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Research Article

Unilateral Global Bifurcation for Fourth-Order Problems and Its Applications

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We will establish unilateral global bifurcation result for a class of fourth-order problems. Under some natural hypotheses on perturbation function, we show that $(\lambda_k, 0)$ is a bifurcation point of the above problems and there are two distinct unbounded continua, C_k^+ and C_k^-, consisting of the bifurcation branch C_k from $(\mu_k, 0)$, where μ_k is the kth eigenvalue of the linear problem corresponding to the above problems. As the applications of the above result, we study the existence of nodal solutions for the following problems: $x'''' + kx'' + lx = rh(t)f(x)$, $0 < t < 1$, $x(0) = x(1) = x'(0) = x'(1) = 0$, where $r \in \mathbb{R}$ is a parameter and k, l are given constants; $h(t) \in C([0,1],[0,\infty))$ with $h(t) \not\equiv 0$ on any subinterval of $[0,1]$; and $f : \mathbb{R} \to \mathbb{R}$ is continuous with $sf(s) > 0$ for $s \neq 0$. We give the intervals for the parameter $r \neq 0$ which ensure the existence of nodal solutions for the above fourth-order Dirichlet problems if $f_0 \in [0,\infty]$ or $f_\infty \in [0,\infty]$, where $f_0 = \lim_{|s|\to 0} f(s)/s$ and $f_\infty = \lim_{|s|\to+\infty} f(s)/s$. We use unilateral global bifurcation techniques and the approximation of connected components to prove our main results.

1. Introduction

The deformations of an elastic beam in equilibrium state with fixed both endpoints can be described by the fourth-order boundary value problem

$$x'''' + kx'' + lx = rh(t)f(x), \quad 0 < t < 1,$$
$$x(0) = x(1) = x'(0) = x'(1) = 0, \tag{1}$$

where $r \in \mathbb{R}$ is a parameter, k, l are given constants, and $f : \mathbb{R} \to \mathbb{R}$ is continuous. When $k = l = 0$, since problem (1) cannot transform into a system of second-order equation, the treatment method of second-order system does not apply to problem (1). Thus, there exists some difficulty studying problem (1) even in the case of $k = l = 0$.

In recent years, there has been considerable interest in the above BVP (1) mainly because of their interesting applications. For example, Agarwal and Chow [1] ($k = l = 0$) first investigated the existence of the solutions of problem (1) by contraction mapping and iterative methods. Subsequently, when $k = l = 0$, by fixed point theory on cones, Ma et al. [2, 3], Yao [4, 5], Zhai et al. [6], and Webb et al. [7] studied the existence of positive solutions of problem (1).

On the other hand, by applying the bifurcation techniques of Rabinowitz [8, 9], Gupta and Mawhin [10], Lazer and McKenna [11], Liu and O'Regan [12], and Ma et al. [13–15] studied the existence of nodal solutions for the fourth-order BVP where both ends were simply supported, and Rynne [16] investigated the nodal properties of the solutions for a general $2m$th-order problem.

Meanwhile, it is well known that the spectrum structure of the linear eigenvalue problems according to (1) plays a key role to study problem (1) by the bifurcation techniques. Kratochvíl and Nečas [17] first studied the spectrum of the p-biharmonic operator together with $x(0) = x(1) = x'(0) = x'(1) = 0$. Subsequently, Benedikt [18–21] also studied the spectral properties of the corresponding eigenvalue problem of the same problems as [17], and Benedikt [22] studied existence and global bifurcation of solutions for the above problems. When $k = l = 0$, by applying the bifurcation techniques, Korman [23] investigated the uniqueness of positive solutions and Rynne [24] studied nodal properties of the solutions for problem (1), respectively. By Elias's theory [25, 26], Xu and Han [27] ($k = 0$), Ma et al. [28] ($k = 0$, $l = l(t)$), and Ma and Gao [29] ($x'''' + kx'' + lx = (p(t)x)'''' - (q(t)x)''$) established the spectrum structure of the linear eigenvalue problems according to (1) and studied the existence of nodal solutions of problem (1) using bifurcation theory [8]. In 2012, Shen [30, 31] established the following spectrum structure by applying disconjugate operator theory [25, 26].

Lemma 1 (see [30, 31]). *Let* $(A1)$ *and* $(A2)$ *hold. The linear eigenvalue problem*

$$x''''(t) + kx''(t) + lx(t) = \lambda h(t)x, \quad 0 < t < 1,$$
$$x(0) = x(1) = x'(0) = x'(1) = 0 \tag{2}$$

has a unique infinite number of positive eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \longrightarrow \infty, \quad \text{as } k \to \infty. \tag{3}$$

Moreover, each eigenvalue is simple. The eigenfunction ψ_k *corresponding to* λ_k *has exactly* $k-1$ *simple zeros in* $(0,1)$. *For each* $k \in \mathbb{N}$, *the algebraic multiplicity of* λ_k *is* 1, *where (A1) one of following conditions holds:*

(i) k, l *satisfying* $(k,l) \in \{(k,l) \mid k \in (-\infty, 0],\ l \in (0,\infty)\} \setminus \{(0, \pi^4/64)\} \cup \{(k,l) \mid k \in (-\infty, \pi^2),\ l \in (-\infty, 0]\}$, *are given constants with*

$$\pi^2\left(k - \pi^2\right) < l \leq \frac{1}{4}\left(k - \frac{\pi^2}{4}\right)^2. \tag{4}$$

(ii) k, l *satisfying* $(k,l) \in \{(k,l) \mid k \in (0, \pi^2/2),\ l \in (0,\infty)\}$, *are given constants with*

$$\frac{1}{4}\left(\pi^2 k - \frac{\pi^4}{4}\right) < l \leq \frac{1}{4}k^2. \tag{5}$$

(A2) $h(t) \in C([0,1],[0,\infty))$ *with* $h(t) \not\equiv 0$ *on any subinterval of* $[0,1]$.

On the basis of Lemma 1, Shen [30, 31] studied the existence of nodal solutions of problem (1) by applying Rabinowitz's global bifurcation theorem [8].

In 2013, when k, l satisfy $(A1)$ and $(A2)$, Shen and He [32] also studied bifurcation from interval and the existence of positive solutions for problem (1) by applying Rabinowitz's global bifurcation theorem [9].

