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Research Article

Dynamic Analysis of Complex Synchronization Schemes between Integer Order and Fractional Order Chaotic Systems with Different Dimensions

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We present new approaches to synchronize different dimensional master and slave systems described by integer order and fractional order differential equations. Based on fractional order Lyapunov approach and integer order Lyapunov stability method, effective control schemes to rigorously study the coexistence of some synchronization types between integer order and fractional order chaotic systems with different dimensions are introduced. Numerical examples are used to validate the theoretical results and to verify the effectiveness of the proposed schemes.

1. Introduction

Nature is intrinsically nonlinear. So, it is not surprising that most of the systems we encounter in the real world are nonlinear. And what is interesting is that some of these nonlinear systems can be described by fractional order differential equations which can display a variety of behaviors including chaos and hyperchaos [1–5]. Recently, study on synchronization of fractional order chaotic systems has started to attract increasing attention of many researchers [6–12], since the synchronization of chaotic systems with integer order is understood well and widely explored [13–15]. Many scientists who are interested in the field of chaos synchronization have struggled to achieve the synchronization between integer order and fractional order chaotic systems.

At present, many schemes of control have been proposed to study the problem of synchronization between integer order and fractional order chaotic systems such as anticipating synchronization [16], function projective synchronization [17], complete synchronization [18], antisynchronization [19], Q-S synchronization [20], and generalized synchronization [21]. Also, different techniques have been introduced to synchronize integer order and fractional order chaotic systems. For example, a nonlinear feedback control method has been introduced in [22]. The idea of tracking control has been applied in [23, 24]. In [25], general control scheme has been described. A new fuzzy sliding mode method has been proposed in [26], and a sliding mode method has been designed in [27, 28]. Synchronization of a class of hyperchaotic systems has been studied in [29]. A practical method, based on circuit simulation, has been presented in [30], and in [31] a robust observer technique has been tackled.

Complete synchronization (CS), projective synchronization (PS), full state hybrid function projective synchronization (FSHFPS), and generalized synchronization (GS) are effective approaches to achieve synchronization and have been used widely in integer order chaotic systems [32–35] and fractional order chaotic systems [36–39]. Studying inverse problems of synchronization is an attractive and important idea. Recently, some interesting types of synchronization have been introduced such as inverse projective synchronization (IPS) [40], inverse full state hybrid projective synchronization (IFSHPS) [41], inverse full state hybrid function projective synchronization (IFSHFPS) [42], and inverse generalized synchronization (IGS) [43]. Coexistence of several types of synchronization produces new complex type of chaos synchronization. Not long ago, many approaches for the problem of coexistence of synchronization types have been proposed in discrete time chaotic systems, integer order chaotic systems, and fractional order chaotic systems [44–47]. The coexistence of different type of synchronization is very useful in secure communication and chaotic encryption schemes.

This paper introduces new approaches to study the coexistence of some types of synchronization between integer order and fractional order chaotic systems with different dimensions. The new results, derived in this paper, are established in the form of simple conditions about the linear parts of the slave system and the master system, respectively, which are very convenient to verify. Using fractional Lyapunov approach, the coexistence of complete synchronization (CS), projective synchronization (PS), full state hybrid function projective synchronization (FSHFPS), and generalized synchronization (GS) between integer order master system and fractional order slave system is investigated. Based on integer order Lyapunov method, the coexistence of inverse projective synchronization (IPS), inverse full state hybrid function projective synchronization (IFSHFPS), and inverse generalized synchronization (IGS) between fractional order master system and integer order slave system is studied. The capability of the approaches is illustrated by numerical examples.

The rest of this paper is arranged as follows. Some theoretical bases of fractional calculus are introduced in Section 2. In Section 3, our main results of the paper are presented. In Section 4, our approaches are applied between some typical chaotic and hyperchaotic systems to show the effectiveness of the derived results. Section 5 is the conclusion of the paper.

2. Theoretical Basis

2.1. Fractional Integration and Derivative. There are several definitions of fractional derivatives [48, 49]. The two most commonly used are the Riemann-Liouville and Caputo definitions. Each definition uses Riemann-Liouville fractional integration and derivatives of whole order. The difference between the two definitions is in the order of evaluation. The Caputo derivative is a time domain computation method [50]. In real applications, the Caputo derivative is more popular since the unhomogenous initial conditions are permitted if such conditions are necessary [51, 52]. Furthermore, these initial values are prone to measure since they all have idiographic meanings. The Riemann-Liouville fractional integral operator of order $q \geq 0$ of the function $f(t)$ is defined as

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau)\, d\tau, \quad q > 0,\ t > 0. \quad (1)$$

Some properties of the operator J^q can be found, for example, in [53]. In this study, Caputo definition is used and the fractional derivative of $f(t)$ is defined as

$$\begin{aligned} D_t^p f(t) &= J^{m-p}\left(\frac{d^m}{dt^m} f(t)\right) \\ &= \frac{1}{\Gamma(m-p)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{p-m+1}}\, d\tau, \end{aligned} \quad (2)$$

for $m-1 < p \leq m$, $m \in \mathbb{N}$, $t > 0$. The fractional differential operator D_t^p is left-inverse (and not right-inverse) to the fractional integral operator J^p; that is, $D_t^p J^p = I$, where I is the identity operator.

2.2. Lyapunov Stability for Integer Order Systems. Consider the following integer order system:

$$\dot{X}(t) = F(X(t)), \quad (3)$$

where $X(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ and $F : \mathbf{R}^n \to \mathbf{R}^n$.

Lemma 1 (see [54]). *If there exists a positive definite Lyapunov function $V(X(t))$ such that $\dot{V}(X(t)) < 0$, for all $t > 0$, then the trivial solution of system (3) is asymptotically stable.*

2.3. Fractional Order Lyapunov Stability. Consider the following fractional order system:

$$D_t^p X(t) = F(X(t)), \quad (4)$$

where $X(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, p is a rational number between 0 and 1 and D_t^p is the Caputo fractional derivative of order p, and $F : \mathbf{R}^n \to \mathbf{R}^n$. For stability analysis of fractional order systems, a fractional extension of Lyapunov direct method has been proposed by the following theorem.

Theorem 2 (see [55]). *If there exists a positive definite Lyapunov function $V(X(t))$ such that $D_t^p V(X(t)) < 0$, for all $t > 0$, then the trivial solution of system (4) is asymptotically stable.*

Now, we present a new lemma which is helpful in the proving analysis of the proposed method.

Lemma 3 (see [56]). $\forall t > 0 : (1/2) D_t^p (X^T(t) X(t)) \leq X^T(t) D_t^p (X(t))$.

3. Main Results

3.1. Synchronization of Integer Oder Master System and Fractional Oder Slave System. The master system is defined by

$$\dot{x}_i(t) = f_i(X(t)), \quad i = 1, 2, 3, \quad (5)$$

where $X(t) = (x_1(t), x_2(t), x_3(t))^T$ is the state vector of the master system and $f_i : \mathbf{R}^3 \to \mathbf{R}$ $(i = 1, 2, 3)$. We consider the slave system as

$$D_t^q y_i(t) = \sum_{j=1}^{4} b_{ij} y_j(t) + g_i(Y(t)) + u_i, \quad i = 1, 2, 3, 4, \quad (6)$$

where $Y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))^T$ is the state vector of the slave system, $(b_{ij}) \in \mathbf{R}^{4\times 4}$, $g_i : \mathbf{R}^4 \to \mathbf{R}$ $(i = 1,2,3,4)$ are nonlinear functions, $0 < q < 1$, and D_t^q is the Caputo fractional derivative of order q, and u_i $(i = 1,2,3,4)$ are controllers.

Definition 4. We say that CS, PS, FSHFPS, and GS coexist in the synchronization of master system (5) and slave system (6), if there exist controllers u_i $(i = 1,2,3,4)$ constant $\alpha \in \mathbf{R}^* - \{1\}$, differentiable functions $\beta_j(t) : \mathbf{R}^+ \to \mathbf{R}$ $(j = 1,2,3)$ and differentiable function $\varphi : \mathbf{R}^3 \to \mathbf{R}$, such that the synchronization errors

$$\begin{aligned} e_1(t) &= y_1(t) - x_1(t), \\ e_2(t) &= y_2(t) - \alpha x_2(t), \\ e_3(t) &= y_3(t) - \sum_{j=1}^{3} \beta_j(t) x_j(t), \\ e_4(t) &= y_4(t) - \varphi(X(t)) \end{aligned} \tag{7}$$

satisfy that $\lim_{t\to+\infty} e_i(t) = 0$, $i = 1,2,3,4$.

Error system (7), between master system (5) and slave system (6), can be differentiated as follows:

$$\begin{aligned} D_t^q e_1(t) &= \sum_{j=1}^{4} b_{1j} y_j(t) + g_1(Y(t)) + u_1 - D_t^q x_1(t), \\ D_t^q e_2(t) &= \sum_{j=1}^{4} b_{2j} y_j(t) + g_2(Y(t)) + u_2 - \alpha D_t^q x_2(t), \\ D_t^q e_3(t) &= \sum_{j=1}^{4} b_{3j} y_j(t) + g_3(Y(t)) + u_3 \\ &\quad - D_t^q \left[\sum_{j=1}^{3} \beta_j(t) x_j(t) \right], \\ D_t^q e_3(t) &= \sum_{j=1}^{4} b_{4j} y_j(t) + g_4(Y(t)) + u_4 \\ &\quad - D_t^q [\varphi(X(t))]. \end{aligned} \tag{8}$$

Error system (8) can be described as

$$D_t^q e_i(t) = \sum_{j=1}^{4} b_{ij} e_j(t) + R_i + u_i, \quad i = 1,2,3,4, \tag{9}$$

where

$$\begin{aligned} R_1 &= \sum_{j=1}^{4} b_{1j} \left(y_j(t) - e_j(t) \right) + g_1(Y(t)) - D_t^q x_1(t), \\ R_2 &= \sum_{j=1}^{4} b_{2j} \left(y_j(t) - e_j(t) \right) + g_2(Y(t)) - \alpha D_t^q x_2(t), \\ R_3 &= \sum_{j=1}^{4} b_{3j} \left(y_j(t) - e_j(t) \right) + g_3(Y(t)) \\ &\quad - D_t^q \left[\sum_{j=1}^{3} \beta_j(t) x_j(t) \right], \\ R_4 &= \sum_{j=1}^{4} b_{4j} \left(y_j(t) - e_j(t) \right) + g_4(Y(t)) \\ &\quad - D_t^q [\varphi(X(t))]. \end{aligned} \tag{10}$$

Rewrite error system (9) in the compact form

$$D_t^q e(t) = Be(t) + R + U, \tag{11}$$

where $D_t^q e(t) = [D_t^q e_1(t), D_t^q e_2(t), D_t^q e_3(t), D_t^q e_4(t)]^T$, $B = (b_{ij})_{4\times 4}$, $R = (R_i)_{1\le i\le 4}$, and $U = (u_i)_{1\le i\le 4}$.

Theorem 5. *There exists a suitable feedback gain matrix $C \in \mathbf{R}^{4\times 4}$ to realize the coexistence of CS, PS, FSHFPS, and GS between master system (5) and slave system (6) under the following control law:*

$$U = -R - Ce(t). \tag{12}$$

Proof. Substituting (12) into (11), one can have

$$D_t^q e(t) = (B - C) e(t). \tag{13}$$

If a Lyapunov function candidate is chosen as $V(e(t)) = (1/2)e^T(t)e(t)$, then, the time Caputo fractional derivative of V along the trajectory of system (13) is as follows:

$$D_t^q V(e(t)) = D_t^q \left(\frac{1}{2} e^T(t) e(t) \right), \tag{14}$$

and using Lemma 3 in (14) we get

$$D_t^q V(e(t)) \le e^T(t) D_t^q e(t) = e^T(t)(B - C) e(t). \tag{15}$$

If we select the feedback gain matrix C such that $B - C$ is a negative definite matrix, then we get $D_t^q V(e(t)) < 0$. From Theorem 2, the zero solution of system (13) is a globally asymptotically stable; that is, $\lim_{t\to+\infty} e_i(t) = 0$, $i = 1,2,3,4$. We conclude that master system (5) and slave system (6) are globally synchronized. □

3.2. Synchronization of Fractional Oder Master System and Integer Oder Slave System. Now, the master system and the slave system can be described in the following forms:

$$D_t^{p_i} x_i(t) = \sum_{i=1}^{3} a_{ij} x_j(t) + f_i(X(t)), \quad i = 1,2,3, \tag{16}$$

$$\dot{y}_i(t) = g_i(Y(t)) + u_i, \quad i = 1,2,3,4, \tag{17}$$

where $X(t) = (x_i(t))_{1\le i\le 3}$, $Y(t) = (y_i(t))_{1\le i\le 4}$ are the states of the master system and the slave system, respectively, $D_t^{p_i}$ is the Caputo fractional derivative of order p_i, $0 < p_i < 1$ $(i = 1,2,3)$, and $(a_{ij}) \in \mathbf{R}^{3\times 3}$, $f_i : \mathbf{R}^3 \to \mathbf{R}$ $(i = 1,2,3)$, are nonlinear functions; for example, $g_i : \mathbf{R}^4 \to \mathbf{R}$ $(i = 1,2,3,4)$ and u_i $(i = 1,2,3,4)$ are controllers.

Definition 6. We say that IPS, IFSHFPS, and IGS coexist in the synchronization of master system (16) and slave system (17); if there exist controllers u_i $(i = 1,2,3,4)$ differentiable function $h(t) : \mathbf{R}^+ \rightarrow \mathbf{R}^*$, differentiable functions $\Lambda_j(t) : \mathbf{R}^+ \rightarrow \mathbf{R}$ $(j = 1,2,3,4)$ and differentiable function $\phi : \mathbf{R}^4 \rightarrow \mathbf{R}$, such that the synchronization errors

$$\begin{aligned} e_1(t) &= x_1(t) - h(t)\, y_1(t), \\ e_2(t) &= x_2(t) - \sum_{j=1}^{4} \Lambda_j(t)\, y_j(t), \\ e_3(t) &= x_3(t) - \phi(Y(t)) \end{aligned} \tag{18}$$

satisfy that $\lim_{t\rightarrow+\infty} e_i(t) = 0,\ i = 1,2,3$.

Error system (18), between master system (16) and slave system (17), can be derived as

$$\begin{aligned} \dot{e}_1(t) &= \dot{x}_1(t) - \dot{h}(t)\, y_1(t) - h(t)\, \dot{y}_1(t), \\ \dot{e}_2(t) &= \dot{x}_2(t) - \sum_{j=1}^{4} \dot{\Lambda}_j(t)\, y_j(t) - \sum_{j=1}^{4} \Lambda_j(t)\, \dot{y}_j(t), \\ \dot{e}_3(t) &= \dot{x}_3(t) - \sum_{j=1}^{4} \frac{\partial\phi}{\partial y_j} \dot{y}_j. \end{aligned} \tag{19}$$

Error system (19) can be written as

$$\begin{aligned} \dot{e}_1(t) &= \sum_{j=1}^{3} a_{1j} e_j(t) + T_1 - h(t)\, u_1, \\ \dot{e}_2(t) &= \sum_{j=1}^{3} a_{2j} e_j(t) + T_2 - \sum_{j=1}^{4} \Lambda_j(t)\, u_j, \\ \dot{e}_3(t) &= \sum_{j=1}^{3} a_{3j} e_j(t) + T_3 - \sum_{j=1}^{4} \frac{\partial\phi}{\partial y_j} u_j, \end{aligned} \tag{20}$$

where

$$\begin{aligned} T_1 &= \dot{x}_1(t) - \sum_{j=1}^{3} a_{1j} e_j(t) - \dot{h}(t)\, y_1(t) \\ &\quad - h(t)\, g_1(Y(t)), \\ T_2 &= \dot{x}_2(t) - \sum_{j=1}^{3} a_{2j} e_j(t) - \sum_{j=1}^{4} \dot{\Lambda}_j(t)\, y_j(t) \\ &\quad - \sum_{j=1}^{4} \Lambda_j(t)\, g_j(Y(t)), \\ T_3 &= \dot{x}_3(t) - \sum_{j=1}^{3} a_{3j} e_j(t) - \sum_{j=1}^{4} \frac{\partial\phi}{\partial y_j} g_j(Y(t)). \end{aligned} \tag{21}$$

Rewrite error system (20) in the compact form

$$\dot{e}(t) = Ae(t) + T - M \times \widehat{U} - u_4 W, \tag{22}$$

where $\dot{e}(t) = [\dot{e}_1(t), \dot{e}_2(t), \dot{e}_3(t)]^T$, $A = (a_{ij})_{3\times3}$, $T = (T_i)_{1\leq i\leq3}$, $\widehat{U} = (u_i)_{1\leq i\leq3}$, $W = (0, \Lambda_4(t), \partial\phi/\partial y_4)^T$, and

$$M = \begin{pmatrix} h(t) & 0 & 0 \\ \Lambda_1(t) & \Lambda_2(t) & \Lambda_3(t) \\ \dfrac{\partial\phi}{\partial y_1} & \dfrac{\partial\phi}{\partial y_2} & \dfrac{\partial\phi}{\partial y_3} \end{pmatrix}. \tag{23}$$

To achieve synchronization between systems (16) and (17), we assume that M is an invertible matrix and M^{-1} its inverse matrix. Hence, we have the following result.

Theorem 7. *IPS, IFSHFPS, and IGS coexist between master system (16) and slave system (17) under the following conditions:*

(i) *M^{-1} is bounded.*

(ii) *$\widehat{U} = M^{-1}(Le(t) + T)$ and $u_4 = 0$.*

(iii) *The feedback gain matrix $L \in \mathbf{R}^{3\times3}$ is selected such that $(A - L)^T + (A - L)$ is a negative definite matrix.*

Proof. By substituting the control law (ii) into (22), the error system can be written as

$$\dot{e}(t) = (A - L)\, e(t). \tag{24}$$

Construct the candidate Lyapunov function in the form $V(e(t)) = e^T(t)e(t)$, and we obtain

$$\begin{aligned} \dot{V}(e(t)) &= \dot{e}^T(t)\, e(t) + e^T(t)\, \dot{e}(t) \\ &= e^T(t)\,(A - L)^T e(t) + e^T(t)\,(A - L)\, e(t) \\ &= e^T(t) \left[(A - L)^T + (A - L)\right] e(t). \end{aligned} \tag{25}$$

Using (iii), then we get $\dot{V}(e(t)) < 0$. From Lemma 1, the zero solution of error system (24) is globally asymptotically stable and therefore, master system (16) the slave system (17) are globally synchronized. □

4. Numerical Examples

In this section, two numerical examples are considered to validate the proposed chaos synchronization schemes.

4.1. Example 1. Her, in this example, as the master system we consider the chaotic Chen system [57] and the controlled fractional hyperchaotic Liu system [58] as the slave system. The master system is defined as

$$\begin{aligned} \dot{x}_1 &= \alpha(x_2 - x_1), \\ \dot{x}_2 &= (\gamma - \alpha)\, x_1 + \gamma x_2 - x_1 x_3, \\ \dot{x}_3 &= \beta x_3 + x_1 x_2, \end{aligned} \tag{26}$$

where $x_i, i = 1,2,3$, are the states of the master system. System (26) has a chaotic behavior when $\alpha = 35$, $\beta = -3$, and $\gamma = 28$. Chaotic attractors of master system (26) are shown in Figure 1.

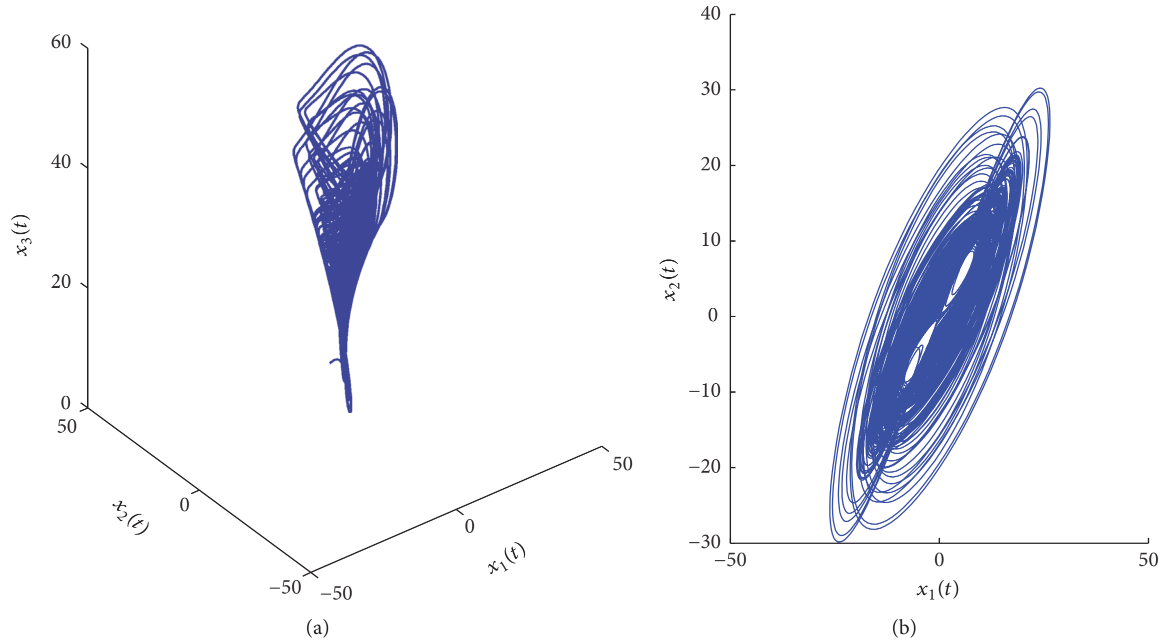

Figure 1: 3D and 2D chaotic attractors of Chen system (26).

The slave system is described by

$$\begin{aligned}
D^q y_1 &= b_1 \left(y_2 - y_1\right) + y_4 + u_1, \\
D^q y_2 &= b_2 y_1 + 0.5 y_4 - y_1 y_3 + u_2, \\
D^q y_3 &= -b_3 y_3 - y_4 + 4 y_1^2 + u_3, \\
D^q y_4 &= -b_4 y_2 - y_4 + u_4,
\end{aligned} \tag{27}$$

where y_i, $i = 1, 2, 3, 4$, are the states of the slave system and $U = (u_i)_{1 \le i \le 4}$ is the vector controller. This system, as shown in [58], exhibits hyperchaotic behavior when $(u_1, u_2, u_3, u_4) = (0, 0, 0, 0)$, $(b_1, b_2, b_3, b_4) = (10, 40, 2.5, 10/15)$, and $q = 0.9$. The projections of the hyperchaotic Lorenz attractor are shown in Figure 2.

Comparing system (27) with system (6), we get

$$B = \left(b_{ij}\right)_{4\times4} = \begin{pmatrix} -10 & 10 & 0 & 1 \\ 40 & 0 & 0 & 0.5 \\ 0 & 0 & -2.5 & -1 \\ 0 & -\dfrac{10}{15} & 0 & -1 \end{pmatrix},$$

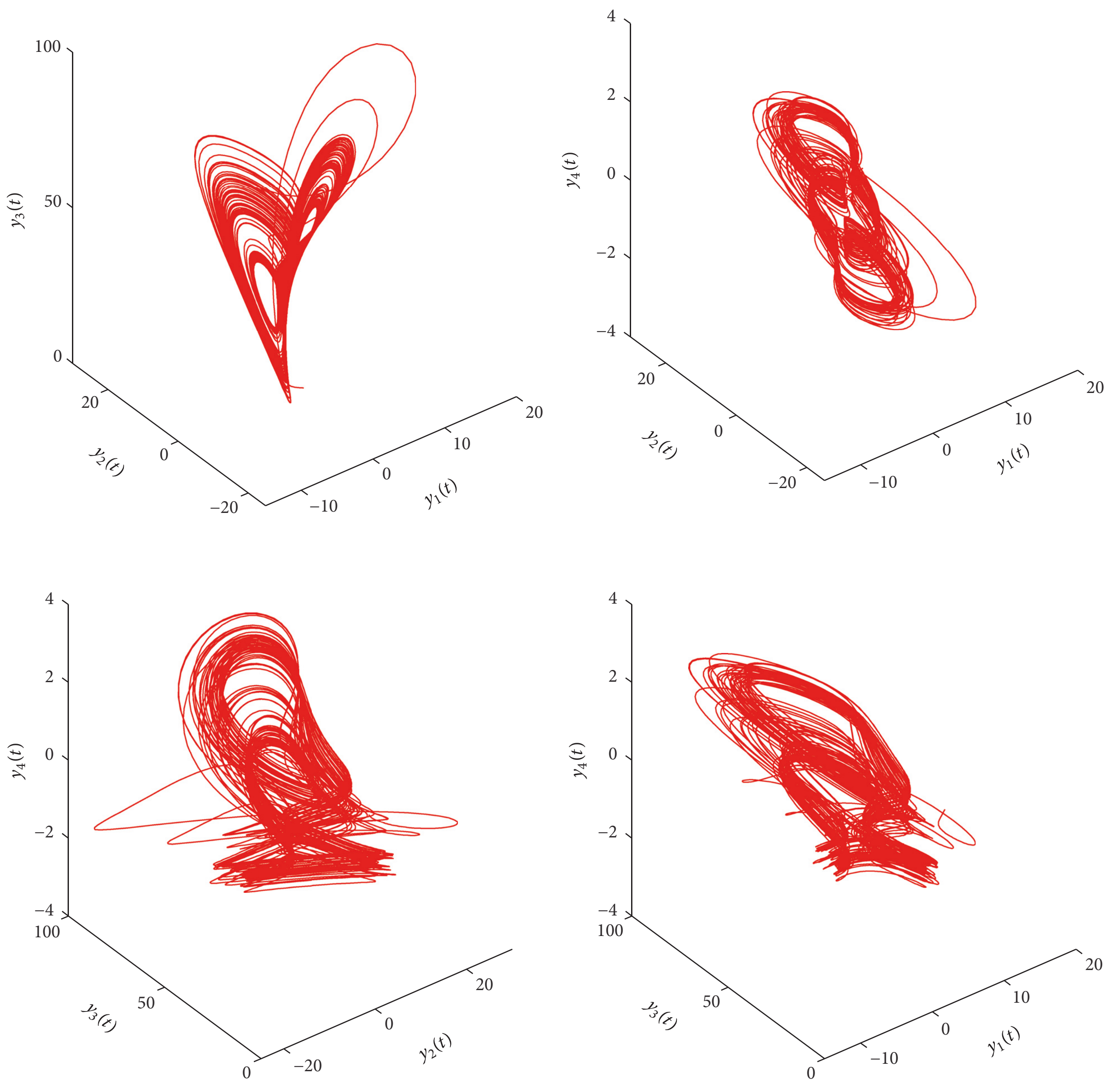

Figure 2: 3D phase portraits of the fractional hyperchaotic Liu system.

$$(g_i)_{1\le i\le 4} = \begin{pmatrix} 0 \\ -y_1 y_3 \\ 4y_1^2 \\ 0 \end{pmatrix}. \tag{28}$$

Using the notations described in Section 3.1, the errors between master system (26) and slave system (27) are defined as

$$\begin{aligned} e_1 &= y_1 - x_1, \\ e_2 &= y_2 - \alpha x_2, \\ e_3 &= y_3 - \sum_{j=1}^{3} \beta_j(t)\, x_j, \\ e_4 &= y_4 - \varphi(x_1, x_2, x_3), \end{aligned} \tag{29}$$

where $\alpha = 2$, $\beta_1(t) = t$, $\beta_2(t) = t^2$, $\beta_3(t) = t^3$, and $\varphi(x_1, x_2, x_3) = x_1 x_2 x_3$. According to Theorem 5, the control law $(u_1, u_2, u_3, u_4)^T$ can be designed as

$$(u_1, u_2, u_3, u_4)^T = -R - C \times (e_1, e_2, e_3, e_4)^T, \tag{30}$$

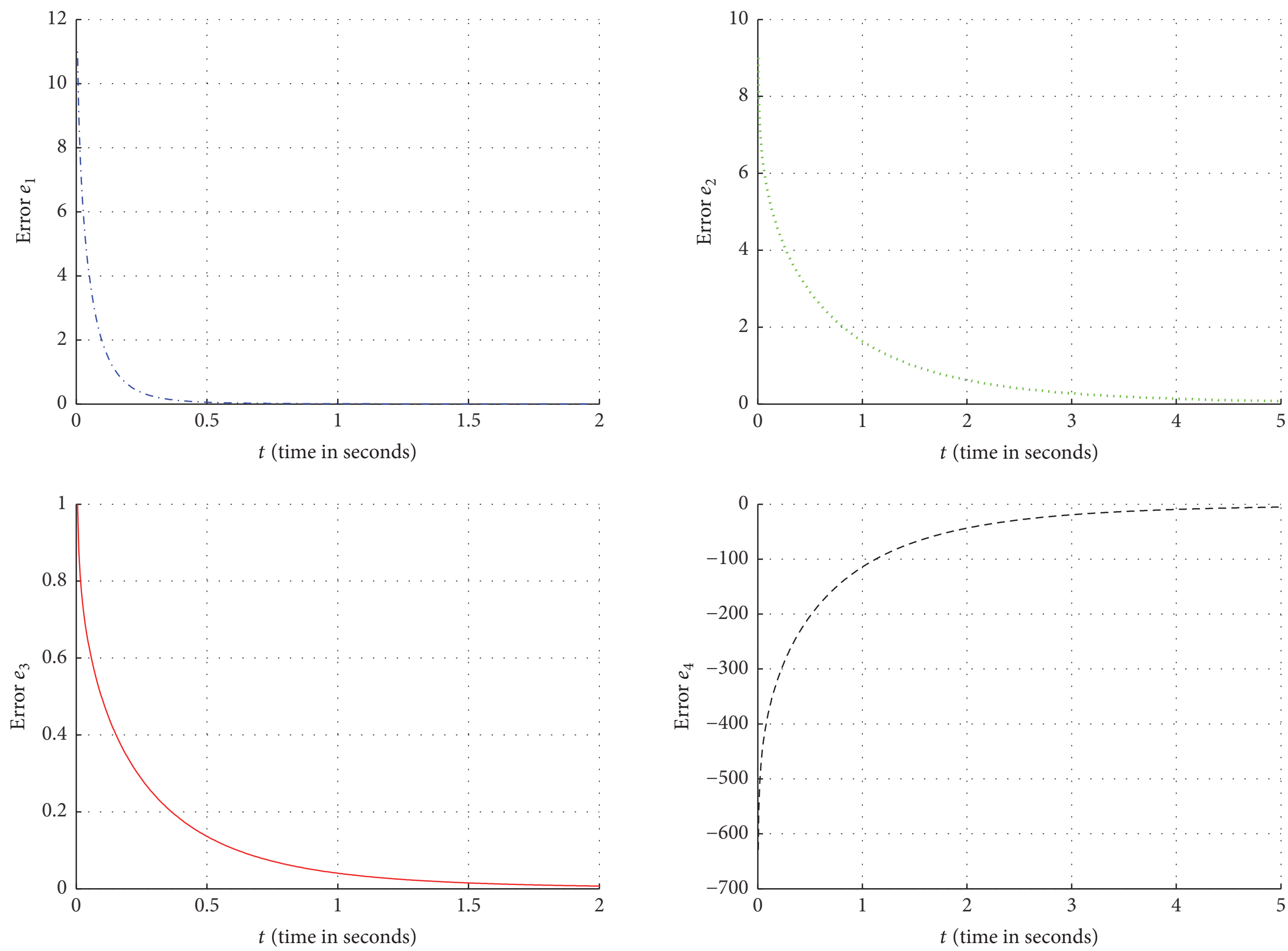

FIGURE 3: Time evolution of the synchronization errors e_1, e_2, e_3, and e_4 between master system (26) and slave system (27).

where

$$\begin{aligned} R_1 &= -10(y_1 - e_1) + 10(y_2 - e_2) + y_4 - e_4 - D^{0.9}x_1, \\ R_2 &= 40(y_1 - e_1) + 0.5(y_4 - e_4) - y_1 y_3 \\ &\quad - 2D^{0.9}x_2(t), \\ R_3 &= -2.5(y_3 - e_3) - (y_4 - e_4) + 4y_1^2 \\ &\quad - D^{0.9}(tx_1 + t^2 x_2 + t^3 x_3), \\ R_4 &= -\frac{10}{15}(y_2 - e_2) - (y_4 - e_4) - D^{0.9}(x_1 x_2 x_3), \end{aligned} \tag{31}$$

and C is a feedback gain matrix selected as

$$C = \begin{pmatrix} 0 & 10 & 0 & 1 \\ 40 & 1 & 0 & 0.5 \\ 0 & 0 & 0 & -1 \\ 0 & -\frac{10}{15} & 0 & 0 \end{pmatrix}. \tag{32}$$

It is easy to show that $B - C$ is a negative definite matrix. Hence, the synchronization between master system (26) and slave system (27) is achieved and the error system can be described as follows:

$$\begin{aligned} D^{0.9}e_1 &= -10e_1, \\ D^{0.9}e_2 &= -e_2, \\ D^{0.9}e_3 &= -2.5e_3, \\ D^{0.9}e_4 &= -e_4. \end{aligned} \tag{33}$$

For the purpose of numerical simulation, the fractional Euler integration method has been used to solve system (33). The initial values of the master and the slave systems are $[x_1(0), x_2(0), x_3(0)] = [-9, -5, 14]$ and $[y_1(0), y_2(0), y_3(0), y_4(0)] = [2, -1, 1, 1]$, respectively, and the initial states of error system (33) are $[e_1(0), e_2(0), e_3(0), e_4(0)] = [11, 9, 1, -629]$. Figure 3 displays the synchronization errors between systems (26) and (27).

4.2. Example 2. Now, in this example, as the master system we consider the fractional Lü system and the controlled

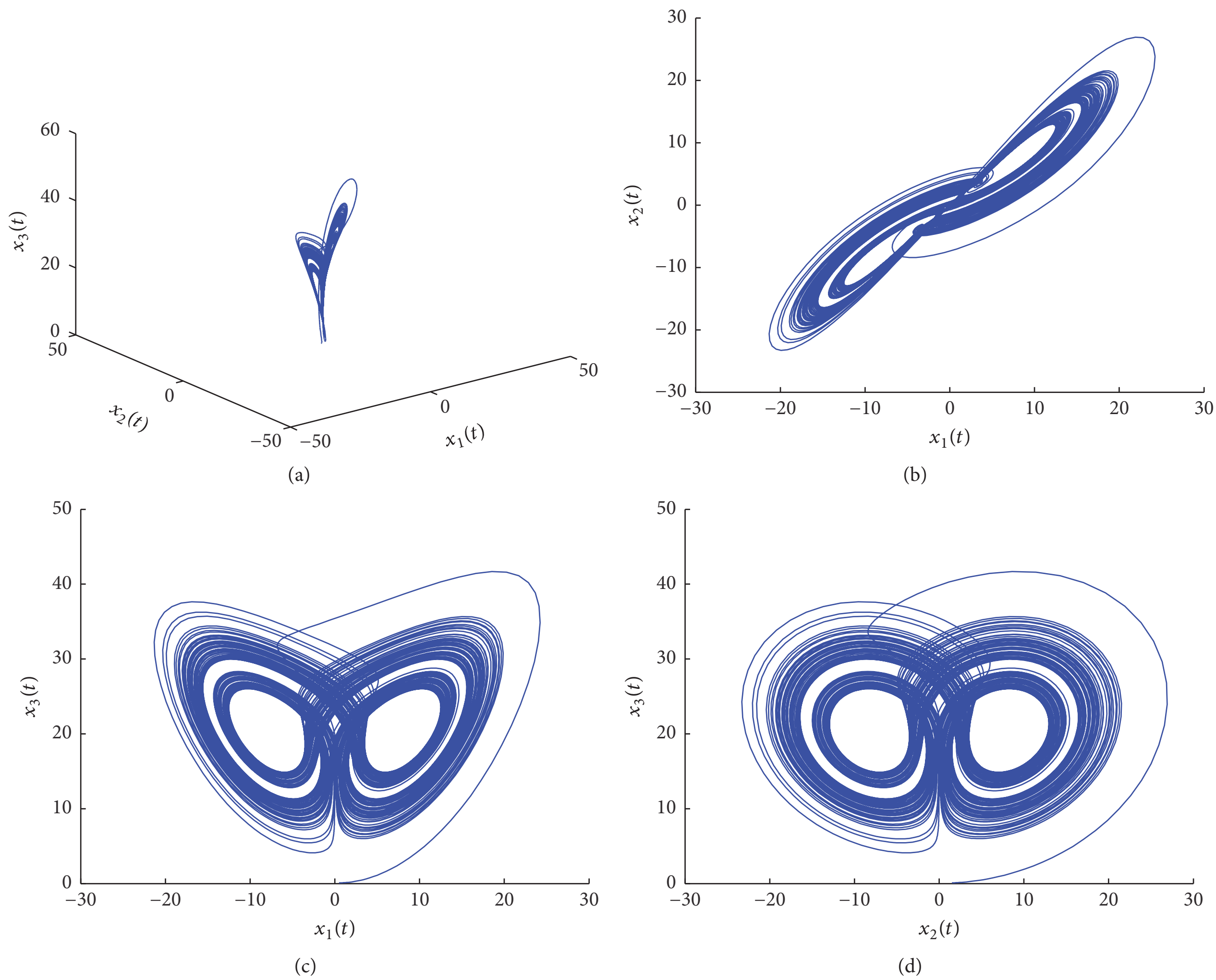

FIGURE 4: 3D and 2D chaotic attractors of fractional system (26).

hyperchaotic system, proposed by Zhang and Shen in [59], as the slave system. The master system is defined as

$$
\begin{aligned}
D^{p_1} x_1 &= \alpha \left(x_2 - x_1\right), \\
D^{p_2} x_2 &= \gamma x_2 - x_1 x_3, \\
D^{p_3} x_3 &= -\beta x_3 + x_1 x_2.
\end{aligned}
\tag{34}
$$

It is found, in [60], that this system displays chaotic attractors when $(\alpha, \beta, \gamma) = (36, 3, 20)$ and $(p_1, p_2, p_3) = (0.985, 0.99, 0.98)$. Chaotic attractors of master system (34) are shown in Figure 4.

