Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 815765, 8 pages
http://dx.doi.org/10.1155/2013/815765

Research Article

Drive-Response Synchronization of a Fractional-Order Hyperchaotic System and Its Circuit Implementation

Darui Zhu,[1,2] Chongxin Liu,[1,2] and Bingnan Yan[1,2]

[1] *State Key Laboratory of Electrical Insulation and Power Equipment, Xi'an Jiaotong University, Xi'an 710049, China*
[2] *School of Electrical Engineering, Xi'an Jiaotong University, Xi'an 710049, China*

Correspondence should be addressed to Darui Zhu; zdarui@163.com

Received 1 June 2013; Accepted 21 July 2013

Academic Editor: Marek Lefik

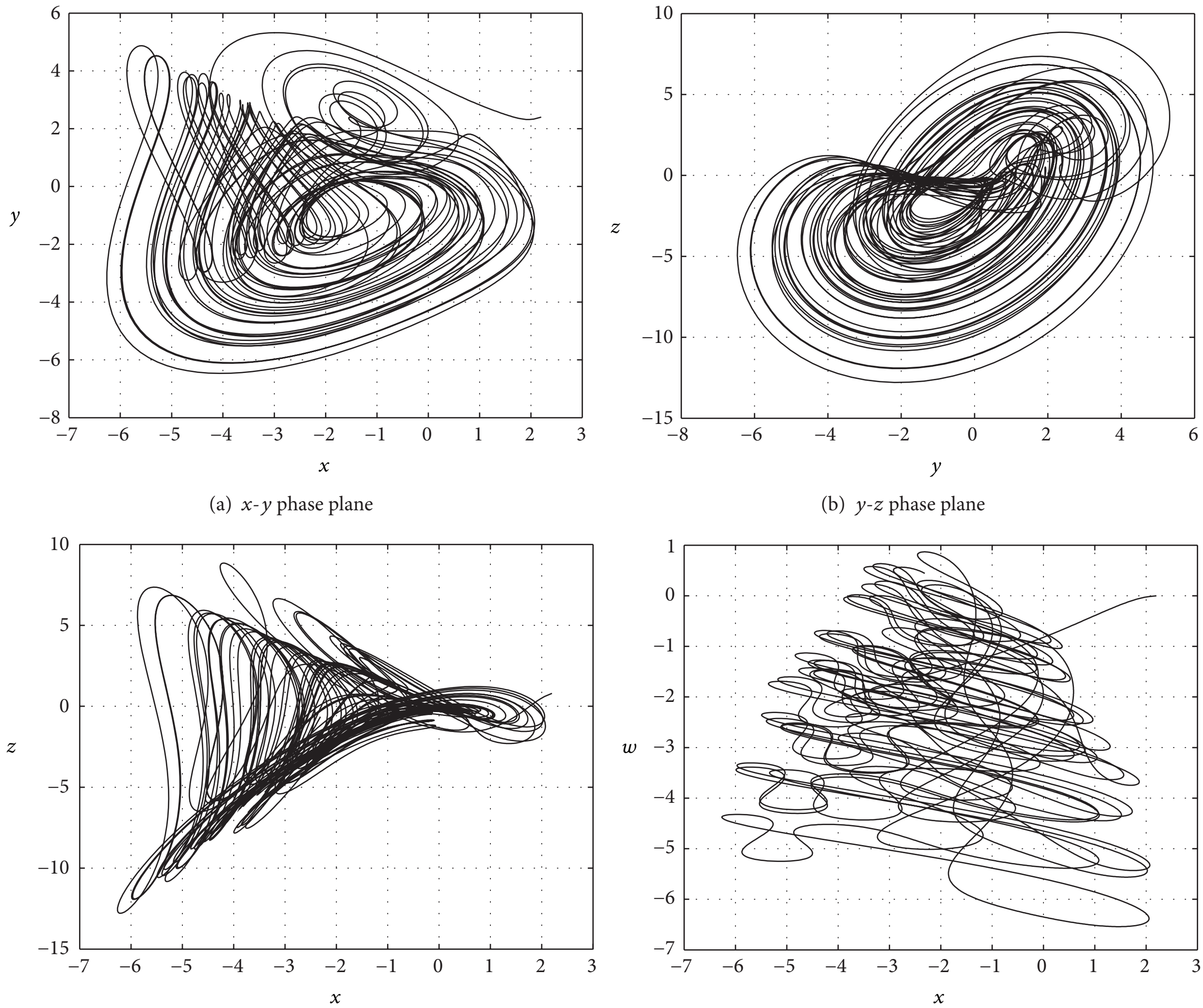
A novel fractional-order hyperchaotic system is proposed; the theoretical analysis and numerical simulation of this system are studied. Based on the stability theory of fractional calculus, we propose a novel drive-response synchronization scheme. In order to achieve this synchronization control, the Adams-Bashforth-Moulton algorithm is studied. And then, a drive-response synchronization controller is designed to realize the synchronization of the drive and response system, and the simulation results are given. At last, the fractional oscillator circuit of the new fractional-order hyperchaotic system is designed based on the EWB software, and it is verified that the simulation results of the fractional-order oscillator circuit are consistent with the numerical simulation results through circuit simulation.

