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Research Article

Exact Solutions of Travelling Wave Model via Dynamical System Method

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By using the method of dynamical system, the exact travelling wave solutions of the coupled nonlinear Schrödinger-Boussinesq equations are studied. Based on this method, the bounded exact travelling wave solutions are obtained which contain solitary wave solutions and periodic travelling wave solutions. The solitary wave solutions and periodic travelling wave solutions are expressed by the hyperbolic functions and the Jacobian elliptic functions, respectively. The results show that the presented findings improve the related previous conclusions. Furthermore, the numerical simulations of the solitary wave solutions and the periodic travelling wave solutions are given to show the correctness of our results.

1. Introduction

In laser and plasma physics, the significant problems under interactions between a nonlinear real Boussinesq field and a nonlinear complex Schrödinger field have been raised [1]. In particular, the investigation of the coupled Schrödinger-Boussinesq equations has attracted much attention of physicists and mathematicians. In [2], the existence of the solutions for the equations was investigated. In [3], Guo and Du studied the local and global well-posedness of the periodic boundary value problem for the nonlinear Schrödinger-Boussinesq system. In [4], the approximate solutions and conservation law for the nonlinear Schrödinger-Boussinesq equations have been studied. The study of the coupled nonlinear Schrödinger-Boussinesq equations plays a crucial role in the study of nonlinear scientific fields.

In this paper, we consider the following coupled nonlinear Schrödinger-Boussinesq equations [5]:

$$\begin{gathered} iE_t + E_{xx} + \gamma E = NE, \\ 3N_{tt} - N_{xxxx} + 3\left(N^2\right)_{xx} + \delta N_{xx} = \left(|E|^2\right)_{xx}, \end{gathered} \tag{1}$$

where γ, δ are real parameters. E is complex function and N is real function. Equations (1) are known to describe various physical processes in laser and plasma physics, such as Langmuir field amplitude, modulational instabilities, and intense electromagnetic waves [6]. The study of travelling wave solutions of the coupled Schrödinger-Boussinesq equations has also attracted much attention of physicists and mathematicians. In [7], Farah and Pastor used the (G'/G)-expansion method to construct travelling wave solutions for the equations. In [8], Chen and Xu used the F-expansion method to obtain some periodic wave solutions for (1). In [9], Cai et al. studied coupled equations (1) by the modified expansion method and so on. However, we notice that the previous authors did not study the nonlinear dynamics of (1) and did not find all possible travelling wave solutions. Therefore, it is essential to study the nonlinear dynamics of (1) and find all possible travelling wave solutions of (1). Here, we use the approach of dynamical system to solve (1) and to give some bounded travelling wave solutions of (1) [10, 11].

The rest of this paper is built up as follows. In Section 2, we give the description of the dynamical system method. In Section 3, we apply this method to solve (1) and numerical simulations are conducted for the solitary wave solutions and the periodic travelling wave solutions to (1) with the aid of the Maple software. Finally, a conclusion is given in Section 4.

2. The Dynamical System Method

In this section, we describe the dynamical system method for finding travelling wave solutions of nonlinear wave equations. Suppose an $(n+1)$-dimensional nonlinear partial differential equation is given as follows:

$$P\left(t, x_i, u_t, u_{x_i}, u_{x_i x_i}, u_{x_i x_j}, u_{tt}, \ldots\right) = 0, \quad i, j = 1, 2, \ldots, n. \tag{2}$$

The main steps of the dynamical system method are as follows.

Step 1 (reduction of (2)). Making a transformation $u(t, x_1, x_2, \ldots, x_n) = \phi(\xi)$, $\xi = \sum_{i=1}^{n} k_i x_i - ct$, (2) can be reduced to a nonlinear ordinary differential equation

$$D\left(\xi, \phi, \phi_\xi, \phi_{\xi\xi}, \phi_{\xi\xi\xi}, \ldots\right) = 0, \tag{3}$$

where k_i are nonzero constant and c is the wave speed. Integrating several times for (3), if it can be reduced to the second-order nonlinear ordinary differential equation

