Hindawi
Advances in Mathematical Physics
Volume 2018, Article ID 5094878, 10 pages
https://doi.org/10.1155/2018/5094878

Research Article

Regularization of the Boundary-Saddle-Node Bifurcation

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Received 10 June 2017; Accepted 9 November 2017; Published 4 January 2018

Academic Editor: Andrei D. Mironov

In this paper we treat a particular class of planar Filippov systems which consist of two smooth systems that are separated by a discontinuity boundary. In such systems one vector field undergoes a saddle-node bifurcation while the other vector field is transversal to the boundary. The boundary-saddle-node (BSN) bifurcation occurs at a critical value when the saddle-node point is located on the discontinuity boundary. We derive a local topological normal form for the BSN bifurcation and study its local dynamics by applying the classical Filippov's convex method and a novel regularization approach. In fact, by the regularization approach a given Filippov system is approximated by a piecewise-smooth continuous system. Moreover, the regularization process produces a singular perturbation problem where the original discontinuous set becomes a center manifold. Thus, the regularization enables us to make use of the established theories for continuous systems and slow-fast systems to study the local behavior around the BSN bifurcation.

1. Introduction

The study of Filippov system was motivated by its considerable applications in mechanical systems exhibiting dry friction [1–4], biological systems [5–7], and control systems [8–10]. In general, Filippov systems, or differential equations with discontinuous right-hand side, model physical processes which experience abrupt transitions between different modes. Modelling in such way we often idealize the transition to be instantaneous. The consequence of doing so is that the established discontinuous model focuses on the overall dynamics of the entire physical process while slightly ignoring the detailed dynamics in the transition stage. An efficient method to learn the dynamics in the transition stage is modelling this process in a small time scale such that the dynamics of one mode smoothly or continuously changes to the other mode. For a given Filippov system this can be realized by a smooth or continuous approximation which removes the discontinuities. This method is referred to as *regularization*. As a consequence, this enables us to make use of the established theories for smooth or continuous systems to study the dynamical properties of Filippov systems.

Different ways have been proposed to regularize a Filippov system; see [11–13]. In this paper we apply the regularization of [12] to a particular class of planar Filippov systems. Such systems consist of two smooth vector fields that are separated by a smooth discontinuity boundary. The vector field on one side undergoes a standard saddle-node bifurcation, while the vector field on the other side intersects the boundary transversally. Here a "standard saddle-node bifurcation" means a saddle-node bifurcation in smooth systems. We refer for a formal definition of Filippov system and the regularization approach of [12] to Sections 2.1 and 2.2, respectively.

Our aim of this work is to describe the local dynamics in the neighborhood of the codimension-2 *boundary-saddle-node* (BSN) bifurcation where two equilibria of one smooth vector field go through a saddle-node bifurcation while they lie on the boundary. We treat this particular class of Filippov systems by applying both Filippov's convex method and the regularization approach to the BSN point to obtain its local bifurcation diagram, to understand how all the codimension-1 bifurcations interact. As we shall see in Section 3 the BSN point studied by Filippov's convex method acts as organizing center for three families of codimension-1 bifurcations appearing in its neighborhood, which are standard saddle-node bifurcation, *equilibrium transition, and* nonsmooth fold bifurcation. However, after regularization, some of the *discontinuity-induced bifurcations* disappear, such as the equilibrium transition. The unique bifurcation that occurs in the regularized system is the saddle-node-like bifurcation; see Section 2.2. Here the extension "like" means that this bifurcation is not the standard one as known from smooth systems. We refer to [15] for a detailed description of the generic bifurcations in piecewise-smooth continuous systems.

We briefly outline the structure of this paper. In the next section we give an overview of Filippov dynamics and the regularization technique of [12]. Moreover, we explain how to transform the regularized system to a singular perturbation problem. In Section 3 we construct a topological normal form for the BSN bifurcation. Subsequently, we study the BSN bifurcation by Filippov's convex method and obtain its bifurcation diagram and phase portraits in Section 4. After that, in Section 5 we apply the regularization approach to the topological normal form and investigate its local dynamics as a piecewise-smooth continuous system. Finally, in Section 6 we summarize the main results.

2. Preliminaries

In this section we introduce Filippov's convex method and the regularization approach of [12].

2.1. Filippov's Convex Method. We first introduce the basic concepts in Filippov systems.

2.1.1. Filippov Dynamics. A triplet (X, Y, Σ) defined on a 2-dimensional manifold M is called a Filippov system, denoted by Z. Both X and Y are $\mathscr{C}^r$ ($r \geq 1$ or $r = \infty$) vector fields, which are extendable over a full neighborhood of the discontinuity boundary Σ. The boundary is given as follows:

$$\Sigma = \{(x, y) \in M : f(x, y) = 0\}, \quad (1)$$

where $f : M \to \mathbb{R}$ is a $\mathscr{C}^r$ function and has 0 as a regular value. Therefore, Σ is a smooth 1-dimensional submanifold of M. As a consequence, Σ separates M to two open subsets M_X and M_Y:

$$\begin{aligned} M_X &= \{(x, y) \in M : f(x, y) < 0\}, \\ M_Y &= \{(x, y) \in M : f(x, y) > 0\}. \end{aligned} \quad (2)$$

For any $(x, y) \in M \backslash \Sigma$, the dynamics of Z is given by

$$Z(x, y) = \begin{cases} X(x, y), & \text{for } (x, y) \in M_X, \\ Y(x, y), & \text{for } (x, y) \in M_Y. \end{cases} \quad (3)$$

According to Filippov's convex method [16], Σ can be divided into the *crossing set*

$$\Sigma_c = \{(x, y) \in \Sigma : Xf(x, y)\, Yf(x, y) > 0\} \quad (4a)$$

and the *sliding set*

$$\Sigma_s = \{(x, y) \in \Sigma : Xf(x, y)\, Yf(x, y) \leq 0\}, \quad (4b)$$

where $Xf(x, y)$ denotes the directional derivative of f with respect to X at the point (x, y). That is, $\Sigma = \Sigma_s \cup \Sigma_c$.

In particular, the dynamics of a point (x, y) at the sliding set is defined by a unique convex linear combination of $X(x, y)$ and $Y(x, y)$, which is tangent to Σ at (x, y). This dynamics is called the *sliding vector field* [16], denoted by Z_s and given as follows:

$$Z_s(x, y) = \frac{Xf(x, y)\, Y(x, y) - Yf(x, y)\, X(x, y)}{Xf(x, y) - Yf(x, y)}, \quad (5)$$

when $Xf(x, y) \neq Yf(x, y)$. We define $Z(x, y) = 0$ when $Xf(x, y) = Yf(x, y)$ for $(x, y) \in \Sigma_s$.