1. Introduction

Fractional-order calculus is a mathematical theory which studies the characteristic and application of the arbitrary-order differential and integral operator. It has the same history as the integer-order calculus and can be dated back to the 17th century [1, 2]. It has attracted more researchers' interest and has more broad application prospects due to its unique advantages. But until the last 20 years, the fractional-order calculus theory has been applied to the practical projects; it has been applied to the chaos system, electromagnetism, signal processing, mechanical engineering, robot control, and so on.

Since hyperchaos was firstly reported by Rossler [3], there have been considerable achievements in the study of hyperchaos. Hyperchaotic systems have more than one positive Lyapunov exponent, and the strange attractor is usually unstable in more than one direction. Hyperchaotic system is a high-dimensional chaotic system. The fractional-order differential operator is introduced into hyperchaotic system which can reflect the hyperchaotic system with complicated nonlinear dynamic characteristics. Fractional-order hyperchaotic system implementation and application have been attracted more researchers' interest and in-depth study. In recent years, the fractional-order chaotic dynamical systems began to attract more researchers' attention, such as the fractional-order Chua system, the fractional-order Lorenz system, Chen's system, and Liu's system and so on [4–9].

In this paper, a novel fractional-order hyperchaotic system is presented. Based on the stability theory of fractional calculus, a novel drive-response synchronization scheme is proposed. Section 4 studied the Adams-Bashforth-Moulton algorithm to achieve this synchronization scheme. The fractional oscillator circuit of the new hyperchaotic system is designed based on the EWB software in Section 5, and the simulation results are given to demonstrate that the fractional-order oscillator is a hyperchaotic system. Finally, conclusions end this paper.



(a) x-y phase plane
(b) y-z phase plane
(c) x-z phase plane
(d) x-w phase plane

Figure 1: 2D strange attractors of the fractional-order hyperchaotic system: (a) x-y phase plane; (b) y-z phase plane; (c) x-z phase plane; (d) x-w phase plane.

2. A Novel Fractional-Order Hyperchaotic System

A novel four-dimensional fractional-order hyperchaotic system is given as follows:

$$\begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= y - x - y^2 - w, \\ \frac{d^\alpha y}{dt^\alpha} &= ay + xz + w, \\ \frac{d^\alpha z}{dt^\alpha} &= -bxy - cz + w, \\ \frac{d^\alpha w}{dt^\alpha} &= -y, \end{aligned} \tag{1}$$

where $(a, b, c) = (2.5, 4, 4)$ and α is the fractional order satisfying $0 < \alpha < 1$.

By calculating, this system has two positive Lyapunov exponents: $\mathrm{LE}_1 = 0.7980$ and $\mathrm{LE}_1 = 0.1841$, and the negative Lyapunov exponents are $\mathrm{LE}_3 = -0.9656$ and $\mathrm{LE}_4 = -2.5164$. According to chaos theory, there are two positive Lyapunov exponents implying that its dynamics is expanded in more than one direction simultaneously and that the system is a hyperchaotic system.

Based on the Lyapunov exponents, we can calculate the Hausdorff dimension (Lyapunov dimension) of the nonlinear autonomous system.

$$D_L = j + \frac{1}{\left|\mathrm{LE}_{j+1}\right|}\sum_{i=1}^{j}\mathrm{LE}_i = 3 + \frac{\mathrm{LE}_1 + \mathrm{LE}_2 + \mathrm{LE}_3}{\left|\mathrm{LE}_4\right|} = 3.007. \tag{2}$$

In this paper, the following simulations are all performed by using $\alpha = 0.9$, $(a, b, c) = (2.5, 4, 4)$. By simulations, we have obtained the 2D phase portraits of the fractional-order system as shown in Figure 1 and the time domain waveform of the $x(t)$ as shown in Figure 2. These figures clearly show

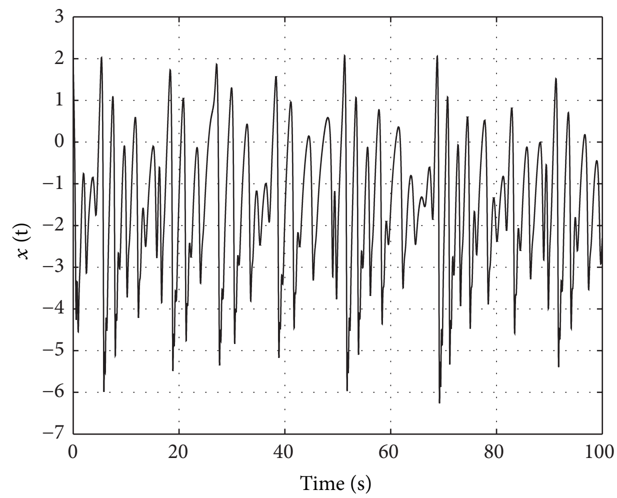


Figure 2: The time domain waveform of the $x(t)$.

that the fractional-order hyperchaotic system exhibits chaotic behaviors.

3. The Theory of the Fractional Drive-Response Synchronization

In this section, the drive-response method is used to realize synchronization control of the fractional-order system, assuming that the fractional order chaotic system is studied in the form of the driving equation

$$\frac{d^{\alpha}x}{dt^{\alpha}} = Ax + Bf(Cx) + D, \tag{3}$$

where $0 < \alpha < 1$, $x \in R^n$ is the column vector, $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{1\times n}$, and $D \in R^{n\times 1}$ are the continuous matrixes, and $Cx \in R$, $f : R^1 \rightarrow R^m$ is the nonlinear function.

It is worth noting that we must select the appropriate driving signal to drive the response system; generally we select the linear term which can be written as follows:

$$y = Cx. \tag{4}$$

The fractional-order response system is defined

$$\frac{d^{\alpha}\tilde{x}}{dt^{\alpha}} = A\tilde{x} + Bf(y) + D + K(y - \tilde{y}), \tag{5}$$

where $\tilde{y} = C\tilde{x}$, $\tilde{x} \in R^n$ are the column vectors and $K = [k_1, k_2, \ldots, k_n]^T$ are the synchronization control parameters. The error system is defined as $e(t) = x(t) - \tilde{x}(t)$. By (3), (4), and (5), the error equation of the two fractional order chaotic systems can be obtained as follows:

$$\frac{d^{\alpha}e}{dt^{\alpha}} = (A - KC)e. \tag{6}$$

Obviously, the system (6) and the following system (7) have the same stability:

$$\frac{d^{\alpha}\hat{e}}{dt^{\alpha}} = \left(A^T - C^T K^T\right)\hat{e}, \tag{7}$$

where $u(t) = -K^T\hat{e}(t)$ is the feedback control signal; it determines the synchronization effect of the fractional-order system. Based on the stability theory of the fractional-order system, we propose a novel synchronization theorem [10, 11].

Theorem 1. *If the fractional order linear system* (7) *is asymptotically stable, the necessary and sufficient condition is that all the eigenvalues* λ_{ncj} *of the coefficient matrix* $(A^T - C^T K^T)$ *are satisfied:*

$$\frac{\alpha\pi}{2} < \left|\arg\left(\lambda_{ncj}\right)\right| < \frac{3\alpha\pi}{2}, \tag{8}$$

where $\arg(\lambda_{ncj})$ *is the explement of the eigenvalues* λ_{ncj}.

Proof. If the coefficient matrix $M = (A^T - C^T K^T)$ of the system (7) has n different eigenvalues λ_i $(i = 1, \ldots, n)$, there exists a nonsingular transformation matrix, which makes the system (7) convert into

$$\frac{d^{\alpha}\bar{e}(t)}{dt^{\alpha}} = \Lambda\bar{e}(t), \tag{9}$$

where $\bar{e}(t) = T\hat{e}(t)$, $\Lambda = TMT^{-1} = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $\sigma_{\max}$ is the maximum singular value of T; and $\sigma_{\min}$ is the minimum singular value of T, hence

$$\sigma_{\min}\|\hat{e}(t)\| \le \|\bar{e}(t)\| \le \sigma_{\max}\|\hat{e}(t)\|. \tag{10}$$

The analytical solution of the system (9) can be expressed as the function $E_{\alpha,\beta}(z)$ as follows:

$$\bar{e}(t) = E_{\alpha,1}\left(\Lambda t^{\alpha}\right)\bar{e}(0). \tag{11}$$

If $\|\bar{e}(t)\| \rightarrow 0$ is established, it is necessary that $E_{\alpha,1}(\lambda_i t^{\alpha}) \rightarrow 0$, and this is equivalent to that all the eigenvalues λ_{ncj} are satisfied: $\alpha\pi/2 < |\arg(\lambda_{\text{ncj}})| < 3\alpha\pi/2$.

Then, the proof is completed. □

4. Drive-Response Synchronization of the Fractional-Order System

The proposed fractional-order hyperchaotic system (1) as the drive system, the controlled fractional-order response system, is described:

$$\begin{aligned} \frac{d^{\alpha}\tilde{x}}{dt^{\alpha}} &= \tilde{y} - \tilde{x} - y^2 - \tilde{w} - k_1(\tilde{x} - x), \\ \frac{d^{\alpha}\tilde{y}}{dt^{\alpha}} &= a\tilde{y} + xz + \tilde{w} - k_2(\tilde{y} - y), \\ \frac{d^{\alpha}\tilde{z}}{dt^{\alpha}} &= -bxy - c\tilde{z} + \tilde{w} - k_3(\tilde{z} - z), \\ \frac{d^{\alpha}\tilde{w}}{dt^{\alpha}} &= -\tilde{y} - k_4(\tilde{w} - w), \end{aligned} \tag{12}$$

where k_1, k_2, k_3, and k_4 are the control parameters of the response system. The error system is defined: $e_1(t) = \tilde{x}(t) - x(t)$, $e_2(t) = \tilde{y}(t) - y(t)$, $e_3(t) = \tilde{z}(t) - z(t)$, and $e_4(t) = \tilde{w}(t) - w(t)$; based on the systems (1) and (12), we can obtain the error system:

$$\begin{aligned} \frac{d^{\alpha}e_1}{dt^{\alpha}} &= e_2 - e_1 - e_4 - k_1 e_1, \\ \frac{d^{\alpha}e_2}{dt^{\alpha}} &= ae_2 + e_4 - k_2 e_2, \\ \frac{d^{\alpha}e_3}{dt^{\alpha}} &= -ce_3 + e_4 - k_3 e_3, \\ \frac{d^{\alpha}e_4}{dt^{\alpha}} &= -e_2 - k_4 e_4, \end{aligned} \tag{13}$$

$$G = \begin{bmatrix} -1-k_1 & 1 & 0 & -1 \\ 0 & a-k_2 & 0 & 1 \\ 0 & 0 & -c-k_3 & 1 \\ 0 & -1 & 0 & -k_4 \end{bmatrix},$$

where $\alpha = 0.9$, $(a, b, c) = (2.5, 4, 4)$.

In order to achieve the synchronization control of the above system, the Adams-Bashforth-Moulton algorithm is applied to the systems (1) and (12). Consider the following differential equations:

$$\begin{aligned} \frac{d^{\alpha}y(t)}{dt^{\alpha}} &= f(t, y(t)), \quad 0 \le t \le T, \\ y^{(k)}(0) &= y_0^{(k)}, \quad k = 0, 1, \ldots, [\alpha] - 1. \end{aligned} \tag{14}$$

They are equivalent to the Volterra integral equation as follows [12, 13]:

$$y(t) = \sum_{k=0}^{[\alpha]-1} \frac{t^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-w)^{\alpha-1} f(w, y(w))\, dw. \tag{15}$$

Here, $h = T/N$, $t_n = nh$, $n = 0, 1, \ldots, N \in Z$, and (15) can be discretized into [6]:

$$\begin{aligned} y_h(t_{n+1}) = &\sum_{k=0}^{[\alpha]-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \left(f\left(t_{n+1}, y_h^*(t_{n+1})\right)\right) \\ &+ \sum_{j=0}^{n} a_{j,n+1} f\left(t_j, y_h(t_j)\right), \end{aligned} \tag{16}$$

$$\begin{aligned} a_{j,n+1} &= \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, & j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} & 1 \le j \le n, \\ \quad -2(n-j+1)^{\alpha+1}, & \\ 1, & j = n+1, \end{cases} \\ y_h^*(t_{n+1}) &= \sum_{k=0}^{[\alpha]-1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f\left(t_j, y_h(t_j)\right), \\ b_{j,n+1} &= \frac{h^{\alpha}}{\alpha}\left((n+1-j)^{\alpha} - (n-j)^{\alpha}\right). \end{aligned} \tag{17}$$

The error equation is, $\max_{j=0,1,\ldots,N} |y(t_j) - y_h(t_j)| = O(h^p)$, where $p = \min(2, 1+\alpha)$.

By (14), (15), and (16), we can obtain the discrete form of the drive system (1) and response system (12):

$$\begin{aligned} x_{n+1} = x_0 &+ \frac{h^{\alpha}}{\Gamma(\alpha+2)} \\ &\cdot \left(\left(y_{n+1}^* - x_{n+1}^* - {y_{n+1}^*}^2 - w_{n+1}^*\right) \right. \\ &\left. + \sum_{j=0}^{n} a_{j,n+1} \cdot \left(y_j - x_j - y_j^2 - w_j\right) \right), \\ y_{n+1} = y_0 &+ \frac{h^{\alpha}}{\Gamma(\alpha+2)} \\ &\cdot \left(\left(a y_{n+1}^* + x_{n+1}^* z_{n+1}^* + w_{n+1}^*\right) \right. \\ &\left. + \sum_{j=0}^{n} a_{j,n+1} \cdot \left(a y_j + x_j z_j + w_j\right) \right), \end{aligned}$$

$$z_{n+1} = z_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \cdot \left(\left(-bx^*_{n+1}y^*_{n+1} - cz^*_{n+1} + w^*_{n+1}\right) + \sum_{j=0}^{n} a_{j,n+1} \cdot \left(-bx_jy_j - cz_j + w_j\right) \right),$$

$$w_{n+1} = w_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \left(-y^*_{n+1} + \sum_{j=0}^{n} a_{j,n+1} \cdot \left(-y_j\right) \right),$$

$$\tilde{x}_{n+1} = \tilde{x}_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \cdot \left(\tilde{y}^*_{n+1} - \tilde{x}^*_{n+1} - {y^*_{n+1}}^2 - \tilde{w}^*_{n+1} - k_1\left(\tilde{x}^*_{n+1} - x^*_{n+1}\right) + \sum_{j=0}^{n} a_{j,n+1} \cdot \left(\tilde{y}_j - \tilde{x}_j - y_j^2 - \tilde{w}_j - k_1\left(\tilde{x}_j - x_j\right)\right) \right),$$

$$\tilde{y}_{n+1} = \tilde{y}_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \cdot \left(a\tilde{y}^*_{n+1} + x^*_{n+1}z^*_{n+1} + \tilde{w}^*_{n+1} - k_2\left(\tilde{y}^*_{n+1} - y^*_{n+1}\right) + \sum_{j=0}^{n} a_{j,n+1} \cdot \left(a\tilde{y}_j + x_jz_j + \tilde{w}_j - k_2\left(\tilde{y}_j - y_j\right)\right) \right),$$

$$\tilde{z}_{n+1} = \tilde{z}_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \cdot \Big(-bx^*_{n+1}y^*_{n+1} - c\tilde{z}^*_{n+1} + \tilde{w}^*_{n+1} - k_3\left(\tilde{z}^*_{n+1} - z^*_{n+1}\right) + \sum_{j=0}^{n} a_{j,n+1} \cdot \left(-bx_jy_j - c\tilde{z}_j + \tilde{w}_j - k_3\left(\tilde{z}_j - z_j\right)\right)\Big),$$

$$\tilde{w}_{n+1} = \tilde{w}_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \cdot \left(-\tilde{y}^*_{n+1} - k_4\left(\tilde{w}^*_{n+1} - w^*_{n+1}\right) + \sum_{j=0}^{n} a_{j,n+1} \cdot \left(-y_j - k_4\left(\tilde{w}_j - w_j\right)\right) \right),$$

$$x^*_{n+1} = x_0 + \frac{1}{\Gamma(\alpha)} \cdot \sum_{j=0}^{n} b_{j,n+1} \cdot \left(y_j - x_j - y_j^2 - w_j\right),$$

$$y^*_{n+1} = y_0 + \frac{1}{\Gamma(\alpha)} \cdot \sum_{j=0}^{n} b_{j,n+1} \cdot \left(ay_j + x_jz_j + w_j\right),$$

$$z^*_{n+1} = z_0 + \frac{1}{\Gamma(\alpha)} \cdot \sum_{j=0}^{n} b_{j,n+1} \cdot \left(-bx_jy_j - cz_j + w_j\right),$$

$$w^*_{n+1} = w_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} \cdot \left(-y_j\right),$$

$$\tilde{x}^*_{n+1} = \tilde{x}_0 + \frac{1}{\Gamma(\alpha)} \cdot \sum_{j=0}^{n} b_{j,n+1} \cdot \left(\tilde{y}_j - \tilde{x}_j - y_j^2 - \tilde{w}_j - k_1\left(\tilde{x}_j - x_j\right)\right),$$

$$\tilde{y}^*_{n+1} = \tilde{y}_0 + \frac{1}{\Gamma(\alpha)} \cdot \sum_{j=0}^{n} b_{j,n+1} \cdot \left(a\tilde{y}_j + x_jz_j + \tilde{w}_j - k_2\left(\tilde{y}_j - y_j\right)\right),$$

$$\tilde{z}^*_{n+1} = \tilde{z}_0 + \frac{1}{\Gamma(\alpha)} \cdot \sum_{j=0}^{n} b_{j,n+1} \cdot \left(-bx_jy_j - c\tilde{z}_j + \tilde{w}_j - k_3\left(\tilde{z}_j - z_j\right)\right),$$

$$\tilde{w}^*_{n+1} = \tilde{w}_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} \cdot \left(-\tilde{y}_j - k_4\left(\tilde{w}_j - w_j\right)\right),$$

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, & 1 \le j \le n, \\ 1, & j = n+1, \end{cases}$$

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}\left[(n-j+1)^\alpha - (n-j)^\alpha\right], \quad 0 \le j \le n+1. \tag{18}$$

When $k_1 = 10$, $k_2 = 20$, $k_3 = 10, k_4 = 40$ and the values of a, b, c and k_1, k_2, k_3, k_4 are substituted into the system (13), and we can obtain the following: $\arg(\lambda_1(G)) = \pi$,

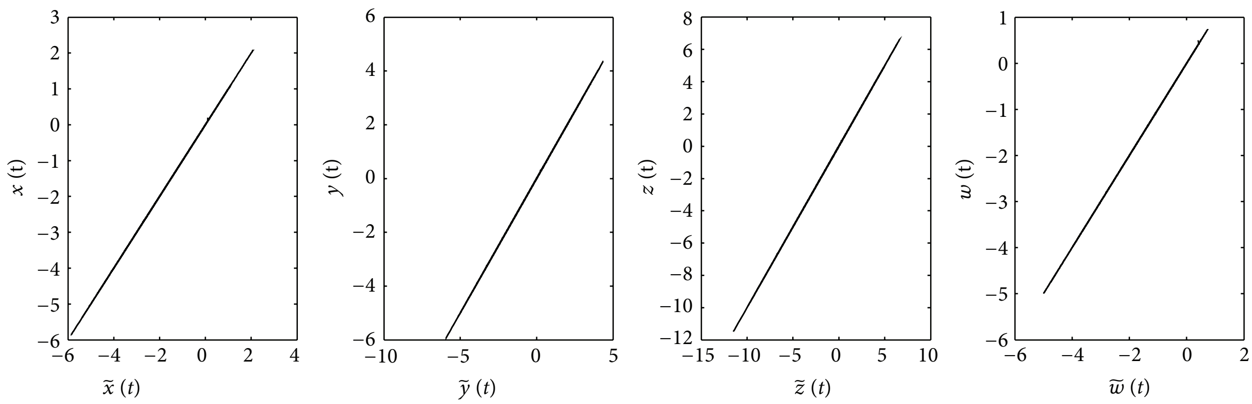


FIGURE 3: The synchronization phase diagram of the systems (1) and (12) when $k_1 = 10$, $k_2 = 20$, $k_3 = 10$, and $k_4 = 40$.

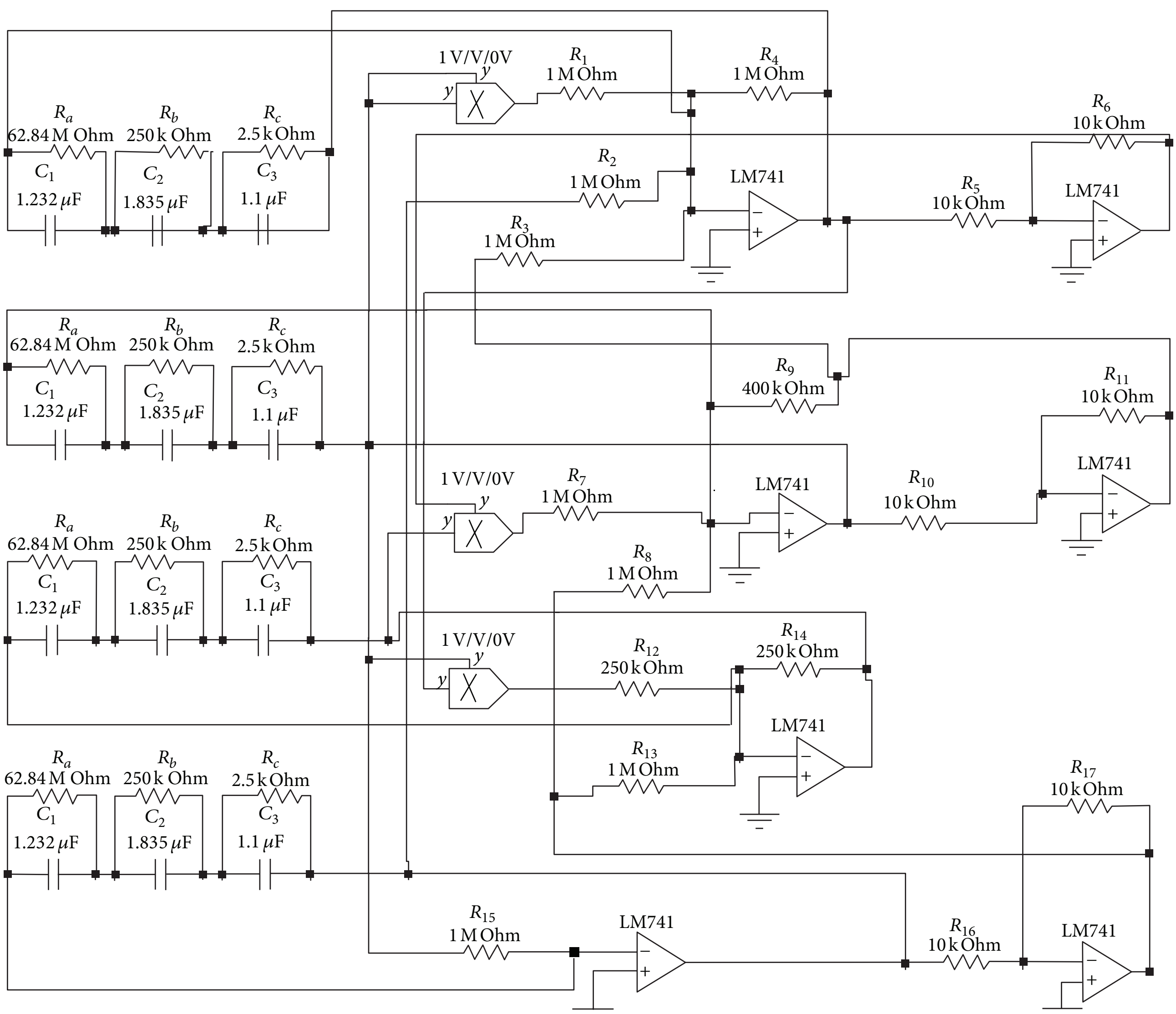
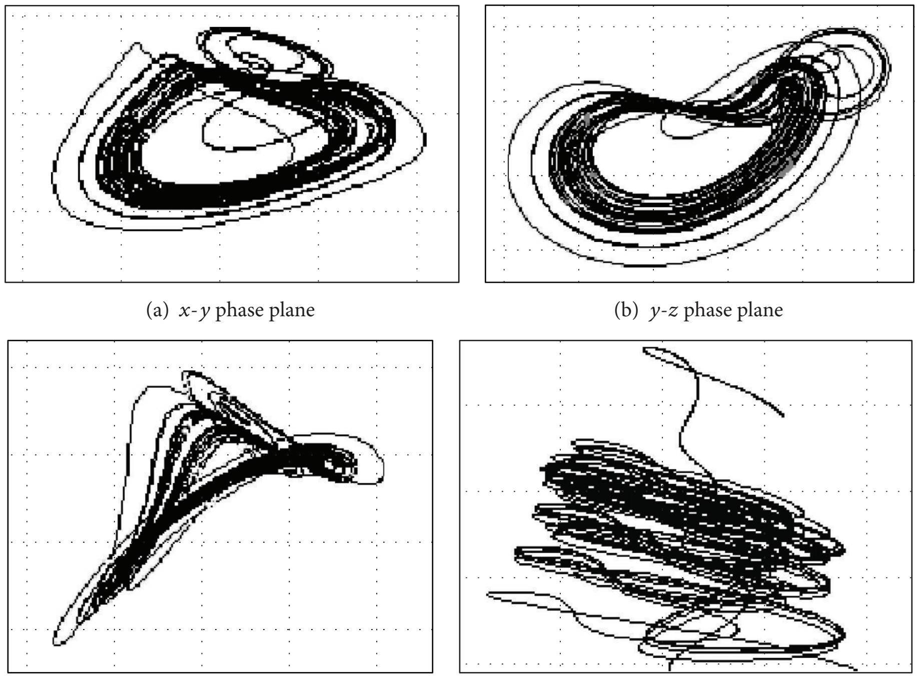