Comparing system (34) with system (16), one can have

$$
A = \left(a_{ij}\right) = \begin{pmatrix} -36 & 36 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & -3 \end{pmatrix},
$$

$$
\left(f_i\right)_{1 \le i \le 3} = \begin{pmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{pmatrix}.
\tag{35}
$$

The slave system is described by

$$
\begin{aligned}
\dot{y}_1 &= a y_1 - y_2 + u_1, \\
\dot{y}_2 &= y_1 - y_2 y_3^2 + u_2, \\
\dot{y}_3 &= b y_3 - y_2 - 6 y_4 + u_3, \\
\dot{y}_4 &= y_3 + c y_4 + u_4.
\end{aligned}
\tag{36}
$$

The 4D Zhang-Shen system, that is, system (36) with $u_1 = u_2 = u_3 = u_4 = 0$, exhibits hyperchaotic behavior when $(a, b, c) = (0.56, -1, 0.8)$. Chaotic attractors in 3D of the uncontrolled system (36) are shown in Figure 5.

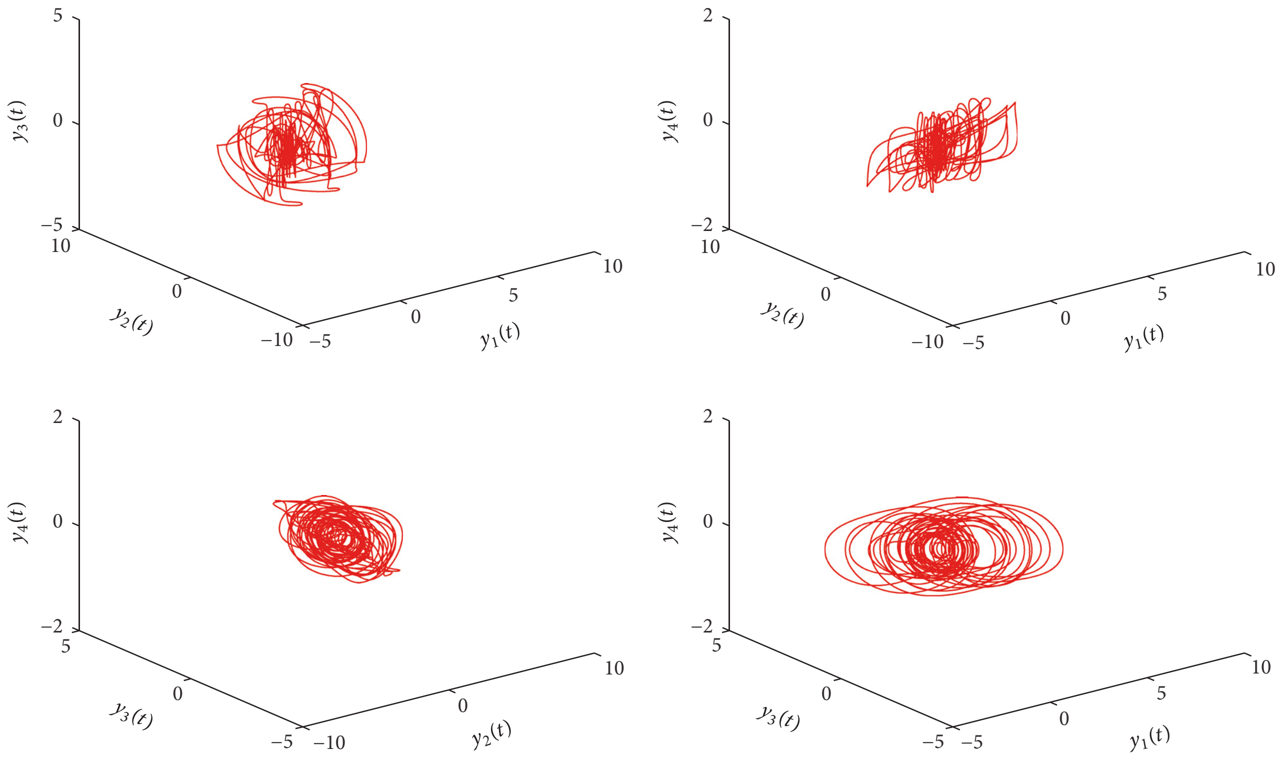

FIGURE 5: 3D phase portraits of the hyperchaotic Zhang-Shen system.

Based on the notations presented in Section 3.2, the errors between master system (34) and slave system (36) are given as

$$\begin{aligned} e_1 &= x_1 - h(t)\, y_1, \\ e_2 &= x_2 - \sum_{j=1}^{4} \Lambda_j(t)\, y_j, \\ e_3 &= x_3 - \phi(y_1, y_2, y_3, y_4), \end{aligned} \tag{37}$$

where $h(t) = t^2 + 1$, $\Lambda_1(t) = 0$, $\Lambda_2(t) = \exp(t)$, $\Lambda_3(t) = 0$, $\Lambda_4(t) = t^4$, and $\phi(y_1, y_2, y_3, y_4) = 3y_3 + y_4^2$. So, the matrix M defined by (23) is

$$M = \begin{pmatrix} t^2+1 & 0 & 0 \\ 0 & \exp(t) & 0 \\ 0 & 0 & 3 \end{pmatrix}. \tag{38}$$