$$E\left(\phi, \phi_\xi, \phi_{\xi\xi}\right) = 0, \tag{4}$$

then let $\phi_\xi = d\phi/d\xi = y$; (4) can be reduced to a two-dimensional dynamical system

$$\begin{aligned} \frac{d\phi}{d\xi} &= y, \\ \frac{dy}{d\xi} &= f(\phi, y), \end{aligned} \tag{5}$$

where $f(\phi, y)$ is an integral expression or a fraction. If $f(\phi, y)$ is a fraction such as $f(\phi, y) = F(\phi, y)/g(\phi)$ and $g(\phi_s) = 0$, $dy/d\xi$ does not exist when $\phi = \phi_s$. Then we will make a transformation $d\xi = g(\phi) d\zeta$; system (5) can be rewritten as

$$\begin{aligned} \frac{d\phi}{d\zeta} &= g(\phi)\, y, \\ \frac{dy}{d\zeta} &= F(\phi, y), \end{aligned} \tag{6}$$

where ζ is a parameter. If (2) can be reduced to system (5) or (6), then we can go on to the next step.

Step 2 (discussion of bifurcations of phase portraits of system (5)). If system (5) is an integral system, systems (5) and (6) can be reduced to the differential equation

$$\begin{aligned} \frac{dy}{d\phi} &= \frac{f(\phi, y)}{y}, \\ \frac{dy}{d\phi} &= \frac{F(\phi, y)}{g(\phi)\, y} = \frac{f(\phi, y)}{y}, \end{aligned} \tag{7}$$

and then systems (5) and (6) have the same first integral (i.e., Hamiltonian) as follows:

$$H(\phi, y) = h, \tag{8}$$

where h is an integral constant. According to the first integral, we can get all kinds of phase portraits in the parametric space. Because the phase orbits that defined the vector fields of system (5) (or system (6)) determine all their travelling wave solutions of (2), we can investigate the bifurcations of phase portraits of system (5) (or system (6)) to seek the travelling wave solutions of (2). Usually, a periodic orbit always corresponds to a periodic wave solution; a homoclinic orbit always corresponds to a solitary wave solution; a heteroclinic orbit (or the so-called connecting orbit) always corresponds to kink (or antikink) wave solution. When we find all phase orbits, we can get the value of h or its range.

Step 3 (calculation of the first equation of system (5)). After h is determined, we can get the following relationship from (8):

$$y = y(\phi, h); \tag{9}$$

that is, $d\phi/d\xi = y(\phi, h)$. If expression (9) is an integral expression, then substituting it into the first term of (5) and integrating we obtain

$$\int_{\phi_0}^{\phi} \frac{d\varphi}{y(\varphi, h)} = \int_0^{\xi} d\tau, \tag{10}$$

where $\phi(0)$ and 0 are initial constants. Usually, the initial constants can be taken by a root of (9) or inflection points of the travelling waves. Taking proper initial constants and integrating (10), through the Jacobian elliptic functions [12], we can obtain the exact travelling wave solutions of (2).

From the above description of the "three-step method," we can see that solutions of (2) can be obtained by studying and solving the dynamical system simplified by (2). Therefore, this approach is called dynamical system method. The different nonlinear wave equations correspond to different dynamical systems. The different dynamical systems correspond to different travelling wave solutions. This is the whole process of the dynamical system method.

3. Travelling Wave Solutions of (1)

3.1. The Reduced Dynamical System of (1). Following the procedure described in Section 2, we solve (1) by using the dynamical system method.

Use the transformation

$$\begin{aligned} E(x, t) &= \phi(\xi)\, e^{i\eta}, \\ N(x, t) &= \psi(\xi), \end{aligned} \tag{11}$$

$$\xi = kx - ct, \quad \eta = px + lt,$$

where k, c, p, and l are travelling wave parameters. Substituting (11) into the first equation of (1), canceling $e^{i\eta}$, and separating the real and imaginary parts, we have

$$\begin{gathered}(2kp-c)\,\phi' = 0,\\ k^2\phi'' - (p^2+l-\gamma)\,\phi - \phi\psi = 0.\end{gathered} \tag{12}$$