FIGURE 4: The circuit realization of the fractional-order system.



(a) x-y phase plane

(b) y-z phase plane

(c) x-z phase plane

(d) x-w phase plane

Figure 5: 2D strange attractors of the fractional-order hyperchaotic system by EWB: (a) x-y phase plane; (b) y-z phase plane; (c) x-z phase plane; (d) x-w phase plane.

$\arg(\lambda_2(G)) = \pi$, $\arg(\lambda_3(G)) = \pi$, and $\arg(\lambda_4(G)) = \pi$, which are satisfied $0.45\pi < |\arg(\lambda_i(G))| < 1.35\pi$. The drive system (1) and response system (12) realized the synchronization. The initial values of the system (1) and the system (12) are $x(0) = 0.1$, $y(0) = 0.2$, $z(0) = 0.3$, $w(0) = 0.4$ and $\tilde{x}(0) = 0.2$, $\tilde{y}(0) = 0.3$, $\tilde{z}(0) = 0.4$, $\tilde{w}(0) = 0.5$, respectively. Based on the above discrete form of the two systems, we can get the synchronization phase diagram by using MATLAB when $k_1 = 10$, $k_2 = 20$, $k_3 = 10$, and $k_4 = 40$ as shown in Figure 3.

5. Circuit Implementation of the Fractional-Order Hyperchaotic System

In the circuit design, we use linear resistor, capacitor, linear operational amplifier (LM741), analog multiplier (AD633), and the fractional unit circuit structure in the fractional-order hyperchaotic circuits. The linear operational amplifier is used for addition and subtraction, and the analog multiplier is used to realize the nonlinear term. Then, we designed the circuit of fractional-order system (6) with $\alpha = 0.9$, as shown in Figure 4.

The resistance values of this circuit are $(R_1, R_2, R_3, R_4, R_7, R_8, R_{13}, R_{15}) = 1\,\text{M}\Omega$, $(R_5, R_6, R_{10}, R_{11}, R_{16}, R_{17}) = 10\,\text{k}\Omega$, $R_9 = 400\,\text{k}\Omega$, and $(R_{12}, R_{14}) = 250\,\text{k}\Omega$. The component parameter values of the fractional order system with $\alpha = 0.9$ are as follows [14, 15]:

$$\begin{aligned} R_a &= 62.84\,\text{M}\Omega, \quad & R_b &= 250\,\text{k}\Omega, \quad & R_c &= 2.5\,\text{k}\Omega, \\ C_1 &= 1.232\,\text{uF}, \quad & C_2 &= 1.835\,\text{uF}, \quad & C_3 &= 1.1\,\text{uF}. \end{aligned} \tag{19}$$

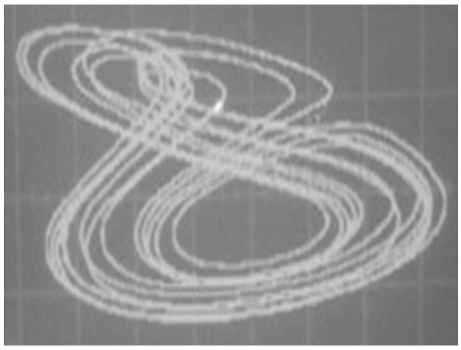
From Figure 5, the fractional-order hyperchaotic system oscillator circuit simulation results are consistent with the MATLAB numerical simulation results. Figure 6 shows phase diagrams of the integer-order hyperchaotic system based on the circuit experimental.