Now, consider the following operator equation:

$$u = \lambda Bu + H(\lambda, u), \tag{6}$$

where B is a compact linear operator and $H : \mathbb{R} \times E \to E$ is compact with $H = o(\|u\|)$ at $u = 0$ uniformly on bounded λ intervals, where E is a real Banach space with the norm $\|\cdot\|$. If the eigenvalue μ of B has multiplicity 1,

$$\mathcal{S} = \overline{\{(\lambda, u) : (\lambda, u) \text{ satisfies (6) and } u \not\equiv 0\}}^{\mathbb{R}\times E}. \tag{7}$$

Dancer [33] has shown that there are two distinct unbounded continua C_μ^+ and C_μ^-, consisting of the bifurcation branch C_μ of $\mathcal{S}$ emanating from $(\mu, 0)$, which satisfy either that C_μ^+ and C_μ^- are both unbounded or $C_\mu^+ \cap C_\mu^- \neq \{(\mu, 0)\}$. This result has been extended to one-dimensional p-Laplacian problem by Dai and Ma [34]. The above results [34] have been improved partially by Dai [35] with nonasymptotic nonlinearity at 0 or ∞. Later, Dancer's result [33] has been also extended to the periodic p-Laplacian problems by Dai et al. [36]. In 2013, Dai and Han [37] established Dancer-type unilateral global bifurcation results for fourth-order problems of the deformations of an elastic beam in equilibrium state where both ends are simply supported by Dancer [33].

In this paper, based the spectral theory of [30, 31], we will establish Dancer-type unilateral global bifurcation results about the continuum of solutions for the following fourth-order eigenvalue problem:

$$x'''' + kx'' + lx = \lambda h(t)x + g(t, x, \lambda), \quad 0 < t < 1,$$
$$x(0) = x(1) = x'(0) = x'(1) = 0, \tag{8}$$

where h satisfies $(A2)$, and the perturbation function $g : (0,1) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous with $g(t,s,0) \equiv 0$ and satisfies the following hypotheses

$$\lim_{|s|\to 0} \frac{g(t,s,\lambda)}{s} = 0 \tag{9}$$

uniformly for $t \in (0,1)$ and λ on bounded sets.

Let $Y = C[0,1]$ with the norm $\|x\|_\infty = \max_{t\in[0,1]}|x(t)|$ and $E = \{x(t) \in C^3[0,1] \mid x(0) = x(1) = x'(0) = x'(1) = 0\}$ with the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty, \|x'''\|_\infty\}$. Let S_k^+ denote the set of functions in E which have exactly $k-1$ interior nodal (i.e., nondegenerate) zeros in $(0,1)$ and are positive near $t = 0$, set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. They are disjoint and open in E. Let $\Phi_k^\nu = \mathbb{R} \times S_k^\nu$, $\nu \in \{+,-\}$, and $\Phi_k = \mathbb{R} \times S_k$ under the product topology. Let $\mathcal{S}$ denote the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of (1) and let $\mathcal{S}^\pm$ denote the subset of $\mathcal{S}$ with $x \in \Phi_k^\pm$ and $\mathcal{S}^\pm = \mathcal{S}^+ \cup \mathcal{S}^-$.

Under condition (9), we will show that $(\mu_k, 0)$ is a bifurcation point of (8) and there are two distinct unbounded continua, C_k^+ and C_k^-, consisting of the bifurcation branch C_k from $(\mu_k, 0)$, where μ_k is the kth eigenvalue of problem (2). Based on the above result, we investigate the existence of nodal solutions for problem (1).

Remark 2. By applying disconjugate operator theory [25, 26], the authors [13, 14, 16] also established the spectrum structure of the corresponding linear eigenvalue problems. On the basis of the above spectrum structure, the authors [13, 14, 16] studied the existence of nodal solutions of the above problem by applying Rabinowitz's global bifurcation theorem [8].

The rest of this paper is arranged as follows. In Section 2, we will establish unilateral global bifurcation results. In Section 3, we will investigate the existence of nodal solutions for problem (1) under the linear growth condition on f.

2. Unilateral Global Bifurcation Results

We define the linear operator $L : D(L) \subset E \to Y$

$$Lx = x'''' + kx'' + lx, \quad x(t) \in D(L) \tag{10}$$

with $D(L) = \{x(t) \in C^4[0,1] \mid x(0) = x(1) = x'(0) = x'(1) = 0\}$.

From [31, p. 93], we consider the following auxiliary problem:

$$\begin{gathered} x'''' + kx'' + lx = e(t), \quad 0 < t < 1, \\ x(0) = x(1) = x'(0) = x'(1) = 0, \end{gathered} \tag{11}$$

for a given $e(t) \in C[0,1]$. We can get that problem (11) can be equivalently written as

$$x(t) = L^{-1}(e)(t) = \int_0^1 G(t,s)\, e(s)\, ds, \tag{12}$$

where $G(t,s) > 0$ was given in (2.29) of [31, p. 93].

Then L is a closed operator and $L^{-1} : Y \to E$ is completely continuous.

Define the operator $H : \mathbb{R} \times E \to E$ by

$$H(\lambda, x)(t) := \lambda L^{-1}(hx) + L^{-1}(g(t,x,\lambda)). \tag{13}$$

Furthermore, it is clear that problem (8) can be equivalently written as

$$x = H(\lambda, x)(t). \tag{14}$$

Clearly, H is completely continuous from $\mathbb{R} \times E \to E$ and $H(\lambda, 0) = 0$, $\forall \lambda \in \mathbb{R}$.

Let

$$\begin{gathered} \overline{g}(t,x,\lambda) = \max_{0 \le |s| \le x} |g(t,s,\lambda)| \\ \text{for } t \in (0,1), \ \lambda \text{ on bounded sets,} \end{gathered} \tag{15}$$

and then $\overline{g}$ is nondecreasing and

$$\lim_{x \to 0^+} \frac{\overline{g}(t,x,\lambda)}{x} = 0 \tag{16}$$

uniformly for $t \in (0,1)$ and λ on bounded sets. Further it follows from (16) that

$$\begin{aligned} \frac{|g(t,x,\lambda)|}{\|x\|} &\le \frac{\overline{g}(t,x,\lambda)}{\|x\|} \le \frac{\overline{g}(t,\|x\|_\infty,\lambda)}{\|x\|} \\ &\le \frac{\overline{g}(t,\|x\|,\lambda)}{\|x\|} \longrightarrow 0 \quad \text{as } \|x\| \longrightarrow 0, \end{aligned} \tag{17}$$

uniformly for $t \in (0,1)$ and λ on bounded sets.

By (17), we have that $\|L^{-1}(g(t,x,\lambda))\|/\|x\| \to 0$ as $\|x\| \to 0$ uniformly for $t \in (0,1)$ and λ on bounded sets. Furthermore, Applying Theorem 2 of [33], we may obtain the following result.

Theorem 3. *Assume that (A1), (A2), and (9) hold. Then $(\lambda_k, 0)$ is a bifurcation point of problem (8) and there exist two distinct unbounded continua C_k^+ and C_k^- of problem (8) emanating from $(\lambda_k, 0)$ such that either they are both unbounded or $C_k^+ \cap C_k^- \neq \{(\lambda_k, 0)\}$.*

Next, we prove that the first choice of the alternative of Theorem 3 is the only possibility. To do it, we give the following lemma.

Lemma 4. *Let $C_k := C_k^+ \cup C_k^-$. If $C_k \subset \Phi_k \cup (\lambda_k, 0)$, then C_k cannot contain a pair $(\overline{\lambda}, 0)$ and $\overline{\lambda} \neq \lambda_k$.*

Proof. Suppose on the contrary that there exists $(\lambda_m, x_m) \to (\lambda_j, 0)$ when $m \to +\infty$ with $(\lambda_m, x_m) \in C_k$, $x_m \neq 0$ and $j \neq k$. Let $y_m = x_m/\|x_m\|$; then y_m should be a solution of problem

$$y_m = L^{-1}\left(\lambda_m h y_m + \frac{g(t, x_m, \lambda)}{\|x_m\|}\right). \tag{18}$$

By (17), (18), and the compactness of L^{-1} we obtain that for some convenient subsequence $y_m \to y_0$ as $m \to +\infty$. Now y_0 verifies the equation

$$Ly_0 = \lambda_j h y_0 \tag{19}$$

and $\|y_0\| = 1$. Hence $y_0 \in S_j$ which is an open set in E, and as a consequence for some m large enough, $y_m \in S_j$, and this is a contradiction. □

Lemma 5. *If (λ, x) is a solution of (9) and $x \in \partial S_k$, then $x \equiv 0$.*

Proof. By the proof of Theorem 3.1 in [16, p. 467] (see also Corollary 1.12 and the proof of Theorem 2.3, together with the remark following that proof, in [16]), we easily obtain the result. □

Connecting Theorem 3 with Lemma 4, we can easily deduce the following Dancer-type unilateral global bifurcation result.

Theorem 6. *Assume that (A1), (A2), and (9) hold; then C_k^+ and C_k^- are unbounded continua. Moreover, we have*

$$\begin{gathered} C_k^+ \subset (\{(\lambda_k, 0)\} \cup (\mathbb{R} \times S_k^+)), \\ C_k^- \subset (\{(\lambda_k, 0)\} \cup (\mathbb{R} \times S_k^-)). \end{gathered} \tag{20}$$

Proof. By Theorem 3 with Lemma 4, we only prove $C_k^\nu \subset \Phi_k^\nu \cup \{(\lambda_k, \theta)\}$ for $\nu \in \{+, -\}$. In the following, we only prove the case of C_k^+ since the proof of C_k^- is similar.

We claim that there exists a neighborhood $B_\delta(\lambda_k, 0)$ of $(\lambda_k, 0)$ such that $(B_\delta(\lambda_k, 0) \cap C_k^+) \subset (\Phi_k^+ \cup \{(\lambda_k, \theta)\})$. Suppose on the contrary that there exists $(\lambda_m, x_m) \to (\lambda_k, 0)$ when $m \to +\infty$ with $(\lambda_m, x_m) \in C_k^+ \setminus (\mathbb{R} \times S_k^+)$ and $x_m \neq 0$. Let $z_m = x_m/\|x_m\|$; then z_m should be a solution of problem

$$z_m = L^{-1}\left(\lambda_m h z_m + \frac{g(t, x_m, \lambda_m)}{\|x_m\|}\right). \tag{21}$$

By (17), (21), and the compactness of L^{-1}, we obtain that for some convenient subsequence $z_m \to z_0$ as $m \to +\infty$. Now z_0 verifies the equation

$$Lz_0 = \lambda_k h z_0 \tag{22}$$

and $\|z_0\| = 1$. Hence $z_0 \in S_k$ which is an open set in E, and as a consequence for some m large enough, $x_m \in S_k^+$, and this is a contradiction.

Suppose that $C_k^+ \not\subseteq \Phi_k^+ \cup \{(\lambda_k, \theta)\}$. Then there exists $(\lambda^*, x) \in C_k^+ \cap (\mathbb{R} \times \partial S_k^+)$ such that $(\lambda^*, x) \neq (\lambda_k, \theta)$ and $(\lambda_n, x_n) \to (\lambda^*, x)$ with $(\lambda_n, x_n) \in C_k^+ \cap (\mathbb{R} \times S_k^+)$. Since $x \in \partial S_k^+$, by Lemma 8, $x \equiv 0$. Let $y_n := x_n/\|x_n\|$; then y_n should be a solution of problem

$$y_n = \lambda_n L^{-1}\left[\lambda h y_n + \frac{g(t, x_n, \lambda_n)}{\|x_n\|}\right]. \tag{23}$$

By (17), (23), and the compactness of L^{-1}, we obtain that for some convenient subsequence $y_n \to y_0 \neq 0$ as $n \to +\infty$. Now y_0 verifies the equation

$$Ly_0 = \lambda^* h(t)\, y_0(t), \quad t \in (0,1) \tag{24}$$

and $\|y_0\| = 1$. Hence $\lambda^* = \lambda_i$, for some $i \neq k$, $i \in \mathbb{N}$. Therefore, $(\lambda_n, x_n) \to (\lambda_i, \theta)$ with $(\lambda_n, x_n) \in C_k^+ \cap (\mathbb{R} \times S_k^+)$. This contradicts Lemma 4.

In order to treat the case $f_0 \notin (0, +\infty)$ or $f_\infty \notin (0, +\infty)$, we will need the following results. □

Definition 7 (see [38]). Let X be a Banach space and let $\{C_n \mid n = 1, 2, \ldots\}$ be a family of subsets of X. Then the superior limit $\mathbb{D}$ of $\{C_n\}$ is defined by

$$\mathbb{D} := \limsup_{n\to\infty} C_n = \left\{x \in X \mid \exists \{n_i\} \subset \mathbb{N},\ x_{n_i} \in C_{n_i},\ \text{such that } x_{n_i} \longrightarrow x\right\}. \tag{25}$$

Lemma 8 (see [38]). *Each connected subset of metric space X is contained in a component, and each connected component of X is closed.*

Lemma 9 (see [39]). *Let X be a Banach space and let $\{C_n \mid n = 1, 2, \ldots\}$ be a family of closed connected subsets of X. Assume that*

(i) *there exist $z_n \in C_n$, $n = 1, 2, \ldots$, and $z^* \in X$, such that $z_n \to z^*$;*

(ii) *$r_n = \sup\{\|x\| \mid x \in C_n\} = \infty$;*

(iii) *for all $R > 0$, $(\bigcup_{n=1}^{\infty} C_n) \cap B_R$ is a relative compact set of X, where*

$$B_R = \{x \in X \mid \|x\| \leq R\}. \tag{26}$$

Then there exists an unbounded component C in $\mathbb{D}$ and $z^* \in C$.

Lemma 10. *Assume (A1) and (A2). Let $g_n \in C([0,1], (0, +\infty))$. Assume that I is a subset of $[0,1]$ with meas$(I) > 0$, and let*

$$\lim_{n\to+\infty} g_n(t) = +\infty \tag{27}$$

uniformly on I. Let y_n be a solution of the equation

$$\begin{gathered} y_n'''' + k y_n'' + l y_n = \lambda h(t)\, g_n(t)\, y_n, \quad 0 < t < 1, \\ y_n(0) = y_n(1) = y_n'(0) = y_n'(1) = 0, \end{gathered} \tag{28}$$

and then y_n must change sign on I as n is large enough.

Proof. After taking a subsequence if necessary, we may assume that

$$h(t)\, g_{n_j}(t) \geq j, \quad t \in I, \tag{29}$$

for j large enough. By [32, Lemma 2.4], $L(x) = 0$ is disconjugate on $[0,1]$, which is a key condition in Elias [25]. Obviously, y_n have the property P. (For the definition of property P, see [25, p. 36].) Now, from the proof of [25, Lemma 4] (see also the remarks in the final paragraph in [25, p. 43]; or see the proof of [16, Lemma 3.7]), it follows that, for all n sufficiently large, y_n must change sign on I. □

3. Main Results

In this section, we first study the following eigenvalue problem:

$$\begin{gathered} x'''' + kx'' + lx = \lambda r h(t)\, f(x), \quad 0 < t < 1, \\ x(0) = x(1) = x'(0) = x'(1) = 0, \end{gathered} \tag{30}$$

where $\lambda > 0$ is a parameter.

In the section, $f \in C(\mathbb{R}, \mathbb{R})$ satisfy the following conditions:

$(H1)$ $sf(s) > 0$ for $s \neq 0$.

$(H2)$ $f_0, f_\infty \in (0, +\infty)$.

$(H3)$ $f_0 \in (0, \infty)$ and $f_\infty = \infty$.

$(H4)$ $f_0 = \infty$ and $f_\infty \in (0, \infty)$.

$(H5)$ $f_0 \in (0, \infty)$ and $f_\infty = 0$.

$(H6)$ $f_0 = 0$ and $f_\infty \in (0, \infty)$.

$(H7)$ $f_0 = 0$ and $f_\infty = \infty$.

$(H8)$ $f_0 = \infty$ and $f_\infty = 0$.

$(H9)$ $f_0 = \infty$ and $f_\infty = \infty$.

$(H10)$ $f_0 = 0$ and $f_\infty = 0$,

where

$$f_0 = \lim_{|x|\to 0} \frac{f(x)}{x}, \qquad f_\infty = \lim_{|x|\to\infty} \frac{f(x)}{x}. \tag{31}$$

Let $\zeta(x), \xi(x) \in C(\mathbb{R}, \mathbb{R})$ be such that

$$f(x) = f_0 x + \zeta(x), \qquad f(x) = f_\infty x + \xi(x) \tag{32}$$

with

$$\lim_{|x|\to 0} \frac{\zeta(x)}{x} = 0, \qquad \lim_{|x|\to\infty} \frac{\xi(x)}{x} = 0. \tag{33}$$

Let us consider

$$x'''' + kx'' + lx = \lambda rh(t) f_0 x + \lambda rh(t) \zeta(x), \quad 0 < t < 1, \tag{34}$$
$$x(0) = x(1) = x'(0) = x'(1) = 0$$

as a bifurcation problem from the trivial solution $x \equiv 0$ and

$$x'''' + kx'' + lx = \lambda rh(t) f_\infty x + \lambda rh(t) \xi(x), \quad 0 < t < 1, \tag{35}$$
$$x(0) = x(1) = x'(0) = x'(1) = 0$$

as a bifurcation problem from infinity.

We add the points $\{(\lambda, \infty) \mid \lambda \in \mathbb{R}\}$ to space $\mathbb{R} \times E$. By [40], we note that problem (34) and problem (35) are the same, and each of them is equivalent to problem (30). By Theorems 3 and 6 and the results of Rabinowitz [41], we have the following Lemma.

Lemma 11. *Let* $(A1)$, $(A2)$, $(H1)$, *and* $(H2)$ *hold.* $(\lambda_k/rf_0, 0)$ *and* $(\lambda_k/rf_\infty, \infty)$ *are bifurcation points for problem (30). Moreover, there are two distinct unbounded subcontinua of solutions to problem (30),* $\mathscr{D}_k^+$ *and* $\mathscr{D}_k^-$, *consisting of the bifurcation branch* $\mathscr{D}_k^\nu$ *emanating from* $(\lambda_k/rf_0, 0)$ *or* $(\lambda_k/rf_\infty, \infty)$. *For* $\nu = +, -$, $\mathscr{D}_k^\nu$ *joins* $(\lambda_k/rf_0, 0)$ *to* $(\lambda_k/rf_\infty, \infty)$, *such that* $\mathscr{D}_k^\nu \subset (\Phi_k^\nu \cup \{(\lambda_k/rf_0, 0)\})$ *and* $\mathscr{D}_k^\nu \subset (\Phi_k^\nu \cup \{(\lambda_k/rf_\infty, \infty)\})$.

Remark 12. Any solution of the problem (30) of the form $(1, x)$ yields a solution x of the problem (1). In order to prove our main results, one will only show that $\mathscr{D}_k^\nu$ crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$.

Theorem 13. *Let* $(A1)$, $(A2)$, $(H1)$, *and* $(H2)$ *hold, and either* $\lambda_k/f_\infty < r < \lambda_k/f_0$ *or* $\lambda_k/f_0 < r < \lambda_k/f_\infty$. *Then problem (1) has two solutions* x_k^+ *and* x_k^-, x_k^+ *has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is positive near* $t = 0$, *and* x_k^- *has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is negative near* $t = 0$.

Proof of Theorem 13. By Lemma 11 and Remark 12, we only prove $\mathscr{D}_k^+$ and $\mathscr{D}_k^-$ crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$. We only prove the case of $\mathscr{D}_k^+$ since the case of $\mathscr{D}_k^-$ is similar.

Case 1. (i) Consider $\lambda_k/f_\infty < r < \lambda_k/f_0$.

In this case, we only need to show that

$$\left(\frac{\lambda_k}{rf_\infty}, \frac{\lambda_k}{rf_0}\right) \subseteq \{\mu \in \mathbb{R} : (\mu, x) \in \mathscr{D}_k^+\}. \tag{36}$$

We divide the proof into two steps.

Let $(\lambda_n, x_n) \in \mathscr{D}_k^+$ satisfy

$$\lambda_n + \|x_n\| \longrightarrow \infty. \tag{37}$$

We note that $\lambda_n > 0$ for all $n \in \mathbb{N}$, since $(0,0)$ is the only solution of the problem (30) for $\lambda = 0$ and $\mathscr{D}_k^+ \cap (\{0\} \times E) = \emptyset$.

Step 1. We show that if there exists a constant number $M > 0$ such that

$$\lambda_n \in (0, M], \tag{38}$$

for $n \in \mathbb{N}$ large enough, then $\mathscr{D}_k^+$ joins $(\lambda_k/rf_0, 0)$ to $(\lambda_k/rf_\infty, \infty)$.

In this case, it follows that

$$\|x_n\| \longrightarrow \infty. \tag{39}$$

By (32) and (33), let

$$\overline{\xi}(x) = \max\{|\xi(s)| : 0 \le |s| \le x\}, \tag{40}$$

and then $\overline{\xi}$ is nondecreasing and

$$\lim_{x \to \infty} \frac{\overline{\xi}(x)}{x} = 0. \tag{41}$$

We divide the equation

$$Lx_n = \lambda_n ra(t) x_n + \lambda_n r\xi(x_n) \tag{42}$$

by $\|x_n\|$ and set $y_n = x_n/\|x_n\|$. Since y_n is bounded in E, after taking a subsequence if necessary, we have that $y_n \to y$ for some $y \in E$ with $\|y\| = 1$. Moreover, from (41) and the fact that $\overline{\xi}$ is nondecreasing, we have that

$$\lim_{n \to \infty} \frac{|\xi(x_n(t))|}{\|x_n\|} = 0, \tag{43}$$

since

$$\frac{|\xi(x_n(t))|}{\|x_n\|} \le \frac{\overline{\xi}(|x_n(t)|)}{\|x_n\|} \le \frac{\overline{\xi}(\|x_n(t)\|_\infty)}{\|x_n\|} \le \frac{\overline{\xi}(\|x_n(t)\|)}{\|x_n\|}. \tag{44}$$

By the continuity and compactness of L^{-1}, it follows that

$$y'''' + ky'' + ly = \overline{\lambda} ra(t) f_\infty y, \tag{45}$$

where $\overline{\lambda} := \lim_{n\to\infty} \lambda_n$, again choosing a subsequence and relabeling if necessary.

We claim that $y \in \mathscr{D}_k^+$.

It is clear that $y \in \overline{C_k^+} \subset C_k^+$ since $\mathscr{D}_k^+$ is closed in $\mathbb{R} \times E$. Thus, $\overline{\lambda} ra(t) f_\infty = \lambda_k$, so that

$$\overline{\lambda} = \frac{\lambda_k}{rf_\infty}. \tag{46}$$

Therefore, $\mathscr{D}_k^+$ joins $(\lambda_k/rf_0, 0)$ to $(\lambda_k/rf_\infty, \infty)$.

Step 2. We show that there exists a constant M such that $\lambda_n \in (0, M]$ for all n. On the contrary, choosing a subsequence and relabeling if necessary, it follows that

$$\lim_{n \to \infty} \lambda_n = \infty. \tag{47}$$

Since $(\lambda_n, x_n) \in \mathscr{D}_k^+$, it follows that

$$x_n'''' + kx_n'' + lx_n = \lambda_n rh(t) f(x_n). \tag{48}$$

Let

$$0 < \tau(1,n) < \tau(2,n) < \cdots < \tau(k-1,n) < 1 \quad (49)$$

denote the simple zeros of $x_n(t)$ in $(0,1)$. Let $\tau(0,n) = 0$ and $\tau(k,n) = 1$. Then, after taking a subsequence if necessary,

$$\lim_{n\to+\infty} \tau(j,n) = \tau(j,\infty), \quad j = 0,1,\ldots,k. \quad (50)$$

We claim that there exists $j_0 \in \{0,1,\ldots,k\}$ such that

$$\tau(j_0,\infty) < \tau(j_0+1,\infty). \quad (51)$$

Otherwise, we have

$$1 = \sum_{j=0}^{k-1} (\tau(j+1,n) - \tau(j,n)) \longrightarrow \sum_{j=0}^{k-1} (\tau(j+1,\infty) - \tau(j,\infty)) = 0. \quad (52)$$

This is a contradiction. Let $(a,b) \subset (\tau(j_0,\infty), \tau(j_0+1,\infty))$ with $a < b$. For all n sufficiently large, we have $(a,b) \subset (\tau(j_0,n), \tau(j_0+1,n))$. So $x_n(t)$ does not change its sign in (a,b).

On the other hand, let

$$x_n'''' + kx_n'' + lx_n = \lambda_n r h(t) \widetilde{f}_n(t), \quad (53)$$

where

$$\widetilde{f}_n(t) = \begin{cases} \dfrac{f(x_n)}{x_n}, & x_n(t) \neq 0, \\ f_0, & x_n(t) = 0. \end{cases} \quad (54)$$

Conditions $(H1)$ and $(H2)$ imply that there exists a positive constant $Q > 0$ such that $h(t)\widetilde{f}_n(t) > Q$ for any $t \in (a,b)$ and all $n \in \mathbb{N}$. By Lemma 10, we get that x_n must change its sign in (a,b) for n large enough, which is the contradicts. Therefore,

$$\lambda_n \leq M \quad (55)$$

for some constant number $M > 0$ and $n \in \mathbb{N}$ sufficiently large.

Case 2. (ii) Consider $\lambda_k/f_0 < r < \lambda_k/f_\infty$.

The proof is similar to that for Case 1, so we omit it. □

Theorem 14. *Let $(A1)$, $(A2)$, $(H1)$, and $(H3)$ hold. Assume condition $r \in (0, \lambda_k/f_\infty)$ holds for some $k \in \mathbb{N}$. Then problem (1) has two solutions x_k^+ and x_k^-, x_k^+ has exactly $k-1$ simple zeros in $(0,1)$ and is positive near $t = 0$, and x_k^- has exactly $k-1$ simple zeros in $(0,1)$ and is negative near $t = 0$.*

Proof. Inspired by the idea of [42], we define the cut-off function of f as the following:

$$f^{[n]}(s) := \begin{cases} ns, & s \in (-\infty,-2n] \cup [2n,+\infty), \\ \dfrac{2n^2 + f(-n)}{n}(s+n) + f(-n), & s \in (-2n,-n), \\ \dfrac{2n^2 - f(n)}{n}(s-n) + f(n), & s \in (n,2n), \\ f(s), & s \in [-n,n]. \end{cases} \quad (56)$$

We consider the following problem:

$$x'''' + kx'' + lx = \lambda r h(t) f^{[n]}(x), \quad 0 < t < 1,$$
$$x(0) = x(1) = x'(0) = x'(1) = 0. \quad (57)$$

Clearly, we can see that $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = f_0$, and $(f^{[n]})_\infty = n$.

