\sum_{n=1}^{\infty} b_n^{-M} \sum_{k=1}^{n} E|X|^M I\Big((k-1)^{1/\alpha}(\log(k-1))^{m+1/\alpha} \\
&\qquad < |X| \le k^{1/\alpha}(\log k)^{m+1/\alpha}\Big) \\
&= C\sum_{k=1}^{\infty} E|X|^M I\Big((k-1)^{1/\alpha}(\log(k-1))^{m+1/\alpha} \\
&\qquad < |X| \le k^{1/\alpha}(\log k)^{m+1/\alpha}\Big) \\
&\quad \times \sum_{n=k}^{\infty} n^{-M/\alpha}(\log n)^{-M/\alpha} \\
&\le C\sum_{k=1}^{\infty} E|X|^M I\Big((k-1)^{1/\alpha}(\log(k-1))^{m+1/\alpha} \\
&\qquad < |X| \le k^{1/\alpha}(\log k)^{m+1/\alpha}\Big) \\
&\quad \times k^{1-M/\alpha}(\log k)^{-M/\alpha} \\
&\le C\sum_{k=1}^{\infty} E|X|^\alpha I\Big((k-1)^{1/\alpha}(\log(k-1))^{m+1/\alpha} \\
&\qquad < |X| \le k^{1/\alpha}(\log k)^{m+1/\alpha}\Big) \\
&\le CE|X|^\alpha < \infty.
\end{aligned} \tag{34}$$

Finally, we will prove that

$$I_2 \triangleq C\sum_{n=1}^{\infty} n^{-1} b_n^{-M}\left(\sum_{i=1}^{n} E\left|X_i^{(n)}\right|^2\right)^{M/2} < \infty. \tag{35}$$

Hence, by C_r inequality, Markov inequality, Lemmas 9–11, and $E|X|^\alpha \log(1+|X|) < \infty$, we have that

$$\begin{aligned}
I_2 &\triangleq C\sum_{n=1}^{\infty} n^{-1} b_n^{-M}\left(\sum_{i=1}^{n} E\left|X_i^{(n)}\right|^2\right)^{M/2} \\
&= C\sum_{n=1}^{\infty} n^{-1}\left(\sum_{i=1}^{n} b_n^{-2} E\left|X_i^{(n)}\right|^2\right)^{M/2} \\
&\le C\sum_{n=1}^{\infty} n^{-1}\left(\sum_{i=1}^{n} P(|a_{ni}X_{ni}| > b_n)\right)^{M/2} \\
&\quad + C\sum_{n=1}^{\infty} n^{-1}\left(\sum_{i=1}^{n} b_n^{-2} E|a_{ni}X_{ni}|^2 (|a_{ni}X_{ni}| \le b_n)\right)^{M/2} \\
&\le C\left(\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P(|a_{ni}X| > b_n)\right)^{M/2} \\
&\quad + C\left(\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} b_n^{-2} E|a_{ni}X|^2 (|a_{ni}X| \le b_n)\right)^{M/2} \\
&\le C\left(E|X|^\alpha \log(1+|X|)\right)^{M/2} < \infty.
\end{aligned} \tag{36}$$

Therefore, the desired result (10) follows from the above statements. This completes the proof of Theorem 4. □

Conflict of Interests

The authors declare that they have no conflict of interests.

Acknowledgments

The authors are most grateful to the referees and to the editor Professor Baodong Zheng for their valuable suggestions and some helpful comments which greatly improved the clarity and readability of this paper. This paper is partially supported by the National Nature Science Foundation of China (71271042), the Fundamental Research Funds for the Central Universities of China (ZYGX2012J119), the Nature Science Foundation of Guangxi Province (2014GXNSFBA118006, 2013GXNSFDA019001), and the Guangxi Provincial Scientific Research Projects (201204LX157, 2013YB104).

References

[1] K. Joag-Dev and F. Proschan, "Negative association of random variables, with applications," *The Annals of Statistics*, vol. 11, no. 1, pp. 286–295, 1983.

[2] A. Volodin, "On the Kolmogorov exponential inequality for negatively dependent random variables," *Pakistan Journal of Statistics*, vol. 18, no. 2, pp. 249–254, 2002.

[3] N. Asadian, V. Fakoor, and A. Bozorgnia, "Rosental's type inequalities for negatively orthant dependent random variables," *Journal of Iranian Statistical Society*, vol. 5, no. 1-2, pp. 66–75, 2006.

[4] M. Amini and A. Bozorgnia, "Complete convergence for negatively dependent random variables," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 16, no. 2, pp. 121–126, 2003.

[5] M. Amini, H. Zarei, and A. Bozorgnia, "Some strong limit theorems of weighted sums for negatively dependent generalized Gaussian random variables," *Statistics & Probability Letters*, vol. 77, no. 11, pp. 1106–1110, 2007.

[6] A. Kuczmaszewska, "On some conditions for complete convergence for arrays of rowwise negatively dependent random variables," *Stochastic Analysis and Applications*, vol. 24, no. 6, pp. 1083–1095, 2006.

[7] H. Zarei and H. Jabbari, "Complete convergence of weighted sums under negative dependence," *Statistical Papers*, vol. 52, no. 2, pp. 413–418, 2011.

[8] Y. Wu and D. Zhu, "Convergence properties of partial sums for arrays of rowwise negatively orthant dependent random variables," *Journal of the Korean Statistical Society*, vol. 39, no. 2, pp. 189–197, 2010.

[9] Q. Wu, "Complete convergence for negatively dependent sequences of random variables," *Journal of Inequalities and Applications*, vol. 2010, Article ID 507293, 10 pages, 2010.