6. Conclusions

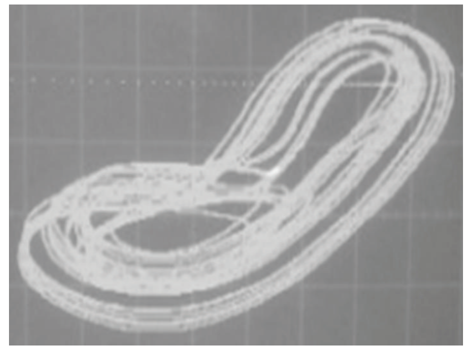
In this paper, we proposed a new fractional-order hyperchaotic system and analyzed the basic properties of this new system. Based on the stability theory of fractional calculus, we proposed a novel drive-response synchronization scheme and studied the Adams-Bashforth-Moulton algorithm. The drive-response synchronization controller is designed to realize synchronization based on the discrete form of the two systems by MATLAB software. And based on the fractional-order unit circuit, we designed the fractional-order oscillator circuit of this system by EWB software. The simulation results of this circuit are consistent with the numerical simulation results. This fractional-order hyperchaotic circuit can be used for the other electronic oscillator and the controller can be used for application in the chaos control due to its simple construction.

Acknowledgments

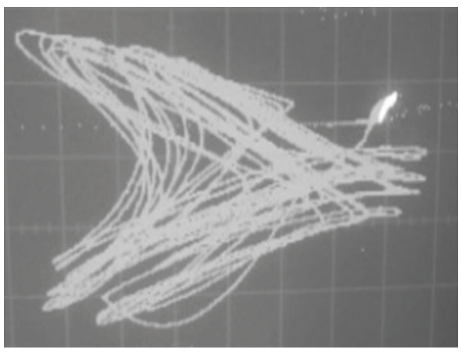
This paper is supported by the National Natural Science Foundation of China (no. 51177117), the Creative Research Groups Fund of the National Natural Science Foundation of



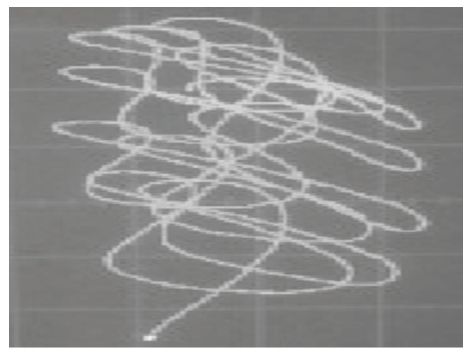
(a) x-y phase plane



(b) y-z phase plane



(c) x-z phase plane



(d) x-w phase plane

FIGURE 6: The circuit experimental phase diagram of the hyperchaotic system.

China (no. 51221005), and the Research Fund for the Doctoral Program of Higher Education of China (no. 20100201110023).

References

[1] P. L. Butzer and U. Westphal, "An introduction to fractional calculus," in *Applications of Fractional Calculus in Physics*, pp. 1–85, World Scientific Publishing, Singapore, 2000.

[2] J. T. Machado, V. Kiryakova, and F. Mainardi, "Recent history of fractional calculus," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1140–1153, 2011.

[3] O. E. Rossler, "An equation for hyperchaos," *Physics Letters A*, vol. 71, no. 2-3, pp. 155–157, 1979.

[4] T. T. Hartley, C. F. Lorenzo, and H. K. Qammer, "Chaos in a fractional order Chua's system," *IEEE Transactions on Circuits and Systems I*, vol. 42, no. 8, pp. 485–490, 1995.

[5] I. Grigorenko and E. Grigorenko, "Chaotic dynamics of the fractional Lorenz system," *Physical Review Letters*, vol. 91, no. 3, Article ID 034101, 2003.

[6] C. Li and G. Peng, "Chaos in Chen's system with a fractional order," *Chaos, Solitons & Fractals*, vol. 22, no. 2, pp. 443–450, 2004.

[7] X.-Y. Wang and M.-J. Wang, "Dynamic analysis of the fractional-order Liu system and its synchronization," *Chaos*, vol. 17, no. 3, Article ID 033106, 6 pages, 2007.

[8] L. M. Tam and W. M. S. Tou, "Parametric study of the fractional-order Chen-Lee system," *Chaos, Solitons & Fractals*, vol. 37, no. 3, pp. 817–826, 2008.

[9] W. H. Deng and C. P. Li, "Chaos synchronization of the fractional Lü system," *Physica A*, vol. 353, no. 1–4, pp. 61–72, 2005.

[10] D. Matignon, "Stability results of fractional differential equations with applications to control processing," in *Proceedings of the IEEE International Conference on System, Man, and Cybernetics*, Lille, France, 1996.

[11] M. S. Tavazoei and M. Haeri, "Chaotic attractors in incommensurate fractional order systems," *Physica D*, vol. 237, no. 20, pp. 2628–2637, 2008.

[12] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[13] K. Diethelm and N. J. Ford, "Analysis of fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 265, no. 2, pp. 229–248, 2002.

[14] C. X. Liu, "A hyperchaotic system and its fractional order circuit simulation," *Acta Physica Sinica*, vol. 56, no. 12, pp. 6865–6873, 2007.

[15] C.-X. Liu and L. Liu, "Circuit implementation of a new hyperchaos in fractional-order system," *Chinese Physics B*, vol. 17, no. 8, pp. 2829–2836, 2008.