Obviously, from (12), we know that if $\phi' = 0$, (1) has the trivial solution. Otherwise, (12) must be satisfied:

$$2kp - c = 0. \tag{13}$$

Substituting (11) into the second equation of (1) and integrating twice (integral constant is zero), we have

$$k^4\psi'' - (\delta k^2 + 3c^2)\,\psi - 3k^2\psi^2 + k^2\phi^2 = 0. \tag{14}$$

Therefore, (1) is reduced to

$$\begin{gathered}2kp - c = 0,\\ k^2\phi'' - (p^2+l-\gamma)\,\phi - \phi\psi = 0,\\ k^4\psi'' - (\delta k^2 + 3c^2)\,\psi - 3k^2\psi^2 + k^2\phi^2 = 0.\end{gathered} \tag{15}$$

It is very difficult to solve these equations by some ordinary methods because of the coupling of (1), so we consider the special transformation in subtle ways:

$$\phi = n\psi. \tag{16}$$

Here, n is a constant to be determined later. Although in the ordinary course of events there is not linear relation between ϕ and ψ, the transformation is a key point to look for the travelling wave solutions of (1). Substituting (16) into (15), (15) is changed into

$$\begin{gathered}2kp - c = 0,\\ k^2\psi'' - (p^2+l)\,\psi - \psi^2 = 0,\\ \beta k^3\psi'' - c\psi + \frac{\alpha k}{2}\psi^2 - m^2 k\psi^2 = 0.\end{gathered} \tag{17}$$

Compared with coefficients of the second equation and the third equation of (17), they follow

$$\begin{gathered}n = \pm\sqrt{2},\\ (p^2+l-\gamma) = \frac{\delta k^2 + 3c^2}{k^2},\\ 2kp - c = 0.\end{gathered} \tag{18}$$

A result of the freedom of these parameters is consistency; under condition (18), (17) is simplified to the following equation:

$$k^2\psi'' - (p^2+l-\gamma)\,\psi - \psi^2 = 0. \tag{19}$$

Suppose that $k \neq 0$ and write that $A = (p^2+l-\gamma)/k^2$, $B = 1/k^2$. Thus, (19) has the following form:

$$\psi'' - A\psi - B\psi^2 = 0, \tag{20}$$

which corresponds to the two-dimensional Hamiltonian system

$$\begin{aligned}\frac{d\psi}{d\xi} &= y,\\ \frac{dy}{d\xi} &= A\psi + B\psi^2.\end{aligned} \tag{21}$$

3.2. The Bifurcations of Phase Portraits of System (21). We divided by (21) and solve the Bernoulli one-order differential equation; we have the first integral

$$H(\psi, y) = \frac{1}{2}y^2 - \frac{1}{2}A\psi^2 - \frac{1}{3}B\psi^3. \tag{22}$$

Let the integral constant be h; that is, $H(\psi, y) = h$. Now we consider the phase portraits of (21). Let the right hand terms of system (21) be zeros; that is, $y = 0$ and $A\psi + B\psi^2 = 0$; we obtain that system (21) has two equilibrium points $S(-A/B, 0)$ and $O(0,0)$. For $H(\psi, y) = (1/2)y^2 - (1/2)A\psi^2 - (1/3)B\psi^3 = h$, we write $h_0 = H(0,0) = 0$, $h_1 = H(-A/B, 0) = -A^3/6B^2$. Because only bounded travelling waves are meaningful to a physical model, we just pay our attention to the bounded solutions of (1) which are physically acceptable. In addition, because of $B = 1/k^2 > 0$, here we just consider the phase portraits of (21) when $B > 0$. With the change of the parameter group of A and B, the system has different phase portraits for (21) which are shown in Figure 1 (drawn by software Maple).

From Figure 1, we summarize crucial conclusions as follows:

(1) when $A = 0$, O is a cusp; when $A > 0$ (< 0), O is a saddle point (center point) and S is a center (saddle point);

(2) if $A \neq 0$, system (21) has a unique homoclinic orbit Γ which is asymptotic to the saddle and enclosing the center;

(3) if $A \neq 0$, there is a family of periodic orbits which are enclosing the center and filling up the interior of the homoclinic orbit Γ.