2.1.2. Equilibria and Bifurcations. Here we give a brief description of the discontinuity-induced bifurcations (bifurcations where the discontinuity of the Filippov systems plays an essential role) of equilibria appearing in planar Filippov system.

Definition 1 (equilibria). A point $(x, y) \in M \backslash \Sigma$ is an *ordinary equilibrium* of X (resp. Y) if $X(x, y) = 0$ and $(x, y) \in M_X$ (resp., $Y(x, y) = 0$ and $(x, y) \in M_Y$). A point $(x, y) \in \Sigma$ is called a *boundary equilibrium* of X (resp. Y) if $X(x, y) = 0$ (resp., $Y(x, y) = 0$). A point $(x, y) \in M$ is an ordinary (resp., boundary) equilibrium of Z if it is an ordinary (resp., boundary) equilibrium of X or Y. Equilibria of X in M_Y and of Y in M_X are called *virtual equilibria*.

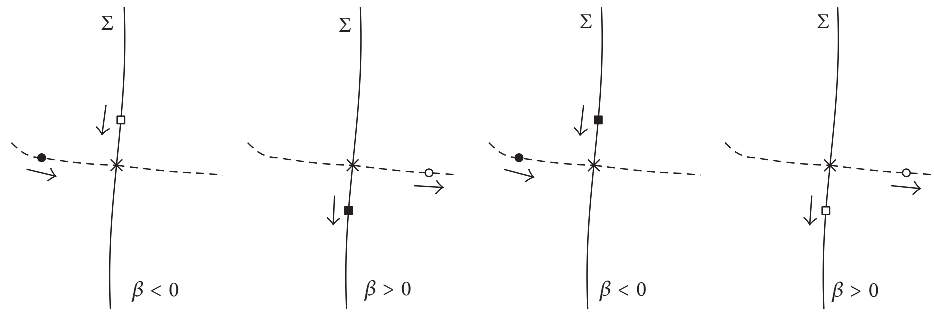
Definition 2 (pseudoequilibria). A point $(x, y) \in \Sigma_s$ is a *pseudoequilibrium* of Z if (x, y) is not a boundary equilibrium and $Z_s(x, y) = 0$. If $Z_s(x, y) = 0$ for $(x, y) \in \Sigma_c$, then one calls (x, y) a *virtual pseudoequilibrium* of Z.

When an ordinary equilibrium of Z collides with the discontinuity boundary Σ, it generically gives either an *equilibrium transition* (ET) or a *nonsmooth fold* (NSF) bifurcation; compare [17]. For the ET an ordinary equilibrium collides with a virtual pseudoequilibrium and then becomes a virtual ordinary equilibrium and a pseudoequilibrium; see Figure 1(a). For the NSF bifurcation an ordinary equilibrium collides with a pseudoequilibrium and then they both become virtual; see Figure 1(b).

The following theorem [14] gives conditions for the occurrence of an equilibrium transition or nonsmooth fold bifurcation in a Filippov system that smoothly depends on a parameter β.

Theorem 3 (equilibrium transition and nonsmooth fold bifurcation [14]). *Assume that $X(0, 0, 0) = 0$ and that $D_{(x,y)}X(0, 0, 0)$ is nonsingular. Moreover, assume that for $\beta = 0$ an equilibrium branch $(x(\beta), y(\beta))$ of the vector field $X(x, y, \beta)$ transversally crosses the discontinuity manifold Σ at $(x, y, \beta) = (0, 0, 0)$; that is,*

$$\left.\frac{d}{d\beta}\right|_{\beta=0} f(x(\beta), y(\beta), \beta) \neq 0. \quad (6)$$



(a) Equilibrium transition (b) Nonsmooth fold

Figure 1: Equilibrium transition and nonsmooth fold bifurcation. The discontinuity boundary Σ is displayed as a solid line. The dashed line represents the motion of the equilibrium as a parameter changes. Round points represent ordinary equilibria (filled) or virtual equilibria (hollow). Square points represent pseudoequilibria (filled) or virtual pseudoequilibria (hollow) [14].

Finally, assume that the nondegeneracy condition

$$\delta := D_{(x,y)} f(0,0,0) \left(D_{(x,y)} X(0,0,0)\right)^{-1} Y(0,0,0) \neq 0 \quad (7)$$

is satisfied. Then, at $\beta = 0$, there is an equilibrium transition if $\delta > 0$ and there is a nonsmooth fold bifurcation if $\delta < 0$.

2.1.3. Equivalence between Filippov Systems. Here we define the equivalence that will be used in this work.

Definition 4 (topological equivalence [18]). Two Filippov systems Z and $\widetilde{Z}$ are *topologically equivalent* if there is a homeomorphism φ that sends the orbits of Z to $\widetilde{Z}$ and the sliding set Σ_s to $\widetilde{\Sigma}_s$.

2.2. Regularization Approach. In this section we introduce the regularization approach of [12] and present its properties about equilibria.

According to [12] piecewise-smooth continuous regularization R_ε of the Filippov system Z is given in the following way. Let

$$M_R = M_X \cup S \cup M_Y \quad (8)$$

be the *extended manifold*, where $S = [-\varepsilon, \varepsilon] \times \Sigma$. That is, M_R is obtained by cutting the manifold M along Σ and then glueing the two pieces together by adding the "strip" S. The regularized vector field is defined on M_R as

$$R_\varepsilon(x,y) = \begin{cases} X(x,y), & \text{for } (x,y) \in M_X, \\ Y(x,y), & \text{for } (x,y) \in M_Y, \\ \left(1 - \alpha\left(\frac{x}{\varepsilon}\right)\right) X(x,y) + \alpha\left(\frac{x}{\varepsilon}\right) Y(x,y), & \text{for } (x,y) = (x,q) \in [-\varepsilon,\varepsilon] \times \Sigma. \end{cases} \quad (9)$$

Here α is a strictly increasing, smooth function defined in $[-1,1]$ with $\alpha(-1) = 0$, $\alpha(1) = 1$. For simplicity we assume $\alpha(0) = 1/2$.