[10] S. H. Sung, "A note on the complete convergence for arrays of dependent random variables," *Journal of Inequalities and Applications*, vol. 2011, article 76, 2011.

[11] X. Wang, S. Hu, and W. Yang, "Complete convergence for arrays of rowwise negatively orthant dependent random variables," *RACSAM*, vol. 106, no. 2, pp. 235–245, 2012.

[12] H. W. Huang and D. C. Wang, "A note on the strong limit theorem for weighted sums of sequences of negatively dependent random variables," *Journal of Inequalities and Applications*, vol. 2012, article 233, 2012.

[13] Z. D. Bai and P. E. Cheng, "Marcinkiewicz strong laws for linear statistics," *Statistics & Probability Letters*, vol. 46, no. 2, pp. 105–112, 2000.

[14] J. Cuzick, "A strong law for weighted sums of i.i.d. random variables," *Journal of Theoretical Probability*, vol. 8, no. 3, pp. 625–641, 1995.

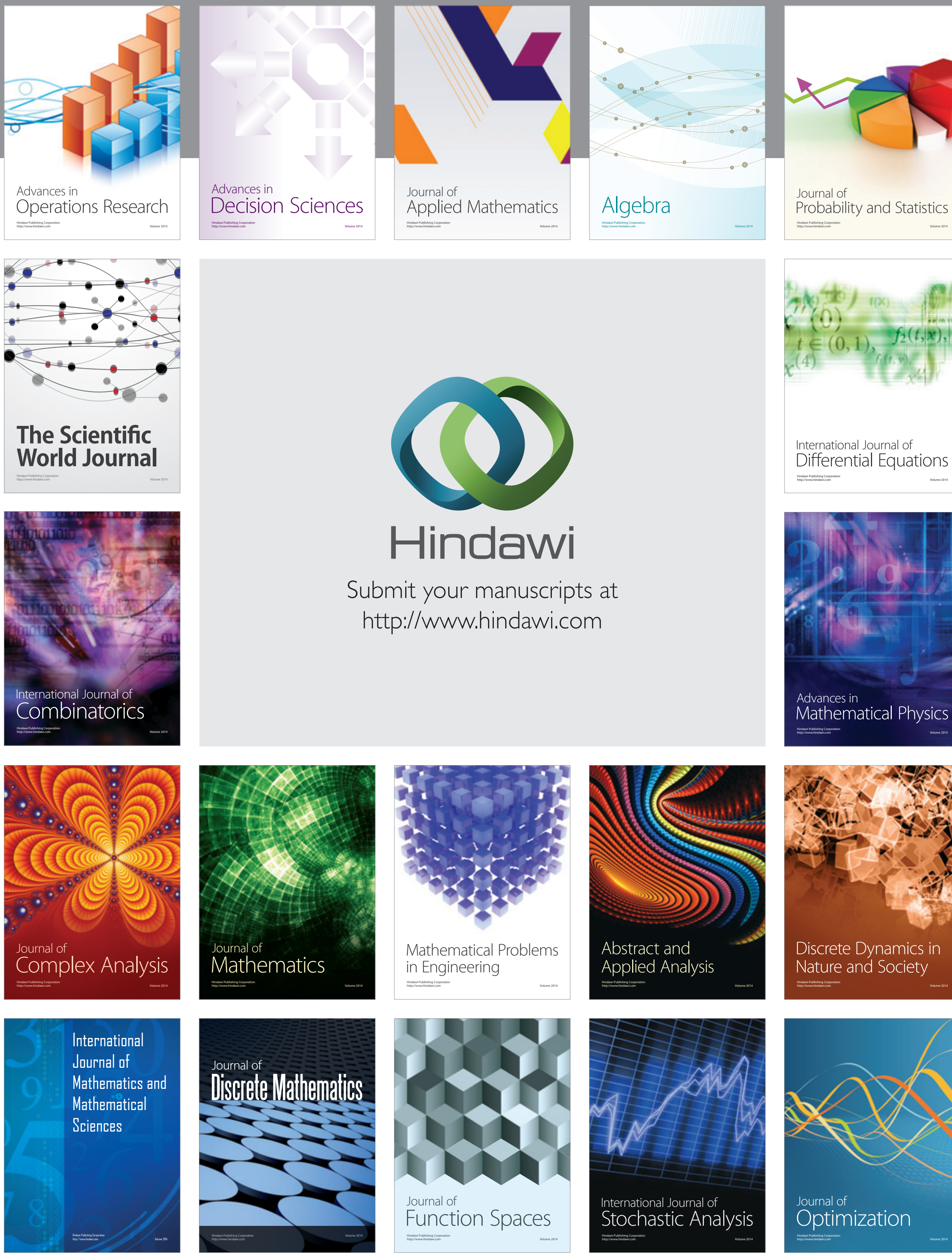
[15] S. H. Sung, "On the strong convergence for weighted sums of random variables," *Statistical Papers*, vol. 52, no. 2, pp. 447–454, 2011.

[16] X. Tang, "Strong convergence results for arrays of rowwise pairwise NQD random variables," *Journal of Inequalities and Applications*, vol. 2013, article 102, 2013.

[17] G. Cai, "Strong laws for weighted sums of NA random variables," *Metrika*, vol. 68, no. 3, pp. 323–331, 2008.

[18] A. Bozorgnia, R. F. Patterson, and R. L. Taylor, "Limit theorems for dependent random variables," in *World Congress of Nonlinear Analysts '92, Vol. I-IV (Tampa, FL, 1992)*, pp. 1639–1650, de Gruyter, Berlin, Germany, 1996.

[19] S. H. Sung, "On the strong convergence for weighted sums of ρ^*- mixing random variables," *Statistical Papers*, vol. 54, pp. 773–781, 2013.


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Differential Equations
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization