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Research Article

Some Identities Involving the Reciprocal Sums of One-Kind Chebyshev Polynomials

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We use the elementary and analytic methods and the properties of Chebyshev polynomials to study the computational problem of the reciprocal sums of one-kind Chebyshev polynomials and give several interesting identities for them. At the same time, we also give a general computational method for this kind of reciprocal sums.

1. Introduction

It is well known that Chebyshev polynomials of the first and second kind $T_n(x)$ and $U_n(x)$ are defined as follows: $T_0(x) = 1$, $T_1(x) = x$, and the recursion formula $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for all integers $n \geq 1$. $U_0(x) = 1$, $U_1(x) = 2x$, and the recursion formula $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$ for all integers $n \geq 1$. The generation functions of these polynomials are

$$\frac{1 - xt}{1 - 2xt + t^2} = \sum_{n=1}^{\infty} T_n(x) t^n, \quad (|x| < 1,\ |t| < 1),$$
$$\frac{1}{1 - 2xt + t^2} = \sum_{n=1}^{\infty} U_n(x) t^n, \quad (|x| < 1,\ |t| < 1). \tag{1}$$

The general term formulae of $T_n(x)$ and $U_n(x)$ are expressed as

$$T_n(x) = \frac{1}{2}\left[\left(x + \sqrt{x^2 - 1}\right)^n + \left(x - \sqrt{x^2 - 1}\right)^n\right],$$
$$U_n(x) = \frac{1}{2\sqrt{x^2 - 1}}\left[\left(x + \sqrt{x^2 - 1}\right)^{n+1} - \left(x - \sqrt{x^2 - 1}\right)^{n+1}\right]. \tag{2}$$

If we take $x = \cos\theta$, then

$$T_n(\cos\theta) = \cos(n\theta),$$
$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}. \tag{3}$$

Since all these definitions and properties of Chebyshev polynomials can be found in any handbook of mathematics, there is no need to list the source everywhere.

Recently, some authors studied the properties of Chebyshev polynomials and obtained many interesting conclusions. For example, Li [1] obtained some identities involving power sums of $T_n(x)$ and $U_n(x)$. As some applications of these results, she obtained some divisibility properties involving Chebyshev polynomials. At the same time, she also proposed the following open problem.

Whether there exists an exact expression for the derivative or integral of the Chebyshev polynomials of the first kind in terms of the Chebyshev polynomials of the first kind (and vice-versa) is the question.

Wang and Zhang [2] and Zhang and Wang [3] partly solved these problems. Some theoretical results related to Chebyshev polynomials can be found in Ma and Zhang [4], Cesarano [5], Babusci et al. [6–8], Lee and Wong [9], and Wang and Han [10]. Doha and others [11–14] and Bircan and Pommerenke [15] also obtained many important applications of the Chebyshev polynomials.

In this paper, we consider the computational problem of the reciprocal sums of Chebyshev polynomials. That is, let q and k be positive integers with $q \geq 3$, for any real number x; if $T_n(x) \neq 0$, then we consider the computational problem of the summations

$$\sum_{a=1}^{q-1} \frac{1}{T_a^{2k}(x)}, \qquad \sum_{a=1}^{q-1} \frac{1}{U_{a-1}^{2k}(x)}. \tag{4}$$

Although there are many results related to Chebyshev polynomials, it seems that none had studied the computational problem of (4). The main reason may be that a computational formula does not exist. But for some special real number x, we can really get the precise value of (4). In this paper, we will illustrate this point. That is, we will use the elementary and analytic methods and the properties of Chebyshev polynomials to prove the following results.

Theorem 1. *Let q be an integer with $q \geq 3$. Then for any integer h with $(h, q) = 1$, one has the identities*

$$\begin{aligned} \sum_{a=1}^{q-1} \frac{1}{U_{a-1}^{2}(\cos(\pi h/q))} &= \frac{\sin^2(\pi h/q)}{3}\left(q^2-1\right); \\ \sum_{a=1}^{q-1} \frac{1}{U_{a-1}^{4}(\cos(\pi h/q))} &= \frac{\sin^4(\pi h/q)}{45}\left(q^2-1\right)\left(q^2+11\right); \\ \sum_{a=1}^{q-1} \frac{1}{U_{a-1}^{6}(\cos(\pi h/q))} &= \frac{\sin^6(\pi h/q)}{945}\left(q^2-1\right)\left(2q^2-11\right)\left(q^2+17\right). \end{aligned} \tag{5}$$

Theorem 2. *Let q be an odd number with $q \geq 3$. Then for any integer h with $(h, q) = 1$, one has the identities*

$$\begin{aligned} \sum_{a=1}^{q-1} \frac{1}{T_a^{2}(\cos(\pi h/q))} &= q^2-1; \\ \sum_{a=1}^{q-1} \frac{1}{T_a^{4}(\cos(\pi h/q))} &= \frac{1}{3}\left(q^2-1\right)\left(q^2+3\right); \\ \sum_{a=1}^{q-1} \frac{1}{T_a^{6}(\cos(\pi h/q))} &= \frac{1}{15}\left(q^2-1\right)\left(2q^4+7q^2-363\right). \end{aligned} \tag{6}$$

Some Notes. First in Theorem 2, we must limit q as an odd number. Otherwise, if q is an even number, then $a = (1/2) \cdot q$ is an integer, $1 \leq a \leq q-1$ and $\cos(\pi a/q) = \cos(\pi/2) = 0$. Therefore, the fraction $1/T_a^{2k}(\cos(\pi a/q))$ is meaningless.

Second, for any positive integer k and $x = \cos(\pi h/q)$ with $(h, q) = 1$, we can give an computational formula for (4). Of course, the calculation is very complicated when k is larger. So we do not give a general conclusion for (4), only give an efficient calculating method. In fact if we use computer MatLab program, and by means of recursive method in Lemma 4, we can also obtain all precise values of (4) for any positive integer k.

2. Several Lemmas

To complete the proofs of our theorems, we need following lemmas. First we have

Lemma 3. *Let $q > 3$ be an integer. Then for variable s with $0 < s < 1$ and function $f(\pi s) = \ln\sin(\pi s)$, one has the identity*

$$\sum_{a=1}^{q-1} f^{(2k)}\left(\frac{\pi a}{q}\right) = \frac{(-1)^k \cdot 2^{2k-1} \cdot B_{2k}}{k} \cdot \left(q^{2k}-1\right), \tag{7}$$

where $f^{(n)}(s)$ denotes the n-order derivative of $f(s)$, B_{2k} is Bernoulli numbers.

Proof. For any real number s, from Corollary 6 (Section 3, Chapter 5) in [16] we have the identity

$$\sin(\pi s) = \pi s \prod_{n=1}^{\infty}\left(1 - \frac{s^2}{n^2}\right). \tag{8}$$

Then from (8), the definition and properties of derivative we have

$$\begin{aligned} \pi f'(\pi s) &= \frac{1}{s} + \sum_{n=1}^{\infty}\left(\frac{1}{n+s} - \frac{1}{n-s}\right), \\ \pi f''(\pi s) &= -\frac{1}{s^2} - \sum_{n=1}^{\infty}\left(\frac{1}{(n+s)^2} + \frac{1}{(n-s)^2}\right). \end{aligned} \tag{9}$$

Generally, for any positive integer k, we have

$$\begin{aligned} \pi^{2k} f^{(2k)}(\pi s) = &-\frac{(2k-1)!}{s^{2k}} \\ &- \sum_{n=1}^{\infty}\left(\frac{(2k-1)!}{(n+s)^{2k}} + \frac{(2k-1)!}{(n-s)^{2k}}\right). \end{aligned} \tag{10}$$

Now taking $s = a/q$ in (10), and summation for all $1 \leq a \leq q-1$ and noting the definition and properties of complete residue system mod q (see [16]) we have