In this paper we are interested in local dynamics of the Filippov system. Since Σ is a submanifold of M, there are local coordinates (x,y) in a neighborhood of p such that locally $\Sigma = \{(x,y) \mid x = 0\}$. In this case the regularized system can be written as

$$R_\varepsilon(x,y) = \begin{cases} X(x+\varepsilon, y), & \text{for } x \leq -\varepsilon, \\ Y(x-\varepsilon, y), & \text{for } x \geq \varepsilon, \\ \left(1 - \alpha\left(\frac{x}{\varepsilon}\right)\right) X(0,y) + \alpha\left(\frac{x}{\varepsilon}\right) Y(0,y), & \text{for } -\varepsilon \leq x \leq \varepsilon. \end{cases} \quad (10)$$

For the regularized system R_ε we have the following theorem about its equilibria and stability type.

Theorem 5. *The ordinary equilibria and pseudoequilibria of Z correspond to equilibria of R_ε with the same type of stability. Boundary equilibria of either X or Y (but not both) correspond to equilibria of R_ε. Common boundary equilibria of X and Y give rise to a curve of equilibria of R_ε in S.*

2.3. Regularization as a Singular Perturbation Problem. In this section we briefly review singular perturbation theory. Consider a general singular perturbation problem with the *slow time system*

$$\frac{d}{dt}\begin{pmatrix} \varepsilon x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y,\varepsilon) \\ g(x,y,\varepsilon) \end{pmatrix} \quad (11)$$

or equivalently the *fast time system*, after time rescaling $\tau = t/\varepsilon$,

$$\frac{d}{d\tau}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y,\varepsilon) \\ \varepsilon g(x,y,\varepsilon) \end{pmatrix}, \quad (12)$$

where $x \in \mathbb{R}^k$, $y \in \mathbb{R}^m$, $0 < \varepsilon \ll 1$.

The *slow manifold* is the set of equilibria of the fast time system for $\varepsilon = 0$ and is given by

$$\mathscr{M} = \{(x, y) : f(x, y, 0) = 0\}. \quad (13)$$

Then $\mathscr{M}$ is said to be *normally hyperbolic* if for every $(x, y) \in \mathscr{M}$ the matrix $D_x f$ has no eigenvalues on the imaginary axis.

For $\varepsilon = 0$ the slow time system (11) is reduced to the following system on $\mathscr{M}$:

$$\begin{pmatrix} 0 \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} f(x, y, 0) \\ g(x, y, 0) \end{pmatrix}, \quad (14)$$

called the *reduced problem*.

The following result of Fenichel (see [19]) on the preservation of normally hyperbolic invariant manifolds plays a central role in geometric singular perturbation theory.

Theorem 6 (Fenichel's invariant manifold theorem). *If $\mathscr{M}$ is a compact set and normally hyperbolic, then, for $\varepsilon > 0$ and sufficiently small, there exists a locally invariant manifold $\mathscr{M}_\varepsilon$ close to $\mathscr{M}$ in the C^1 topology. The manifold $\mathscr{M}_\varepsilon$ is diffeomorphic to $\mathscr{M}$, and the fast flow on $\mathscr{M}_\varepsilon$ is close to the flow of the reduced equation on $\mathscr{M}$.*

Now consider the Filippov system $Z = (X, Y, \Sigma)$ on $\mathbb{R}^2$. Recall that given a point $p \in \Sigma$ we can find a local coordinate system (x, y) such that Σ is given by $x = 0$ and the regularized system is given by (10). We now focus on the region $S = [-\varepsilon, \varepsilon] \times \Sigma$ and we perform the scaling transformation $u = x/\varepsilon$. This scaling induces in $S' = [-1, 1] \times \Sigma$ the system

$$\frac{d}{dt}\begin{pmatrix} \varepsilon u \\ y \end{pmatrix} = \begin{pmatrix} (1 - \alpha(u)) X_1(0, y) + \alpha(u) Y_1(0, y) \\ (1 - \alpha(u)) X_2(0, y) + \alpha(u) Y_2(0, y) \end{pmatrix}. \quad (15)$$

Rescaling time by $\tau = t/\varepsilon$ we get

$$\frac{d}{d\tau}\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} (1 - \alpha(u)) X_1(0, y) + \alpha(u) Y_1(0, y) \\ \varepsilon\left[(1 - \alpha(u)) X_2(0, y) + \alpha(u) Y_2(0, y)\right] \end{pmatrix}. \quad (16)$$

System (15) is the *slow time system* while (16) is the *fast time system*. The slow manifold is given by

$$\mathscr{M} = \{(u, y) : (1 - \alpha(u)) X_1(0, y) + \alpha(u) Y_1(0, y) = 0\}. \quad (17)$$

The slow time system (15) for $\varepsilon = 0$ defines a dynamical system on $\mathscr{M}$ with dynamics given by

$$\frac{dy}{dt} = (1 - \alpha(u)) X_2(0, y) + \alpha(u) Y_2(0, y). \quad (18)$$

Accordingly the slow time system (15) for $\varepsilon = 0$; that is,

$$\begin{pmatrix} 0 \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} (1 - \alpha(u)) X_1(0, y) + \alpha(u) Y_1(0, y) \\ (1 - \alpha(u)) X_2(0, y) + \alpha(u) Y_2(0, y) \end{pmatrix} \quad (19)$$

is called the *reduced problem* of the singularly perturbed system.

Comparing the singular perturbation problem (15) and (16) with the original Filippov system Z, we directly derive the following result.

Lemma 7. *Suppose $X_1(x, y) \neq Y_1(x, y)$. Then there is a diffeomorphism $h : \Sigma_s \to \mathscr{M}$, having the form $h(0, y) = (a(y), y)$, where $\alpha'(u) > 0$ at $u = a(y)$, that maps the flow of the sliding vector field Z_s to the flow of the reduced problem on $\mathscr{M}$.*

Remark 8. The slow manifold and the sliding segment of the boundary manifold are homeomorphic and carry the same dynamics but their geometry is different.

3. The Boundary-Saddle-Node Bifurcation

Here we first define the boundary-saddle-node (BSN) bifurcation.

Definition 9 (boundary-saddle-node bifurcation). Consider a planar Filippov system $Z = (X, Y, \Sigma)$ which smoothly depends on parameters $\mu \in \mathbb{R}^2$. In particular, for $\mu = \mu_0$ the vector field X has an equilibrium (x_0, y_0). Then the Filippov system Z undergoes a *boundary-saddle-node (BSN) bifurcation* at $\mu = \mu_0$ when the following conditions hold true:

(1) the equilibrium (x_0, y_0) is located on the boundary; that is, $f(x_0, y_0; \mu_0) = 0$;

(2) $D_x X_1(x_0, y_0; \mu_0) = 0$, $D_{xx} X_1(x_0, y_0; \mu_0) \neq 0$, and $D_\mu X_1(x_0, y_0; \mu_0) \neq 0$, where X_1 is the first component of X;

(3) $Y(x_0, y_0; \mu_0)$ is transversal to Σ_{μ_0} at (x_0, y_0); that is, $Yf(x_0, y_0; \mu_0) \neq 0$;

(4) $\delta(\mu_0) = D_{(x,y)} f(x_0, y_0; \mu_0)(D_{(x,y)} X(x_0, y_0; \mu_0))^{-1} Y(x_0, y_0; \mu_0) = 0$.

Remark 10. Condition (2) implies that the vector field X goes through a saddle-node bifurcation at $\mu = \mu_0$; condition (4) defines a degenerate situation for the boundary equilibrium (x_0, y_0) at $\mu = \mu_0$. From Theorem 3 we know that the system goes through a nonsmooth fold bifurcation for $\delta < 0$, while it goes through an equilibrium transition for $\delta > 0$.

Theorem 11 (topological normal form). *Consider a general Filippov system Z which undergoes a BSN bifurcation at μ_0. Then at the equilibrium point (x_0, y_0) there exist a local homeomorphism φ depending on parameters μ and a smooth invertible reparameterization $\mu \mapsto \lambda = (\rho, \nu)$ such that Z is topologically equivalent to the following system:*

$$Z(x, y; \lambda) = \begin{cases} X(x, y; \lambda), & \text{for } f(x, y; \lambda) < 0, \\ Y(x, y; \lambda), & \text{for } f(x, y; \lambda) > 0, \end{cases} \quad (20a)$$

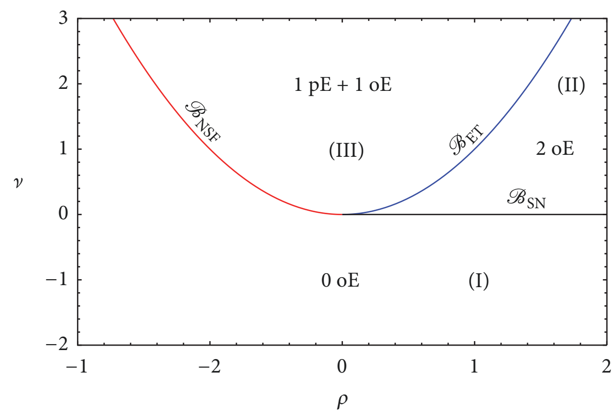
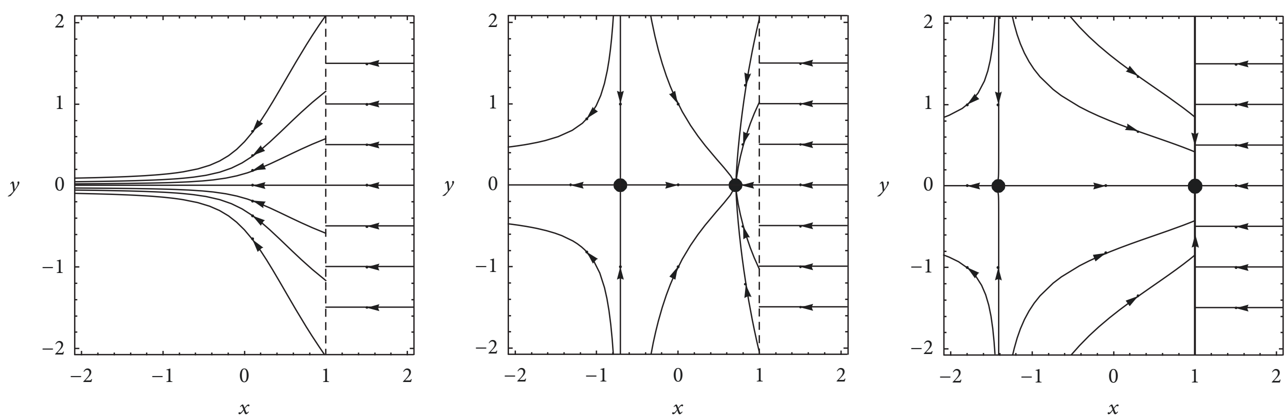


FIGURE 2: Bifurcation diagram of Z (20a), (20b), (20c), and (20d) in the (ρ, ν)-plane when $\delta = \sigma = -1$.



(a) $\rho = 1, \nu = -0.5$ (b) $\rho = 1, \nu = 0.5$ (c) $\rho = 1, \nu = 2$

FIGURE 3: From left to right phase portraits corresponding to the open regions I, II, and III in Figure 2, respectively.

with

$$X(x, y; \lambda) = \begin{pmatrix} \nu + \kappa x^2 \\ -y \end{pmatrix}, \tag{20b}$$

$$Y(x, y; \lambda) = \begin{pmatrix} \sigma \\ 0 \end{pmatrix}, \tag{20c}$$

$$f(x, y; \lambda) = x - \rho, \tag{20d}$$

where $\kappa = \pm 1$ *and* $\sigma = \operatorname{sgn}(Yf(x_0, y_0; \mu_0)) = \pm 1$. *All higher order terms in* X, Y, *and* f *smoothly depend on* λ.

Note that the vector filed X undergoes a saddle-node bifurcation at $\nu = 0$. The parameter ρ determines the passage of the equilibrium of X through the boundary Σ.

Proof. The method of proving this theorem is similar to [5]. Thus we omit it here. □

4. Bifurcation Diagram and Phase Portraits

System (20a), (20b), (20c), and (20d) undergoes 3 different codimension-1 bifurcations: saddle-node bifurcation, equilibrium transition, and nonsmooth fold bifurcation. Next we will only describe these three bifurcations in detail for the case $\kappa = \sigma = -1$. The other cases follow in the same way.