Similar to the proof of Theorem 13, by Lemma 11 and Remark 12, there are two distinct unbounded subcontinua of solutions to problem (57), $\mathscr{D}_k^{+[n]}$ and $\mathscr{D}_k^{-[n]}$ emanating from $(\lambda_k/rf_0, 0)$, and joins to $(\lambda_k/rn, \infty)$.

Taking $z_n = (\lambda_k/rn, \infty)$ and $z^* = (0,\infty)$, we have that $z_n \to z^*$.

So condition (i) in Lemma 9 is satisfied with $z^* = (0,\infty)$.

Obviously

$$r_n = \sup\left\{\lambda + \|u\| \mid (\lambda,u) \in \mathscr{D}_k^{\nu[n]}\right\} = \infty, \quad (58)$$

and accordingly, (ii) in Lemma 9 holds. (iii) in Lemma 9 can be deduced directly from the Arzela-Ascoli Theorem and the definition of $f^{[n]}$.

Therefore, by Lemma 9, $\limsup_{n\to\infty} \mathscr{D}_k^{\nu[n]}$ contains an unbounded component $\mathscr{D}_k^\nu$ emanating from $(\lambda_k/rf_0, 0)$, and $\mathscr{D}_k^\nu$ joins $(\lambda_k/rf_0, 0)$ to $(0,\infty)$.

From $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, (57) can be converted to the equivalent equation (30). Thus, $\mathscr{D}_k^\nu$ is an unbounded component of solutions of problem (30) emanating from $(\lambda_k/rf_0, 0)$, and $\mathscr{D}_k^\nu$ joins $(\lambda_k/rf_0, 0)$ to $(0,\infty)$. We can prove the result. □

Theorem 15. *Let $(A1)$, $(A2)$, $(H1)$, and $(H4)$ hold. Assume that condition $r \in (0, \lambda_k/f_0)$ holds for some $k \in \mathbb{N}$. Then problem (1) has two solutions x_k^+ and x_k^-, x_k^+ has exactly $k-1$ simple zeros in $(0,1)$ and is positive near $t = 0$, and x_k^- has exactly $k-1$ simple zeros in $(0,1)$ and is negative near $t = 0$.*

Proof. If (λ, x) is any nontrivial solution of problem (30), dividing problem (30) by $\|x\|^2$ and setting $y = x/\|x\|^2$ yield

$$\begin{gathered} y'''' + ky'' + ly = \lambda rh(t)\frac{f(x)}{\|x\|^2}, \quad 0 < t < 1, \\ y(0) = y(1) = y'(0) = y'(1) = 0. \end{gathered} \tag{59}$$

Define

$$\widetilde{f}(y) := \begin{cases} \|y\|^2 f\left(\dfrac{y}{\|y\|^2}\right), & \text{if } y \neq 0, \\ 0, & \text{if } y = 0. \end{cases} \tag{60}$$

Evidently, problem (59) is equivalent to

$$\begin{gathered} y'''' + ky'' + ly = \lambda rh(t)\widetilde{f}(y), \quad 0 < t < 1, \\ y(0) = y(1) = y'(0) = y'(1) = 0. \end{gathered} \tag{61}$$

It is obvious that $(\lambda, 0)$ is always the solution of problem (59). By simple computation, we can show that $\widetilde{f}_0 = f_\infty \in (0,\infty)$ and $\widetilde{f}_\infty = f_0 = \infty$. Now, applying Theorem 13, there are two distinct unbounded subcontinua of solutions to problem (61), $\mathscr{C}_k^+$ and $\mathscr{C}_k^-$ emanating from $(\lambda^\nu/r\widetilde{f}_0, 0)$, and joins to $(0,\infty)$.

Under the inversion $y \to y/\|y\|^2 = x$, we obtain $\mathscr{C}_k^\nu \to \mathscr{D}_k^\nu$ being an unbounded component of solutions of problem (30) emanating from $(0, 0)$, and joins to $(\lambda^\nu/rf_\infty, \infty)$.

Moreover, by Remark 12 and the problem (1), we can obtain that $\mathscr{D}_k^\nu \subset \mathscr{S}_k^\nu$.

Thus, $\mathscr{D}_k^\nu$ is an unbounded component of solutions of problem (1) such that $\mathscr{D}_k^\nu$ joins $(0, 0)$ to $(\lambda^\nu/rf_\infty, \infty)$. □

Theorem 16. *Let* $(A1)$, $(A2)$, $(H1)$, *and* $(H5)$ *hold. Assume that condition* $r \in (\lambda_k/f_0, +\infty)$ *holds. Then problem (1) has two solutions* x_k^+ *and* x_k^-, x_k^+ *has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is positive near* $t = 0$, *and* x_k^- *has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is negative near* $t = 0$.

Proof. In view of the proof to prove Theorem 13, we only need to show that $\mathscr{D}_k^\nu$ joins $(\lambda_k/rf_0, 0)$ to (∞,∞). To do this, it is enough to prove that $[\lambda_k/rf_0, +\infty) \subset \mathrm{Proj}_R\mathscr{D}_k^\nu$.

Assume on the contrary that $\sup\{\lambda \mid (\lambda, u) \in \mathscr{D}_k^\nu\} < +\infty$, and then there exists a sequence $(\mu_n, x_n) \in \mathscr{D}_k^\nu$ such that

$$\begin{gathered} \lim_{n\to\infty} \|x_n\| = +\infty, \\ \mu_n \le c_0 \end{gathered} \tag{62}$$

for some positive constant c_0 depending not on n.

By $(H3)$, let $\overline{f}(x) = \max_{0\le|s|\le x}|f(s)|$, and then $\overline{f}$ is nondecreasing and

$$\lim_{x\to+\infty} \frac{\overline{f}(x)}{x} = 0. \tag{63}$$

We consider the equation

$$\begin{gathered} x_n'''' + kx_n'' + lx_n = \mu_n rh(t) f(x_n), \quad 0 < t < 1, \\ x_n(0) = x_n(1) = x_n'(0) = x_n'(1) = 0. \end{gathered} \tag{64}$$

Let $y_n = x_n/\|x_n\|$, and y_n should be the solutions of problem

$$\begin{gathered} y_n'''' + ky_n'' + ly_n = \mu_n rh(t)\frac{f(x_n)}{\|x_n\|}, \quad 0 < t < 1, \\ y_n(0) = y_n(1) = y_n'(0) = y_n'(1) = 0. \end{gathered} \tag{65}$$

Since y_n is bounded in E, choosing a subsequence and relabeling if necessary, we have that $y_n \to y$ for some $y \in E$ and $\|y\| = 1$.