$$\begin{aligned}
&\pi^{2k}\sum_{a=1}^{q-1} f^{(2k)}\left(\frac{\pi a}{q}\right) \\
&= -(2k-1)!\sum_{a=1}^{q-1}\frac{q^{2k}}{a^{2k}} \\
&\quad - \sum_{a=1}^{q-1}\sum_{n=1}^{\infty}\left(\frac{(2k-1)!}{(n+a/q)^{2k}} + \frac{(2k-1)!}{(n-a/q)^{2k}}\right) \\
&= -q^{2k}\sum_{a=1}^{q-1}\sum_{n=0}^{\infty}\frac{(2k-1)!}{(qn+a)^{2k}} - (2k-1)!\sum_{a=1}^{q-1}\frac{q^{2k}}{(q-a)^{2k}} \\
&\quad - q^{2k}\sum_{a=1}^{q-1}\sum_{n=2}^{\infty}\frac{(2k-1)!}{(qn-a)^{2k}} \\
&= -2q^{2k}\sum_{a=1}^{q-1}\sum_{n=0}^{\infty}\frac{(2k-1)!}{(qn+a)^{2k}} \\
&= -2(2k-1)!\left(\sum_{n=1}^{\infty}\frac{q^{2k}}{n^{2k}} - \sum_{n=1}^{\infty}\frac{1}{n^{2k}}\right) \\
&= -2(2k-1)!\left(q^{2k}-1\right)\zeta(2k).
\end{aligned} \tag{11}$$

Note that Riemann ζ-function $\zeta(2k) = \sum_{n=1}^{\infty}(1/n^{2k}) = (-1)^{k+1}((2\pi)^{2k}B_{2k}/2(2k)!)$ (see [17], Theorem 12.17). Then from (11) we have

$$\begin{aligned}
\pi^{2k}\sum_{a=1}^{q-1} f^{(2k)}\left(\frac{\pi a}{q}\right) &= -2(2k-1)!\left(q^{2k}-1\right) \\
&\quad \cdot (-1)^{k+1}\frac{(2\pi)^{2k}B_{2k}}{2(2k)!} \\
&= \frac{(-1)^k\cdot(2\pi)^{2k}\cdot B_{2k}}{2k}\cdot\left(q^{2k}-1\right).
\end{aligned} \tag{12}$$

This proves Lemma 3. □

Lemma 4. *Let $f(s) = \ln\sin(\pi s)$, $\alpha = \alpha(s) = \cot(\pi s)$. Then one has*

$$\begin{aligned}
f''(s) &= -\pi^2\left(1+\alpha^2\right); \\
f^{(4)}(s) &= -\pi^4\left(6\alpha^4 + 8\alpha^2 + 2\right), \\
f^{(6)}(s) &= -\pi^6\left(120\alpha^6 + 240\alpha^4 + 136\alpha^2 + 16\right).
\end{aligned} \tag{13}$$

Proof. Noting that the identity $1+\cot^2 s = 1/\sin^2 s$, from the definition and properties of derivative we have $f'(s) = \pi\cot(\pi s) = \pi\alpha$ and

$$\begin{aligned}
f''(s) &= -\frac{\pi^2}{\sin(\pi s)} = -\pi^2\left(1+\cot^2(\pi s)\right) \\
&= -\pi^2\left(1+\alpha^2\right).
\end{aligned} \tag{14}$$

This proves the first formula of Lemma 4.

Similarly, we have

$$\begin{aligned}
f^{(3)}(s) &= -2\pi^2\alpha\alpha' = 2\pi^3\alpha + 2\pi^3\alpha^3, \\
f^{(4)}(s) &= 2\pi^3\alpha' + 6\pi^3\alpha^2\alpha' = -\pi^4\left(6\alpha^4 + 8\alpha^2 + 2\right).
\end{aligned} \tag{15}$$

This is the second formula of Lemma 4.

It is easy to prove that

$$f^{(6)}(s) = -\pi^6\left(120\alpha^6 + 240\alpha^4 + 136\alpha^2 + 16\right). \tag{16}$$

This completes the proof of Lemma 4. □

Lemma 5. *Let q be an integer with $q \geq 3$. Then for any integer h with $(h,q) = 1$, one has the identities*

$$\begin{aligned}
&\sum_{a=1}^{q-1}\frac{1}{\sin^2(\pi ha/q)} = \frac{1}{3}(q-1)(q+1); \\
&\sum_{a=1}^{q-1}\frac{1}{\sin^4(\pi ha/q)} = \frac{1}{45}\left(q^2-1\right)\left(q^2+11\right); \\
&\sum_{a=1}^{q-1}\frac{1}{\sin^6(\pi ha/q)} \\
&\quad = \frac{1}{945}\left(q^2-1\right)\left(q^2+17\right)\left(2q^2-11\right).
\end{aligned} \tag{17}$$