The vector field X undergoes a saddle-node bifurcation at $\nu = 0$. It has a saddle at $(-\sqrt{\nu}, 0)$ and a stable node at $(\sqrt{\nu}, 0)$ for $\nu > 0$, while it has no equilibrium for $\nu < 0$. The sliding set is given by

$$\Sigma_s = \left\{(x, y) : x = \rho, \rho^2 - \nu \leq 0\right\}. \tag{21}$$

Therefore, the sliding set is empty for $\nu < \rho^2$, while $\Sigma_s = \Sigma$ for $\nu \geq \rho^2$. According to (5), at the sliding set Σ_s, the sliding vector field Z_s is defined as

$$Z_s(\rho, y) = \frac{1}{\rho^2 - \nu - 1} \begin{pmatrix} 0 \\ y \end{pmatrix}. \tag{22}$$

It is direct to check that $(\rho, 0)$ is the unique attractive pseudoequilibrium.

Now we check that system (20a), (20b), (20c), and (20d) satisfies all the conditions of Theorem 3. For (20a), (20b), (20c), and (20d) it is direct to check that the equilibria

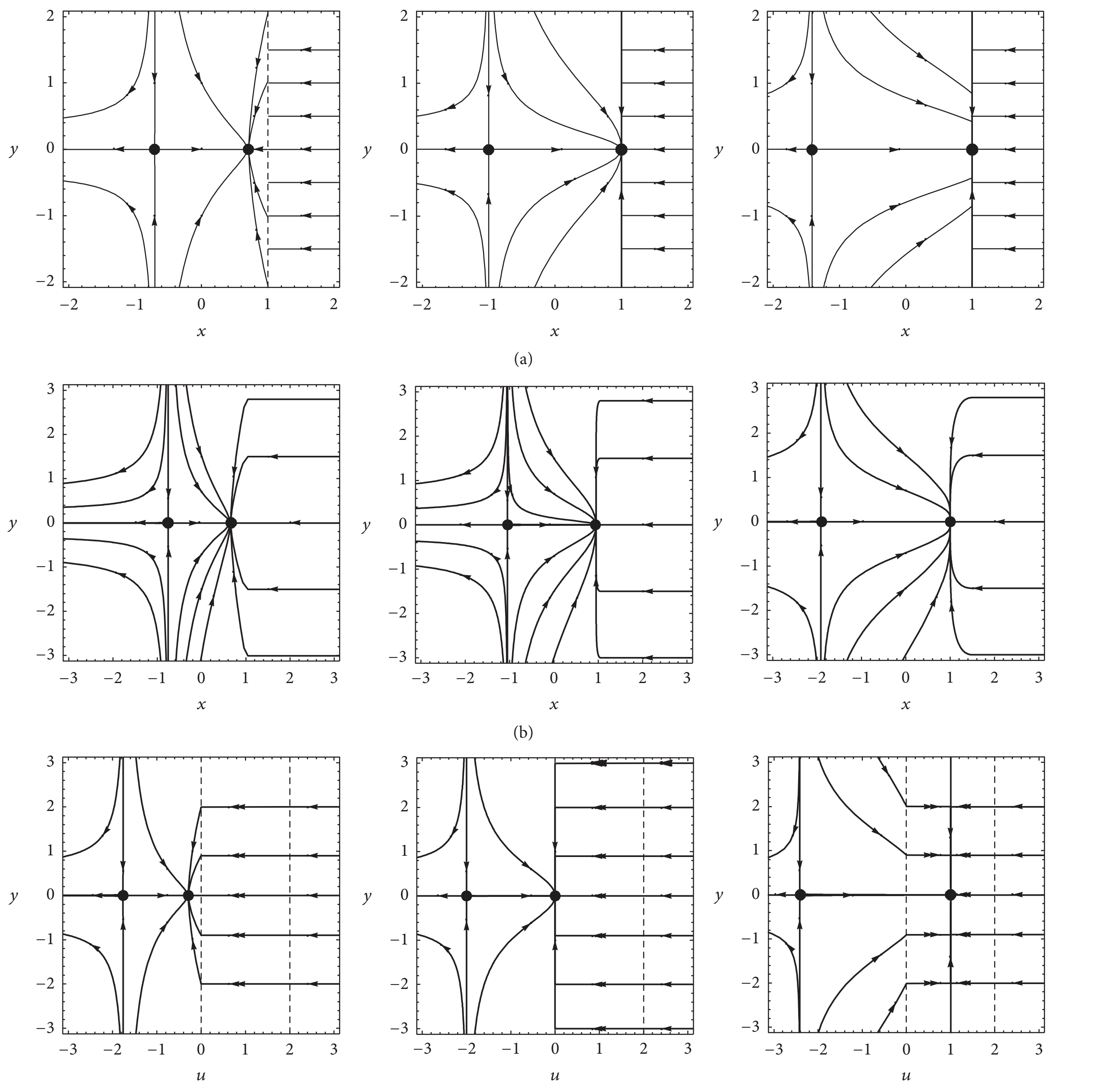


FIGURE 4: (a) Equilibrium transition between an ordinary node and a pseudonode. (b) The corresponding regularization. (c) The corresponding slow-fast flow.

$(\pm\sqrt{\nu}, 0)$ transversally meets Σ at $\rho = \pm\sqrt{\nu}$ or equivalently $\nu = \rho^2$. Moreover,

$$\begin{aligned}\delta(\lambda) &= D_{(x,y)} f(0,0,0)\left(D_{(x,y)} X(0,0;\lambda)\right)^{-1} Y(0,0;\lambda) \\ &= \pm\frac{1}{2\sqrt{\nu}}.\end{aligned} \tag{23}$$

By Theorem 3, this implies that the equilibrium $(\sqrt{\nu}, 0)$ goes through an equilibrium transition for $\rho > 0$, while the equilibrium $(-\sqrt{\nu}, 0)$ goes through a nonsmooth fold bifurcation for $\rho < 0$. Apart from these two boundary bifurcations, the line $\nu = 0$ for $\rho > 0$ in the parameter plane corresponds to a standard saddle-node bifurcation; see Figure 2. The phase portraits corresponding to the open regions I, II, and III labelled in Figure 2 are given in Figure 3.