In the first image of Figure 1, we take $A = 0$, $B > 0$. In the second image of Figure 1, we take $A < 0$, $B > 0$. In the third image of Figure 1, we take $A > 0$, $B > 0$.

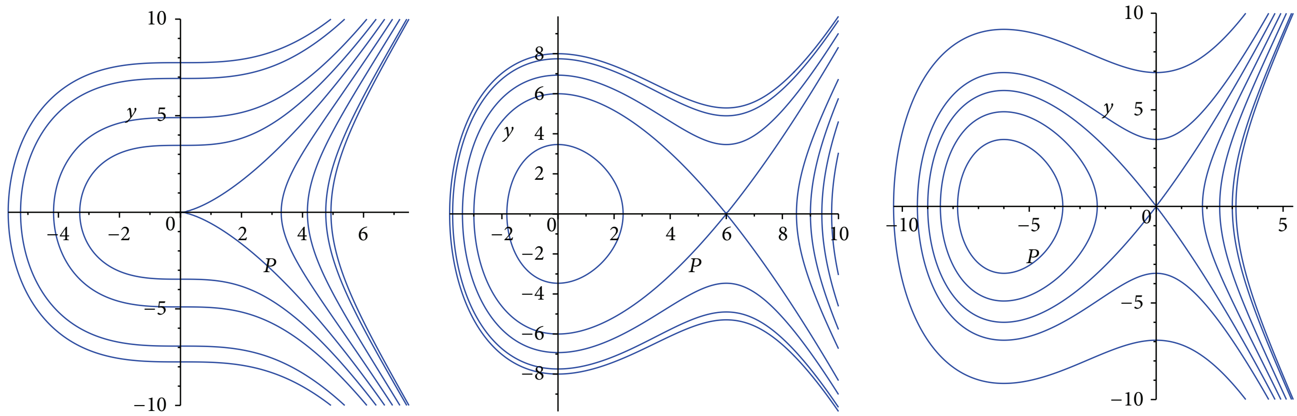

FIGURE 1: The bifurcations of phase portraits of (21).

3.3. The Exact Travelling Wave Solutions of (1). We can get the following relationship from (22):

$$y = \sqrt{A\psi^2 + \frac{2}{3}B\psi^3 + 2h}; \tag{23}$$

then substituting it into the first term of (21) and integrating we use the Maple software and the Jacobian elliptic functions; we have the following parametric representation.

(1) When $A > 0$, $B > 0$, and $h = h_0 = 0$ (see the third image of Figure 1), there exists a smooth solitary solution which corresponds to a smooth homoclinic orbit Γ of (21) defined by $H(\psi, y) = h_0 = 0$; we have the parametric representation:

$$\psi(\xi) = \frac{-3A + 3A\tanh^2\left(\left(\sqrt{A}/2\right)\xi\right)}{2B}. \tag{24}$$

(2) When $A < 0$, $B > 0$, and $h = h_1 = -A^3/6B^2$ (see the second image of Figure 1), there exists a smooth solitary solution which corresponds to a smooth homoclinic orbit Γ of (21) defined by $H(\psi, y) = h_1$; we have the parametric representation:

$$\psi(\xi) = \frac{|A|}{B} - \frac{3|A|}{2B}\operatorname{sech}^2\left(\frac{\sqrt{|A|}}{2}\xi\right). \tag{25}$$

(3) When $A > 0$ $(A < 0)$, $B > 0$, and $h \in (h_1, 0)$ $(h \in (0, h_1))$ (see the second image and third image of Figure 1), similarly, there exist periodic travelling wave solutions which correspond to the family of periodic orbits Γ^h of (21) defined by $H(\psi, y) = h, h \in (h_1, 0)$ $(h \in (0, h_1))$; we have the following parametric representation:

$$\psi(\xi) = z_3 + (z_2 - z_3)\operatorname{sn}^2\left(\frac{\sqrt{6B(z_1 - z_3)}}{6}\xi, \sqrt{\frac{z_2 - z_3}{z_1 - z_3}}\right), \tag{26}$$

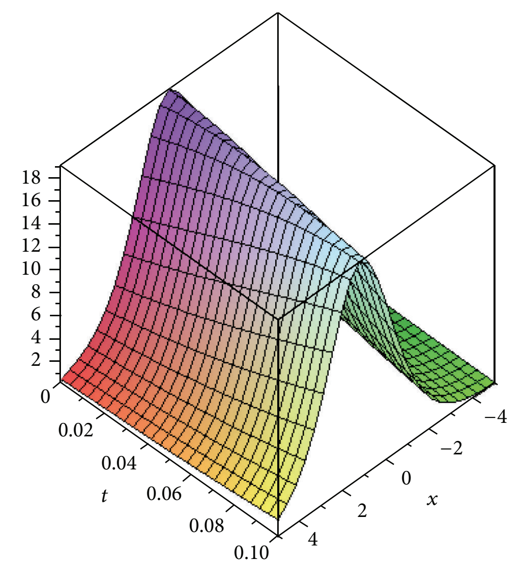

FIGURE 2: The 3D graphics of $|E_1|$ ($n = \sqrt{2}, k = p = 1, c = 2, l = -17, \gamma = -25, \delta = -3, -5 \le x \le 5$, and $0 \le t \le 0.1$).

where the parameters z_1, z_2, and z_3 and $z_1 > z_2 > z_3$ are defined by $y^2 = 2h + A\psi^2 + (2/3)B\psi^3 = (2/3)B(z_1 - \psi)(z_2 - \psi)(\psi - z_3)$.

By using the above results and considering condition (18), according to (11), we obtain the travelling wave solutions of (1) as follows. By using the numerical simulation method, the 3D graphics of bounded solutions of (1) are also shown in Figures 2–7.

(1) When $A > 0$, $B > 0$, and $h = h_0 = 0$,

$$\begin{aligned} E_1(x,t) &= -\frac{3An}{2B} + \frac{3An}{2B}\tanh^2\left(\frac{\sqrt{A}}{2}\xi\right)e^{i\eta}, \\ N_1(x,t) &= -\frac{3An}{2B} + \frac{3An}{2B}\tanh^2\left(\frac{\sqrt{A}}{2}\xi\right). \end{aligned} \tag{27}$$

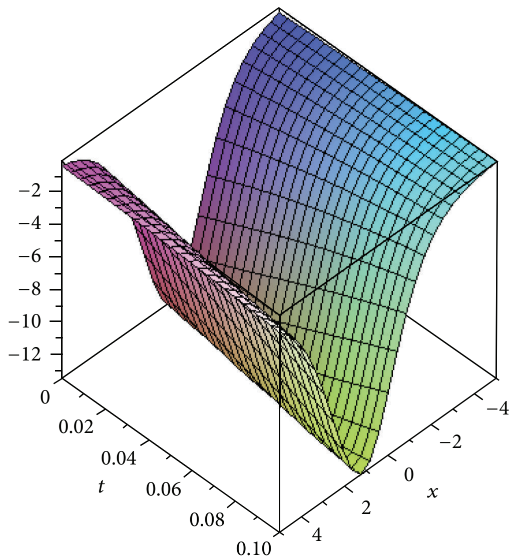

FIGURE 3: The 3D graphics of N_1 ($n = \sqrt{2}, k = p = 1, c = 2, l = -17, \gamma = -25, \delta = -3, -5 \le x \le 5$, and $0 \le t \le 0.1$).

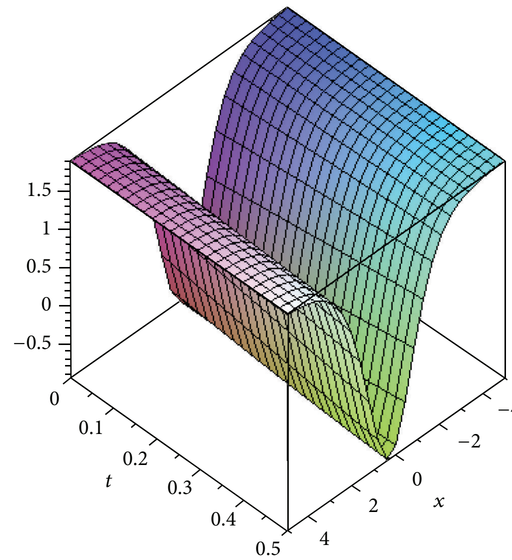

FIGURE 5: The 3D graphics of N_2 ($n = \sqrt{2}, k = 1, p = \sqrt{1/11}, l = -27, c = 2\sqrt{1/11}, \gamma = -25, \delta = -3, -5 \le x \le 5$, and $0 \le t \le 0.5$).

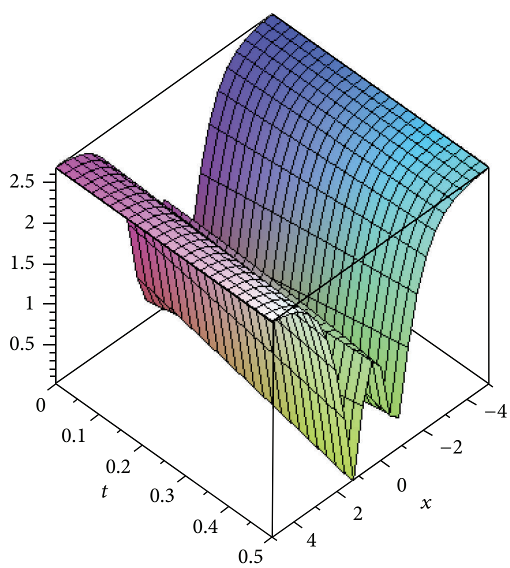

FIGURE 4: The 3D graphics of $|E_2|$ ($n = \sqrt{2}, k = 1, p = \sqrt{1/11}, l = -27, c = 2\sqrt{1/11}, \gamma = -25, \delta = -3, -5 \le x \le 5$, and $0 \le t \le 0.5$).

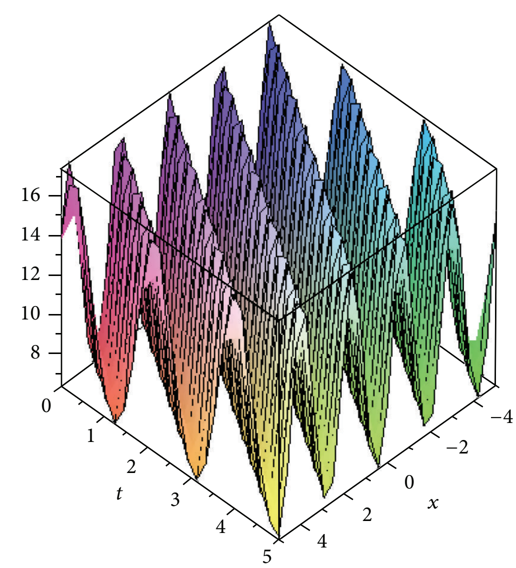

FIGURE 6: The 3D graphics of $|E_3|$ ($n = \sqrt{2}, k = p = 1, c = 2, l = -17, \gamma = -25, \delta = -3, h_1 = -243/4, -5 \le x \le 5$, and $0 \le t \le 5$).

(2) When $A < 0$, $B > 0$, and $h = h_1 = -A^3/6B^2$,

$$E_2(x,t) = \frac{n|A|}{B} - \frac{3n|A|}{2B}\operatorname{sech}^2\left(\frac{\sqrt{|A|}}{2}\xi\right)e^{i\eta},$$
$$N_2(x,t) = \frac{n|A|}{B} - \frac{3n|A|}{2B}\operatorname{sech}^2\left(\frac{\sqrt{|A|}}{2}\xi\right). \tag{28}$$

(3) When $A > 0$ ($A < 0$), $B > 0$, and $h \in (h_1, 0)$ ($h \in (0, h_1)$),

$$E_3(x,t) = nz_3 + n(z_2 - z_3) \cdot \operatorname{sn}^2\left(\frac{\sqrt{6B(z_1 - z_3)}}{6}\xi, \sqrt{\frac{z_2 - z_3}{z_1 - z_3}}\right)e^{i\eta},$$