Proof. Since $(h,q) = 1$, if a pass through a complete residue system mod q, then ha also pass through a complete residue system mod q. Therefore, without loss of generality we can assume that $h = 1$. Noting that $B_2 = 1/6$ and $1+\cot^2(x) = 1/\sin^2(x)$, from Lemmas 3 and 4 with $k = 1$ we have

$$\begin{aligned}
-\sum_{a=1}^{q-1}\left(1+\cot^2\left(\frac{\pi a}{q}\right)\right) &= -\sum_{a=1}^{q-1}\frac{1}{\sin^2(\pi a/q)} \\
&= -\frac{1}{3}\left(q^2-1\right)
\end{aligned} \tag{18}$$

or

$$\sum_{a=1}^{q-1}\frac{1}{\sin^2(\pi a/q)} = \frac{1}{3}(q-1)(q+1). \tag{19}$$

Similarly, noting that $B_4 = -1/30$, from Lemmas 3 and 4 with $k = 2$ and applying (19) we have

$$\begin{aligned}
&-\sum_{a=1}^{q-1}\left(6\cot^4\left(\frac{\pi a}{q}\right) + 8\cot^2\left(\frac{\pi a}{q}\right) + 2\right) \\
&\quad = -\frac{2}{15}\left(q^4-1\right),
\end{aligned} \tag{20}$$

which implies that

$$\sum_{a=1}^{q-1}\left[6\left(\frac{1}{\sin^2(\pi a/q)}-1\right)^2+8\left(\frac{1}{\sin^2(\pi a/q)}-1\right)+2\right]=\frac{2}{15}\left(q^4-1\right) \tag{21}$$

or

$$\sum_{a=1}^{q-1}\frac{1}{\sin^4(\pi a/q)}=\frac{1}{45}\left(q^2-1\right)\left(q^2+11\right). \tag{22}$$

Noting that $B_6 = 1/42$, from Lemmas 3 and 4 with $k = 3$, applying (19) and (22) we have

$$-\sum_{a=1}^{q-1}\left(120\cot^6\left(\frac{\pi a}{q}\right)+240\cot^4\left(\frac{\pi a}{q}\right)+136\cot^2\left(\frac{\pi a}{q}\right)+16\right)=-\frac{16}{63}\left(q^6-1\right) \tag{23}$$

or

$$\sum_{a=1}^{q-1}\left[15\left(\frac{1}{\sin^2(\pi a/q)}-1\right)^3+30\left(\frac{1}{\sin^2(\pi a/q)}-1\right)^2+17\left(\frac{1}{\sin^2(\pi a/q)}-1\right)+2\right]=\frac{2}{63}\left(q^6-1\right), \tag{24}$$

which implies that

$$\sum_{a=1}^{q-1}\frac{1}{\sin^6(\pi a/q)}=\frac{1}{945}\left(q^2-1\right)\left(q^2+17\right)\left(2q^2-11\right). \tag{25}$$

Now Lemma 5 follows from (19), (22), and (25).

In fact, by using Lemma 4 and the method of proving Lemma 5 we can give a computational formula for

$$\sum_{a=1}^{q-1}\frac{1}{\sin^{2k}(\pi a/q)} \tag{26}$$

with all positive integer k. Here just in order to meet the demands of main results we only calculated $k = 1, 2$, and 3. □

3. Proofs of the Theorems

In this section, we shall complete the proofs of our theorems. First we prove Theorem 1. For any integer $q \geq 3$, taking $x = \cos(\pi h/q)$ with $(h, q) = 1$, from (3) we have

$$\sum_{a=1}^{q-1}\frac{1}{U_{a-1}^{2k}(x)}=\sum_{a=1}^{q-1}\frac{1}{U_{a-1}^{2k}(\cos(\pi h/q))}=\sin^{2k}\left(\frac{\pi h}{q}\right)\sum_{a=1}^{q-1}\frac{1}{\sin^{2k}(\pi h a/q)}. \tag{27}$$

Now Theorem 1 follows from (27) and Lemma 5 with $k = 1, 2$, and 3.