Furthermore, from (63) and the fact that $\overline{f}$ is nondecreasing, we have that

$$\lim_{n\to\infty} \frac{f(x_n)}{\|x_n\|} = 0, \tag{66}$$

since

$$\frac{f(x_n)}{\|x_n\|} \le \frac{\overline{f}(|x_n|)}{\|x_n\|} \le \frac{\overline{f}(\|x_n\|_\infty)}{\|x_n\|} \le \frac{\overline{f}(\|x_n\|)}{\|x_n\|} \longrightarrow 0, \quad n \longrightarrow +\infty. \tag{67}$$

By (12), (65), (66), and the compactness of L^{-1}, we obtain that $y(t) \equiv 0$, $\forall t \in [0,1]$.

This contradicts $\|y(t)\| = 1$. □

Theorem 17. *Let* $(A1)$, $(A2)$, $(H1)$, *and* $(H6)$ *hold. Assume that condition* $r \in (\lambda_k/f_\infty, +\infty)$ *holds. Then problem (1) has two solutions* x_k^+ *and* x_k^-, x_k^+ *has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is positive near* $t = 0$, *and* x_k^- *has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is negative near* $t = 0$.

Proof. Similar to the method of the proof of Theorem 15 and the conclusions of Theorem 16, we can prove the conclusion. □

Theorem 18. *Let* $(A1)$, $(A2)$, $(H1)$, *and* $(H7)$ *hold. Assume that condition* $r \in (0, +\infty)$ *holds. Then problem (1) has two solutions* x_k^+ *and* x_k^-, x_k^+ *has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is positive near* $t = 0$, *and* x_k^- *has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is negative near* $t = 0$.

Proof. Define

$$f^{[n]}(s) := \begin{cases} ns, & s \in (-\infty, -2n] \cup [2n, +\infty), \\ \dfrac{2n^2 + f(-n)}{n}(s+n) + f(-n), & s \in (-2n, -n), \\ \dfrac{2n^2 - f(n)}{n}(s-n) + f(n), & s \in (n, 2n), \\ f(s), & s \in \left[-n, -\dfrac{2}{n}\right] \cup \left[\dfrac{2}{n}, n\right], \\ -\left[f\left(-\dfrac{2}{n}\right) + \dfrac{1}{n^2}\right](ns+2) + f\left(-\dfrac{2}{n}\right), & s \in \left(-\dfrac{2}{n}, -\dfrac{1}{n}\right), \\ \left[f\left(\dfrac{2}{n}\right) - \dfrac{1}{n^2}\right](ns-2) + f\left(\dfrac{2}{n}\right), & s \in \left(\dfrac{1}{n}, \dfrac{2}{n}\right), \\ \dfrac{1}{n}s, & s \in \left[-\dfrac{1}{n}, \dfrac{1}{n}\right]. \end{cases} \quad (68)$$

We consider the following problem:

$$x'''' + kx'' + lx = \lambda r h(t) f^{[n]}(x), \quad 0 < t < 1,$$
$$x(0) = x(1) = x'(0) = x'(1) = 0. \quad (69)$$

Clearly, we can see that $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = 1/n$, and $(f^{[n]})_\infty = n$.

Applying the similar method used in the proof of Theorem 13, by Lemma 11 and Remark 12, there are two distinct unbounded subcontinua of solutions to problem (69), $\mathscr{D}_k^{+[n]}$ and $\mathscr{D}_k^{-[n]}$ emanating from $(n\lambda_k/r, 0)$ or $(\lambda_k/rn, \infty)$, and joins $(n\lambda_k/r, 0)$ to $(\lambda_k/rn, \infty)$.

Taking $z_n = (n\lambda_k/r, 0)$ and $z^* = (\infty, 0)$ or $z_n = (\lambda_k/rn, \infty)$ and $z^* = (0, \infty)$, we have that $z_n \to z^*$. By Lemma 9, we obtain that $\limsup_{n\to\infty} \mathscr{D}_k^{\nu[n]}$ contains an unbounded component $\mathscr{D}_k^{\nu}$ with $(\infty, 0), (0, \infty) \in \mathscr{D}_k^{\nu}$.

From $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, (69) can be converted to the equivalent equation (30). Thus, $\mathscr{D}_k^{\nu}$ is an unbounded component of solutions of problem (30) emanating from $(\infty, 0)$ or $(0, \infty)$ and joins $(\infty, 0)$ to $(0, \infty)$.

Moreover, by Remark 12 and (1), we can obtain that $\mathscr{D}_k^{\nu} \subset \mathscr{S}_k^{\nu}$.

Thus, $\mathscr{D}_k^{\nu}$ is an unbounded component of solutions of problem (1) emanating from $(\infty, 0)$ or $(0, \infty)$ and joins $(\infty, 0)$ to $(0, \infty)$. □

Theorem 19. *Let (A1), (A2), (H1), and (H8) hold. Assume that condition $r \in (0, +\infty)$ holds. Then problem (2) has two solutions x_k^+ and x_k^-, x_k^+ has exactly $k-1$ simple zeros in $(0,1)$ and is positive near $t = 0$, and x_k^- has exactly $k-1$ simple zeros in $(0,1)$ and is negative near $t = 0$.*

Proof. Similar to the method of the proof of Theorem 14 and the conclusions of Theorem 18, we can obtain the desired results. □

Theorem 20. *Let (A1), (A2), (H1), and (H9) hold. There exists $\lambda_k^+ > 0$ such that $r \in (0, \lambda_k^+)$. Then problem (1) has two solutions x_k^+ and x_k^-, x_k^+ has exactly $k-1$ simple zeros in $(0,1)$ and is positive near $t = 0$, and x_k^- has exactly $k-1$ simple zeros in $(0,1)$ and is negative near $t = 0$.*

Proof. Define

$$f^{[n]}(s) := \begin{cases} ns, & s \in (-\infty, -2n] \cup [2n, +\infty), \\ \dfrac{2n^2 + f(-n)}{n}(s+n) + f(-n), & s \in (-2n, -n), \\ \dfrac{2n^2 - f(n)}{n}(s-n) + f(n), & s \in (n, 2n), \\ f(s), & s \in \left[-n, -\dfrac{2}{n}\right] \cup \left[\dfrac{2}{n}, n\right], \\ -\left[f\left(-\dfrac{2}{n}\right) + 1\right](ns+2) + f\left(-\dfrac{2}{n}\right), & s \in \left(-\dfrac{2}{n}, -\dfrac{1}{n}\right), \\ \left[f\left(\dfrac{2}{n}\right) - 1\right](ns-2) + f\left(\dfrac{2}{n}\right), & s \in \left(\dfrac{1}{n}, \dfrac{2}{n}\right), \\ ns, & s \in \left[-\dfrac{1}{n}, \dfrac{1}{n}\right]. \end{cases} \quad (70)$$

We consider the following problem:

$$x'''' + kx'' + lx = \lambda rh(t) f^{[n]}(x), \quad 0 < t < 1, \\ x(0) = x(1) = x'(0) = x'(1) = 0. \tag{71}$$

It is of no difficulty to verify that $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = n$, and $(f^{[n]})_\infty = n$.