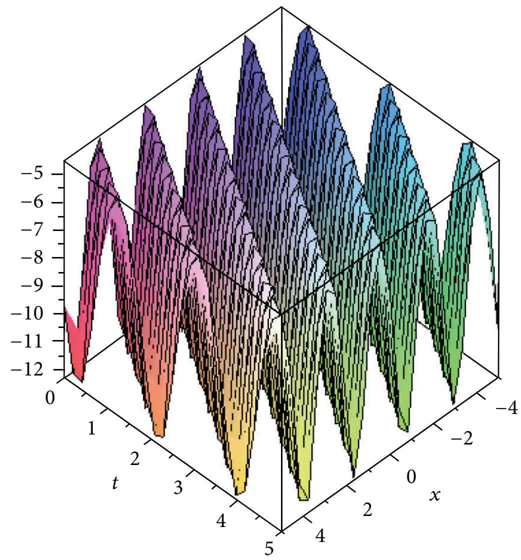

Figure 7: The 3D graphics of N_3 ($n = \sqrt{2}, k = p = 1, c = 2, l = -17, \gamma = -25, \delta = -3, h_1 = -243/4, -5 \le x \le 5$, and $0 \le t \le 5$).

$$N_3(x,t) = z_3 + (z_2 - z_3) \cdot \mathrm{sn}^2\left(\frac{\sqrt{6B(z_1 - z_3)}}{6}\xi, \sqrt{\frac{z_2 - z_3}{z_1 - z_3}}\right). \tag{29}$$

From Figures 2–7, it is easy to see that E_1, N_1, E_2, and N_2 are solitary wave solutions which are expressed by the hyperbolic functions. E_3 and N_3 are periodic travelling wave solution which is expressed by Jacobian elliptic functions. Note that our solutions are different from the given ones in [5, 8, 9].

4. Conclusion

To summarize, by using the dynamical system method, the bounded exact travelling wave solutions (solitary wave solutions and periodic wave solutions) have been obtained for the coupled nonlinear Schrödinger-Boussinesq equations. The dynamical system method is a good method to obtain exact solutions, which can not only obtain exact solutions but also understand nonlinear dynamics of travelling wave equations. We show that the hyperbolic function solutions and the Jacobian elliptic function solutions we found in this paper are different from the solutions presented by other authors before. The results enrich the diversity of wave structures of the coupled nonlinear Schrödinger-Boussinesq equations.

Furthermore, if there is not the condition that $B > 0$, system (21) has another case. When $A > 0$ ($A < 0$), $B < 0$, there exist periodic travelling wave solutions corresponding to the family of periodic orbits Γ^h of (21) defined by $H(\psi, y) = h, h \in (h_1, 0)$ ($h \in (0, h_1)$); we have following parametric representation:

$$\psi(\xi) = z_1 - (z_1 - z_2)\,\mathrm{sn}^2\left(\frac{\sqrt{6|B|(z_1 - z_3)}}{6}\xi, \sqrt{\frac{z_1 - z_2}{z_1 - z_3}}\right), \tag{30}$$

where the parameters z_1, z_2, and z_3 and $z_1 > z_2 > z_3$ are defined by $y^2 = 2h + A\psi^2 + (2/3)B\psi^3 = (2/3)|B|(z_1 - \psi)(\psi - z_2)(\psi - z_3)$. Then we obtain additional travelling wave solutions of (1) as follows:

$$\begin{aligned} E(x,t) &= nz_1 - n(z_1 - z_2) \cdot \mathrm{sn}^2\left(\frac{\sqrt{6|B|(z_1 - z_3)}}{6}\xi, \sqrt{\frac{z_1 - z_2}{z_1 - z_3}}\right)e^{i\eta}, \\ N(x,t) &= z_1 - (z_1 - z_2) \cdot \mathrm{sn}^2\left(\frac{\sqrt{6|B|(z_1 - z_3)}}{6}\xi, \sqrt{\frac{z_1 - z_2}{z_1 - z_3}}\right). \end{aligned} \tag{31}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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