According to Theorem 7, the control law $(u_1, u_2, u_3, u_4)^{\mathrm{T}}$ can be constructed as

$$\begin{aligned} (u_1, u_2, u_3)^T &= M^{-1} \left[T + L (e_1, e_2, e_3)^T \right], \\ u_4 &= 0, \end{aligned} \tag{39}$$

where

$$\begin{aligned} M^{-1} &= \begin{pmatrix} \dfrac{1}{t^2+1} & 0 & 0 \\ 0 & \exp(-t) & 0 \\ 0 & 0 & \dfrac{1}{3} \end{pmatrix}, \\ T_1 &= \dot{x}_1 + 36e_1 - 36e_2 - 2ty_1 \\ &\quad - \left(t^2+1\right)\left(0.56y_1 - y_2\right), \\ T_2 &= \dot{x}_2 - 20e_2 - \exp(t)\, y_2 - 4t^3 y_4 \\ &\quad - \exp(t)\left(y_1 - y_2 y_3^2\right) - t^4 \left(y_3 + 0.8y_4\right), \\ T_3 &= \dot{x}_3 + 3e_3 + 3\left(y_3 + y_2 + 6y_4\right) \\ &\quad - 2y_4\left(y_3 + 0.8y_4\right), \\ L &= \begin{pmatrix} 0 & 36 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{40}$$

We can see that $(A-L)^T + (A-L)$ is a negative definite matrix. Hence, the synchronization between master system

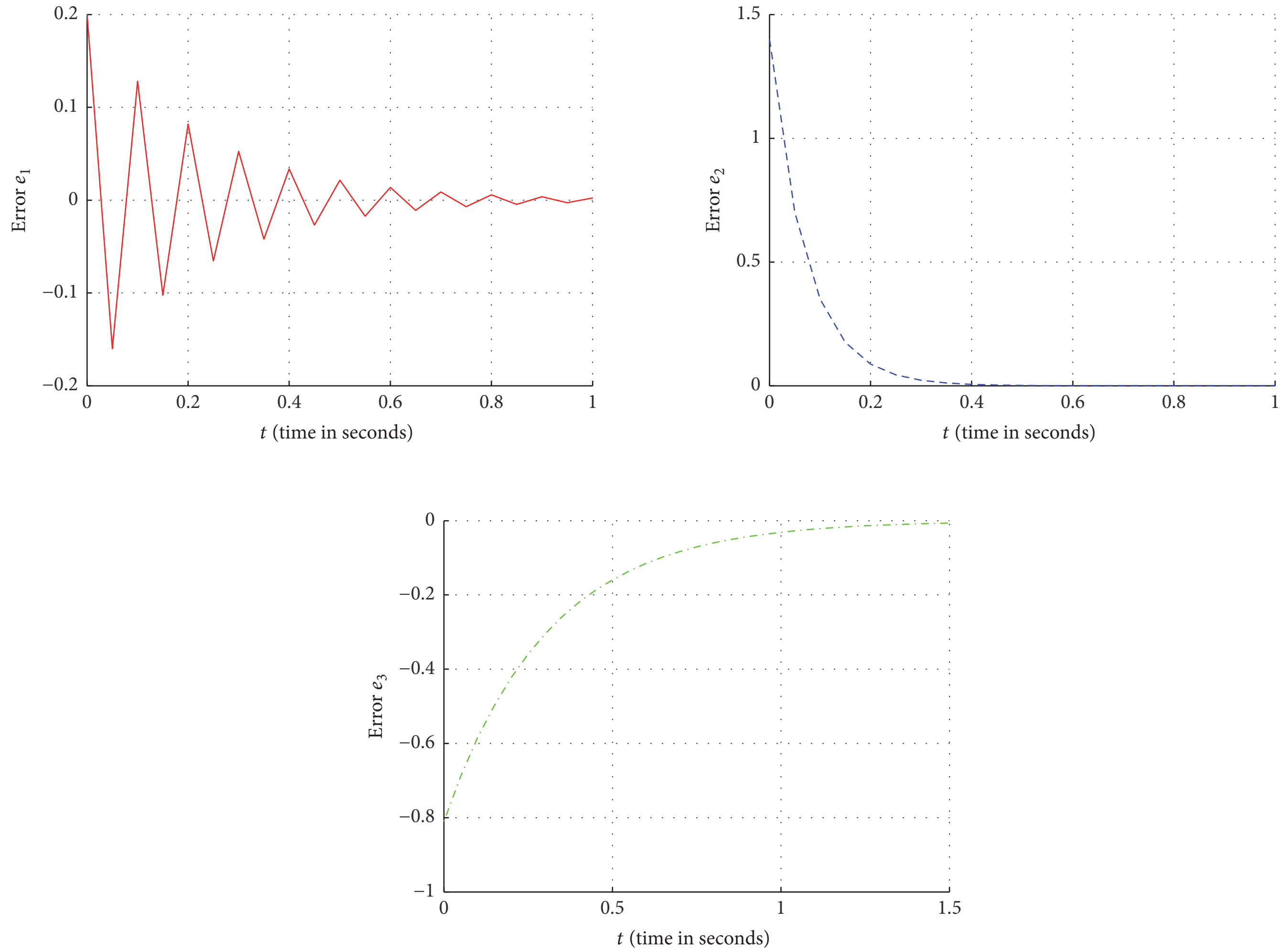

FIGURE 6: Time evolution of the synchronization errors e_1, e_2, and e_3 between master system (34) and slave system (36).

(34) and slave system (36) is achieved and the error system can be described as

$$\dot{e}_1 = -36e_1,$$
$$\dot{e}_2 = -10e_2, \quad (41)$$
$$\dot{e}_3 = -3e_3.$$

For the purpose of numerical simulation, the fourth order Runge–Kutta integration method has been used to solve system (41). The initial values of the master and the slave systems are $[x_1(0), x_2(0), x_3(0)] = [0.5, 1.5, 0.1]$ and $[y_1(0), y_2(0), y_3(0), y_4(0)] = [0.7, 0.1, 0.3, 0.1]$, respectively, and the initial states of the error system are $[e_1(0), e_2(0), e_3(0)] = [0.2, 1.4, -0.81]$. Figure 6 displays the synchronization errors between systems (34) and (36).

5. Conclusion

This paper has presented new schemes to study the coexistence of some types of chaos synchronization between nonidentical and different dimensional master and slave systems described by integer order and fractional order differential equations. The first scheme was constructed by combining CS, PS, FSHFPS, and GS in the synchronization of 3D integer order master system and 4D fractional order slave system. The second one was proposed when IPS, IFSHFPS, and IGS coexist between 3D fractional order master system and 4D integer order slave system. By exploiting fractional order Lyapunov approach and integer order Lyapunov method, the proposed synchronization approaches were rigorously proved to be achievable. The capability of the methods was illustrated by numerical examples and computer simulations.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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