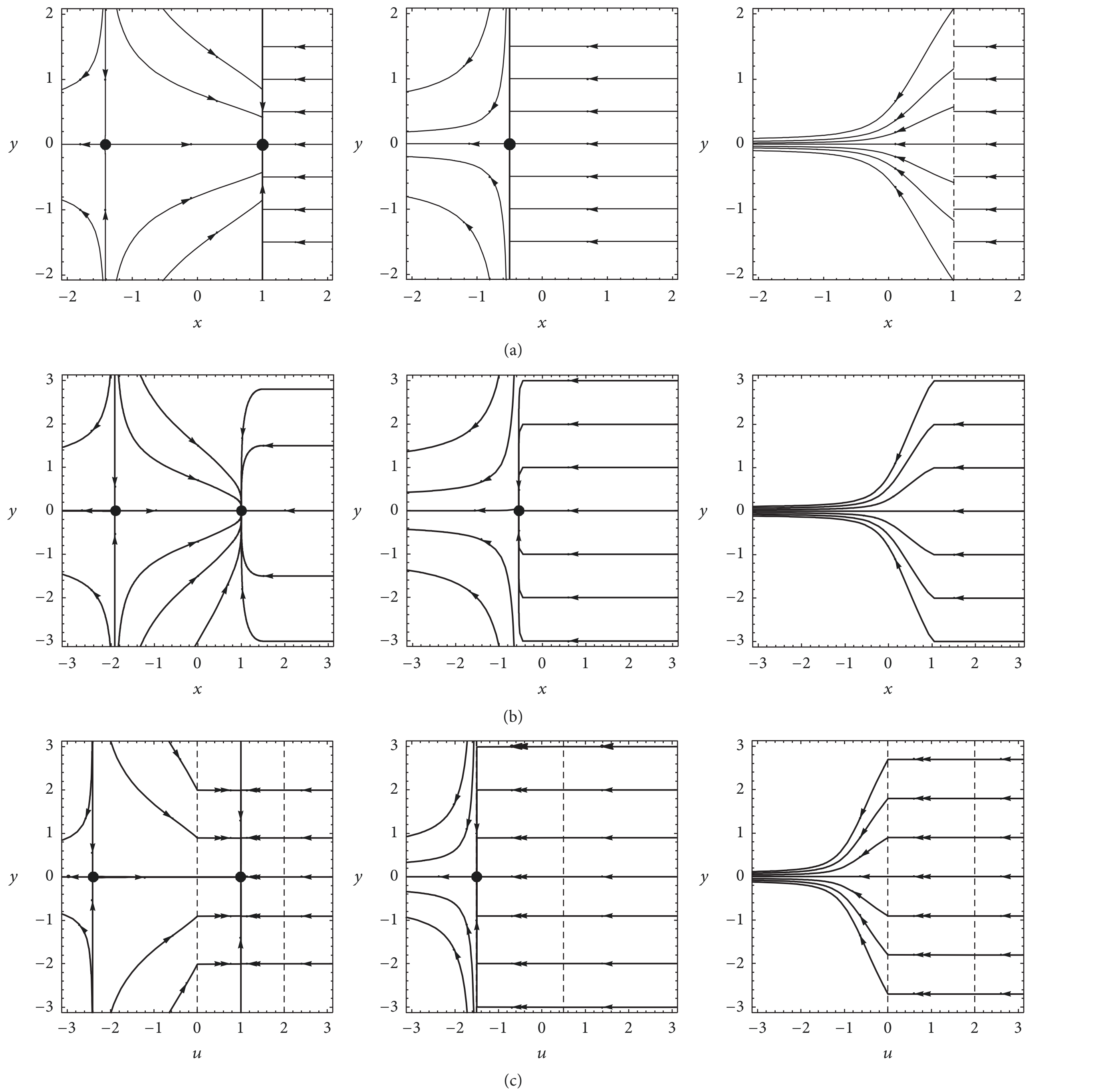


FIGURE 5: (a) Nonsmooth fold bifurcation in the case of boundary node. (b) The corresponding regularization. (c) The corresponding slow-fast flow.

5. Regularization of the BSN Bifurcation

In this section we apply the regularization approach (9) to the normal form of the BSN bifurcation and study the bifurcations of the regularized system.

Following (9) the regularization of (20a), (20b), (20c), and (20d) for $\varepsilon > 0$ is defined as follows:

$$R_{\varepsilon}(x, y) = \begin{cases} X(x+\varepsilon, y), & \text{for } x \le \rho - \varepsilon, \\ Y(x-\varepsilon, y), & \text{for } x \ge \rho + \varepsilon, \\ \left(1 - \alpha\left(\dfrac{x}{\varepsilon}\right)\right) X(\rho, y) + \alpha\left(\dfrac{x}{\varepsilon}\right) Y(\rho, y), & \text{for } \rho - \varepsilon \le x \le \rho + \varepsilon, \end{cases} \tag{24}$$