To prove Theorem 2, we note that, for any odd number $q \geq 3$, if a pass through a complete residue system mod q, then $2a$ also pass through a complete residue system mod q. So from the properties of trigonometric functions we have

$$\begin{aligned}\sum_{a=1}^{q-1}\frac{1}{\sin^2(\pi a/q)}&=\sum_{a=1}^{q-1}\frac{1}{\sin^2(2\pi a/q)}\\&=\sum_{a=1}^{q-1}\frac{\sin^2(\pi a/q)+\cos^2(\pi a/q)}{4\sin^2(\pi a/q)\cos^2(\pi a/q)}\\&=\frac{1}{4}\sum_{a=1}^{q-1}\frac{1}{\sin^2(\pi a/q)}+\frac{1}{4}\sum_{a=1}^{q-1}\frac{1}{\cos^2(\pi a/q)}.\end{aligned} \tag{28}$$

From (3), (28), and Lemma 5 we may immediately deduce that

$$\sum_{a=1}^{q-1}\frac{1}{T_a^2(\cos(\pi h/q))}=\sum_{a=1}^{q-1}\frac{1}{\cos^2(\pi h a/q)}=\sum_{a=1}^{q-1}\frac{3}{\sin^2(\pi a/q)}=q^2-1. \tag{29}$$

Similarly, we also have

$$\begin{aligned}\sum_{a=1}^{q-1}\frac{1}{\sin^4(\pi a/q)}&=\sum_{a=1}^{q-1}\frac{1}{\sin^4(2\pi a/q)}\\&=\sum_{a=1}^{q-1}\frac{\left(\sin^2(\pi a/q)+\cos^2(\pi a/q)\right)^2}{16\sin^4(\pi a/q)\cos^4(\pi a/q)}\\&=\frac{1}{16}\sum_{a=1}^{q-1}\frac{1}{\sin^4(\pi a/q)}+\frac{1}{16}\sum_{a=1}^{q-1}\frac{1}{\cos^4(\pi a/q)}+\frac{1}{2}\sum_{a=1}^{q-1}\frac{1}{\sin^2(2\pi a/q)}.\end{aligned} \tag{30}$$

So from (30) and Lemma 5 we have

$$\begin{aligned}\sum_{a=1}^{q-1}\frac{1}{\cos^4(\pi a/q)}&=\sum_{a=1}^{q-1}\frac{15}{\sin^4(\pi a/q)}-\sum_{a=1}^{q-1}\frac{8}{\sin^2(\pi a/q)}\\&=\frac{1}{3}\left(q^2-1\right)\left(q^2+3\right).\end{aligned} \tag{31}$$

Combining (3) and (31) we have the identity

$$\sum_{a=1}^{q-1}\frac{1}{T_a^4(\cos(\pi h/q))}=\sum_{a=1}^{q-1}\frac{1}{\cos^4(\pi ha/q)}=\frac{1}{3}\left(q^2-1\right)\left(q^2+3\right). \tag{32}$$

From the method of proving (30) we also have

$$\begin{aligned}\sum_{a=1}^{q-1}\frac{1}{\sin^6(\pi a/q)}&=\sum_{a=1}^{q-1}\frac{1}{\sin^6(2\pi a/q)}\\&=\sum_{a=1}^{q-1}\frac{\left(\sin^2(\pi a/q)+\cos^2(\pi a/q)\right)^3}{64\sin^6(\pi a/q)\cos^6(\pi a/q)}\\&=\frac{1}{64}\sum_{a=1}^{q-1}\frac{1}{\sin^6(\pi a/q)}\\&\quad+\frac{1}{64}\sum_{a=1}^{q-1}\frac{1}{\cos^6(\pi a/q)}\\&\quad+\frac{3}{4}\sum_{a=1}^{q-1}\frac{1}{\sin^4(2\pi a/q)}.\end{aligned} \tag{33}$$

From (3), Lemma 5, and (33) we can deduce that

$$\begin{aligned}\sum_{a=1}^{q-1}\frac{1}{T_a^6(\cos(\pi h/q))}&=\sum_{a=1}^{q-1}\frac{1}{\cos^6(\pi ha/q)}\\&=\sum_{a=1}^{q-1}\frac{63}{\sin^6(\pi ha/q)}\\&\quad-\sum_{a=1}^{q-1}\frac{48}{\sin^4(\pi ha/q)}\\&=\frac{\left(q^2-1\right)\left(2q^4+7q^2-363\right)}{15}.\end{aligned} \tag{34}$$

Now Theorem 2 follows from (29), (32), and (34).

This completes all proofs of our results.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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