Theorem 18 implies that there exist two sequences of unbounded continua of solutions to problem (71), $\mathscr{D}_k^{+[n]}$ and $\mathscr{D}_k^{-[n]}$ emanating from $(\lambda_k/rn, 0)$ or $(\lambda_k/rn, \infty)$.

By making use of Lemma 9 again, we obtain that there exist two unbounded components $\mathscr{D}_k^+$ and $\mathscr{D}_k^-$ of $\limsup_{n\to\infty} \mathscr{D}_k^{\nu[n]}$ such that $(0,0) \in \mathscr{D}_k^\nu$ and $(0,\infty) \in \mathscr{D}_k^\nu$. □

Theorem 21. *Let* $(A1)$, $(A2)$, $(H1)$, *and* $(H10)$ *hold. There exists* $\lambda_k^+ > 0$, *such that* $r \in (\lambda_k^+, +\infty)$. *Then problem (1) has two solutions* x_k^+ *and* x_k^-, x_k^+ *has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is positive near* $t = 0$, *and* x_k^- *has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is negative near* $t = 0$.

Proof. Similar to the method of the proof of Theorem 14 and the conclusions of Theorem 20, we can obtain the desired results.

Using the similar proof with that of Theorems 13–16, we can obtain the result. □

Theorem 22. *Let* $(A1)$, $(A2)$, $(H1)$, *and* $(H3)$ *hold. Assume that the following condition holds for some* $k, n \in \mathbb{N}$ *with* $k \le n$:

$$r < \frac{\lambda_k}{f_\infty} < \frac{\lambda_n}{f_\infty}. \tag{72}$$

Then problem (1) possesses $n-k+1$ *pairs solutions* x_i^+ *and* x_i^- *for* $i \in \{k,\ldots,n\}$ *such that* x_i^+ *has exactly* $i-1$ *zeros in* $(0,1)$ *and is positive near* 0, *and* x_i^- *has exactly* $i-1$ *zeros in* $(0,1)$ *and is negative near* 0.

Theorem 23. *Let* $(A1)$, $(A2)$, $(H1)$, *and* $(H4)$ *hold. Assume that the following condition holds for some* $k, n \in \mathbb{N}$ *with* $k \le n$:

$$r < \frac{\lambda_k}{f_0} < \frac{\lambda_n}{f_0}. \tag{73}$$

Then problem (1) possesses $n-k+1$ *pairs solutions* u_i^+ *and* x_i^- *for* $i \in \{k,\ldots,n\}$ *such that* x_i^+ *has exactly* $i-1$ *zeros in* $(0,1)$ *and is positive near* 0, *and* x_i^- *has exactly* $i-1$ *zeros in* $(0,1)$ *and is negative near* 0.

Theorem 24. *Let* $(A1)$, $(A2)$, $(H1)$, *and* $(H5)$ *hold. Assume that the following condition holds for some* $k, n \in \mathbb{N}$ *with* $k \le n$:

$$\frac{\lambda_k}{f_0} < \frac{\lambda_n}{f_0} < r < +\infty. \tag{74}$$

Then problem (1) possesses $n-k+1$ *pairs solutions* x_i^+ *and* x_i^- *for* $i \in \{k,\ldots,n\}$ *such that* x_i^+ *has exactly* $i-1$ *zeros in* $(0,1)$ *and is positive near* 0, *and* x_i^- *has exactly* $i-1$ *zeros in* $(0,1)$ *and is negative near* 0.

Theorem 25. *Let* $(A1)$, $(A2)$, $(H1)$, *and* $(H6)$ *hold. Assume that the following condition holds for some* $k, n \in \mathbb{N}$ *with* $k \le n$:

$$\frac{\lambda_k}{f_\infty} < \frac{\lambda_n}{f_\infty} < r < +\infty. \tag{75}$$

Then problem (1) possesses $n-k+1$ *pairs solutions* x_i^+ *and* x_i^- *for* $i \in \{k,\ldots,n\}$ *such that* x_i^+ *has exactly* $i-1$ *zeros in* $(0,1)$ *and is positive near* 0, *and* x_i^- *has exactly* $i-1$ *zeros in* $(0,1)$ *and is negative near* 0.

Remark 26. When $k = l = 0$, the authors of [2–7] studied the existence of the solutions of the problem (1) by fixed point theory on cones, while we study the unilateral global bifurcation and the existence of nodal solutions of problem (1) for the cases of $f_0, f_\infty \in [0,\infty]$ by Dancer [33]. Thus, in this sense, our results partially extend and improve the corresponding results of [2–7].

Remark 27. The main methods used in this work are unilateral global bifurcation techniques and the approximation of connected components, which are different from the methods used in [13–15, 27–30, 34, 37]. Moreover, we consider the cases of $f_0 \notin (0,+\infty)$ or $f_\infty \notin (0,+\infty)$, while the authors of [13–15, 27–30, 34, 37] only studied the cases of $f_0, f_\infty \in (0,\infty)$.

Remark 28. Since Benedikt [18–22] has established the spectrum structure of the corresponding eigenvalue problems, in the following, one can study the existence of nodal solutions for the problems in Benedikt [18–22] by applying the bifurcation techniques in this paper.

Remark 29. When $k = l = 0$, Korman [23] and Rynne [24] investigated the nodal properties of the solutions for problem (1) by applying the bifurcation techniques. Thus, in this sense, our results partially extend and improve the corresponding results of Korman [23] and Rynne [24].

Remark 30. We consider the cases of $f_0, f_\infty \notin (0,\infty)$ by Dancer [33], while the authors of [27] (when $k = 0$, $l \ne 0$) only studied the cases of $f_0 = \infty$, $f_\infty \in (0,\infty)$ by Rabinowitz [8]. Hence, Theorems 13–20 extend and improve Theorem 3.1 of [27] in some sense.

Remark 31. When $k = 0$, $l = \beta(t)$, Ma et al. [28] only studied the cases of $f_0, f_\infty \in (0,\infty)$. Furthermore, one can study the cases of $f_0, f_\infty \notin (0,\infty)$ of [28] by using similar methods in this paper.

Remark 32. The authors of [30, 31] only studied the cases of $f_0, f_\infty \in (0,\infty)$ and $f_0 = \infty$, $f_\infty \in (0,\infty)$ by Rabinowitz [8], respectively, while we consider the cases of $f_0, f_\infty \notin (0,\infty)$ by Dancer [33]. Hence, Theorems 13–20 extend and improve the corresponding results of [30, 31] in some sense.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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