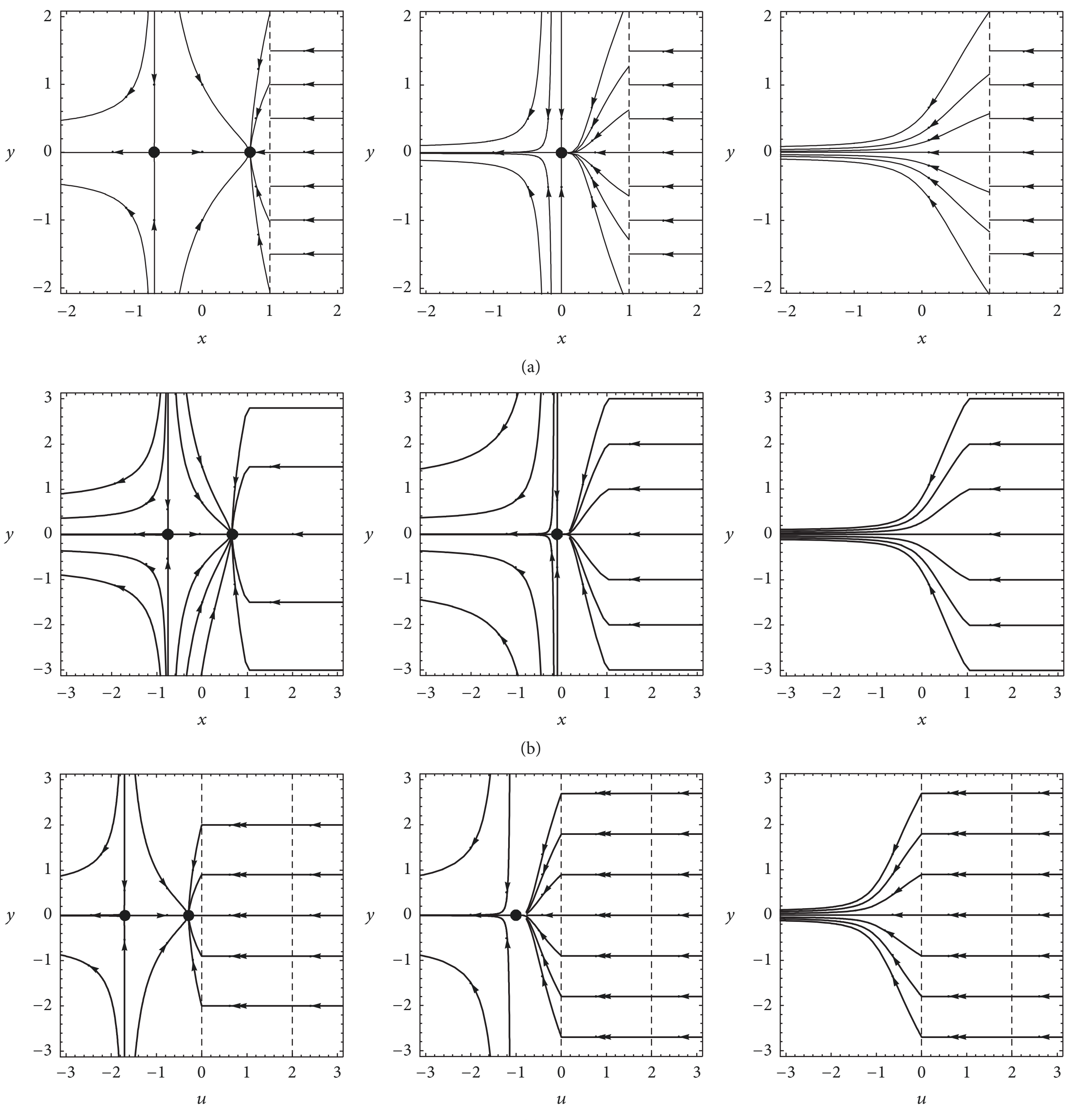


Figure 6: (a) A standard saddle-node bifurcation. (b) The corresponding regularization. (c) The corresponding slow-fast flow.

where the smooth function α is defined in $[\rho - \varepsilon, \rho + \varepsilon]$ with $\alpha(\rho - \varepsilon) = 0$, $\alpha(\rho + \varepsilon) = 1$. For our computations we have chosen $\alpha(x) = (x - \rho + 1)/2$.

Now we consider the bifurcations that occur in the regularized system (24). Before that we first define the following bifurcation for piecewise-smooth continuous system.

Saddle-Node-Like Bifurcation (SNL). A "saddle" and a "node" collide at a certain parameter value and then both disappear.

We note that the "saddle" and "node" in the definition are not restricted to a standard saddle and a standard node. They can also be pseudosaddle and pseudonode. The extension "like" means that these bifurcations are not the standard ones as known from smooth systems. A detailed description of bifurcations in piecewise-smooth continuous systems is referred to in [15].

5.1. Equilibrium Transition. Recall that $(\sqrt{\nu}, 0)$ is an ordinary stable node for $\nu < \rho^2$, $\rho > 0$. It hits Σ for $\nu = \rho^2$ and becomes

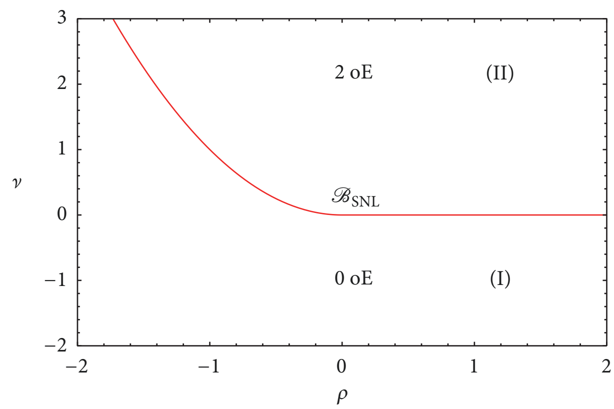


FIGURE 7: Bifurcation diagram of the regularization of system (20a), (20b), (20c), and (20d) in (ρ, ν) plane when $\delta = \sigma = -1$.

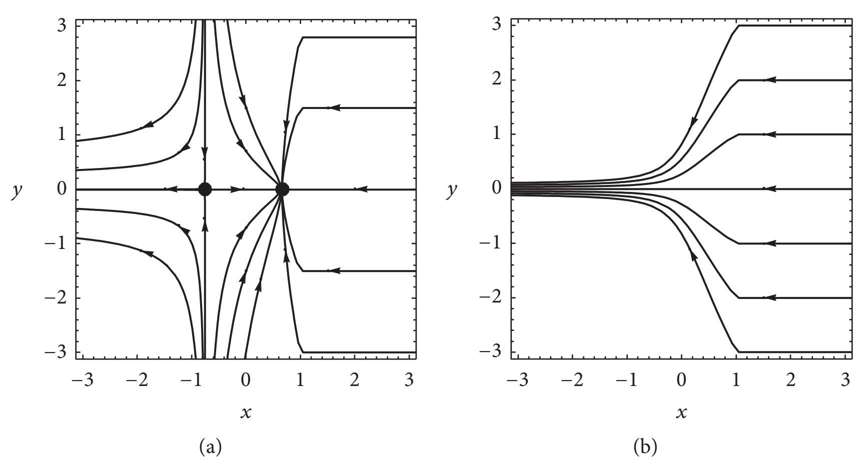


FIGURE 8: From left to right phase portraits corresponding to the open regions I and II in Figure 7, respectively.

a boundary node. Then it is replaced by a stable pseudonode on Σ for $\nu > \rho^2$. By Theorem 5, this equilibrium is still an equilibrium of R_ε after regularization with different position. It is located in M_X for $\nu < \rho^2$, on the line $x = -\varepsilon$ for $\nu = \rho^2$, and in the region $(-\varepsilon, \varepsilon)$ for $\nu > \rho^2$. Thus, the regularized system R_ε does not experience any qualitative change as ν varies. This type of discontinuity-induced bifurcation disappears after regularization, see Figure 4.

5.2. Nonsmooth Fold. Recall that for $\nu > \rho^2, \rho < 0$, an ordinary saddle $(-\sqrt{\nu}, 0)$ and a stable pseudonode $(\rho, 0)$ coexist, colliding on Σ for $\nu = \rho^2$ and then both disappear for $\nu < \rho^2, \rho < 0$. By Theorem 5 these two equilibria both preserve in their regularization R_ε with the same type of stability. To be precise, the saddle of R_ε is located in M_X, while the stable node is located in the region $(-\varepsilon, \varepsilon)$ for $\nu > \rho^2$. The unique equilibrium of R_ε is located on the line $x = -\varepsilon$ for $\nu = \rho^2$. However, there is no equilibrium in the regularized system for $\nu < \rho^2$. Notice that R_ε is a piecewise-smooth continuous system; then a saddle-node-like bifurcation occurs in R_ε; see Figure 5.

5.3. Standard Saddle-Node Bifurcation. Finally, the discontinuous system (20a), (20b), (20c), and (20d) has a standard saddle at $(-\sqrt{\nu}, 0)$ and a standard stable node at $(\sqrt{\nu}, 0)$ for any $\nu > 0$, colliding for $\nu = 0$, and then both disappear for $\nu < 0$. By Theorem 5 the equilibria for different ν persist after regularization. Thus a saddle-node-like bifurcation occurs in the regularized system R_ε; see Figure 6.

From the above results we find that the saddle-node-like bifurcation is the unique bifurcation occurring in the regularized system, which has codimension-1; see Figure 7. The phase portraits corresponding to the open regions I and II are presented in Figure 8.

6. Conclusions

This paper investigated the boundary-saddle-node (BSN) bifurcation by Filippov's convex method and the regularization approach. This is a codimension-2 bifurcation of Filippov systems. We derived the topological normal form of the BSN bifurcation and gave a detailed description of its bifurcations and phase portraits. After regularization the Filippov system becomes a piecewise-smooth continuous system. Some of the discontinuity-induced bifurcations disappear, such as equilibrium transition. The unique bifurcation that occurs in the regularized system is the saddle-node-like bifurcation. The BSN point becomes a codimension-1 bifurcation in the piecewise-smooth continuous system, which makes our discussion easier. The regularization approach enables us to make use of the established theories for continuous systems and slow-fast systems to study the local behavior around the BSN bifurcation. We will apply the regularization approach to more complicated Filippov systems involving limit cycles in the future work, where more advantages of our approach in proving the existence and uniqueness of limit cycles will be presented.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The author greatly appreciates the fruitful discussions with Ying Gao for her valuable suggestions. This research is supported by the Research Foundation of Education Bureau of Hebei Province under Grant no. QN2016026 and the Doctoral Starting up Foundation of Hebei GEO University under Grant no. B201612.

References

[1] P. Glendinning, P. Kowalczyk, and A. Nordmark, "Multiple attractors in grazing-sliding bifurcations in filippov-type flows," *IMA Journal of Applied Mathematics*, vol. 81, no. 4, pp. 711–722, 2016.

[2] P. Kowalczyk and M. di Bernardo, "Two-parameter degenerate sliding bifurcations in Filippov systems," *Physica D: Nonlinear Phenomena*, vol. 204, no. 3-4, pp. 204–229, 2005.

[3] R. I. Leine, *Bifurcations in discontinuous mechanical systems of Filippov's type [Ph.D. thesis]*, Technische Universiteit Eindhoven, Eindhoven, Netherlands, 2000.

[4] A. Nordmark, H. Dankowicz, and A. Champneys, "Discontinuity-induced bifurcations in systems with impacts and friction: Discontinuities in the impact law," *International Journal of Non-Linear Mechanics*, vol. 44, no. 10, pp. 1011–1023, 2009.

[5] F. Dercole, F. Della Rossa, A. Colombo, and Y. A. Kuznetsov, "Two degenerate boundary equilibrium bifurcations in planar Filippov systems," *SIAM Journal on Applied Dynamical Systems*, vol. 10, no. 4, pp. 1525–1553, 2011.

[6] F. Dercole, A. Gragnani, and S. Rinaldi, "Bifurcation analysis of piecewise smooth ecological models," *Theoretical Population Biology*, vol. 72, no. 2, pp. 197–213, 2007.

[7] J.-L. Gouze and T. Sari, "A class of piecewise linear differential equations arising in biological models," *Dynamical Systems*, vol. 17, no. 4, pp. 299–316, 2002.

[8] F. Angulo, G. Olivar, and M. di Bernardo, "Two-parameter discontinuity-induced bifurcation curves in a ZAD-strategy-conrolled DC-DC buck converter," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 55, no. 8, pp. 2392–2401, 2008.

[9] M. Di Bernardo, K. H. Johansson, and F. Vasca, "Self-oscillations and sliding in relay feedback systems: Symmetry and bifurcations," *International Journal of Bifurcation and Chaos*, vol. 11, no. 4, pp. 1121–1140, 2001.

[10] P. Kowalczyk and J. Sieber, "Robustness of Grazing-sliding Bifurcations in Hybrid Systems," *IFAC Proceedings Volumes*, vol. 42, no. 7, pp. 71–75, 2009.

[11] R. I. Leine and H. Nijmeijer, *Dynamics and Bifurcation of Non-Smooth Mechanical Systems*, vol. 18 of *Lecture Notes in Applied and Computational Mechanics*, Springer, Berlin, Germany, 2004.

[12] X. Liu, *The discontinuous Hopf-transversal system and its geometric regularization [Ph.D. thesis]*, University of Groningen, Groningen, Netherlands, 2013.

[13] J. Sotomayor and M. A. Teixeira, "Regularization of discontinuous vector fields," in *Proceedings of the International Conference on Differential Equations*, pp. 207–233, Lisboa, Portugal, 1996.

[14] K. Efstathiou, X. Liu, and H. W. Broer, "The boundary-Hopf-fold bifurcation in Filippov systems," *SIAM Journal on Applied Dynamical Systems*, vol. 14, no. 2, pp. 914–941, 2015.

[15] D. J. W. Simpson, *Bifurcations in Piecewise-Smooth Continuous Systems*, vol. 70 of *Nonlinear Science*, World Scientific, 2010.

[16] A. F. Filippov, *Differential Equations with Discontinuous Right-hand Sides*, Kluwer Academic, 1988.

[17] M. di Bernardo, D. J. Pagano, and E. Ponce, "Nonhyperbolic boundary equilibrium bifurcations in planar Filippov systems: a case study approach," *International Journal of Bifurcation and Chaos*, vol. 18, no. 5, pp. 1377–1392, 2008.

[18] M. Guardia, T. M. Seara, and M. A. Teixeira, "Generic bifurcations of low codimension of planar Filippov systems," *Journal of Differential Equations*, vol. 250, no. 4, pp. 1967–2023, 2011.

[19] N. Fenichel, "Geometric singular perturbation theory for ordinary differential equations," *Journal of Differential Equations*, vol. 31, no. 1, pp. 53–98, 1979.