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Research Article

Existence and Uniqueness of Positive Solutions for a Fractional Switched System

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We discuss the existence and uniqueness of positive solutions for the following fractional switched system: $({}^cD_{0+}^{\alpha}u(t)+f_{\sigma(t)}(t,u(t))+g_{\sigma(t)}(t,u(t))=0, t\in J=[0,1]); (u(0)=u''(0)=0, u(1)=\int_0^1 u(s)\,ds)$, where ${}^cD_{0+}^{\alpha}$ is the Caputo fractional derivative with $2<\alpha\le 3$, $\sigma(t):J\to\{1,2,\ldots,N\}$ is a piecewise constant function depending on t, and $\mathbb{R}^+=[0,+\infty)$, $f_i, g_i \in C[J\times\mathbb{R}^+,\mathbb{R}^+]$, $i=1,2,\ldots,N$. Our results are based on a fixed point theorem of a sum operator and contraction mapping principle. Furthermore, two examples are also given to illustrate the results.

1. Introduction

Fractional differential equations arise in various areas of science and engineering. Due to their applications, fractional differential equations have gained considerable attention (cf., e.g., [1–15] and references therein). Moreover, the theory of boundary value problems with integral boundary conditions has various applications in applied fields. For example, heat conduction, chemical engineering, underground water flow, thermoelasticity, and population dynamics can be reduced to the nonlocal problems with integral boundary conditions. In [2], Cabada and Wang considered the following m-point boundary value problem for fractional differential equation

$$\begin{gathered}{}^cD_{0+}^{\alpha}u(t)+f(t,u(t))=0,\quad 0<t<1,\\ u(0)=u''(0)=0,\qquad u(1)=\lambda\int_0^1 u(s)\,ds,\end{gathered}\tag{1}$$

where $2<\alpha\le 3$, ${}^cD_{0+}^{\alpha}$ is the Caputo fractional derivative, and $f:[0,1]\times[0,\infty)\to[0,\infty)$ is a continuous function.

On the other hand, a switched system consists of a family of subsystems described by differential or difference equations, which has many applications in traffic control, switching power converters, network control, multiagent consensus, and so forth (see [16–18]). When we consider a switched system, we always suppose that the solution is unique. So it is important to study the uniqueness of solution for a switched system. In [1], Li and Liu investigated the uniqueness of positive solution for the following switched system:

$$\begin{gathered}x''(t)+f_{\sigma(t)}(t,x(t))=0,\quad t\in J=[0,1],\\ x(0)=0,\qquad x(1)=\int_0^1 a(s)\,x(s)\,ds,\end{gathered}\tag{2}$$

where $\sigma(t):J\to\{1,2,\ldots,N\}$ is a piecewise constant function depending on t, and $\mathbb{R}^+=[0,+\infty)$, $f_i\in C[J\times\mathbb{R}^+,\mathbb{R}^+]$, $i=1,2,\ldots,N$.

In this paper, we discuss the existence and uniqueness of positive solutions for the following fractional switched system:

$$\begin{gathered}{}^cD_{0+}^{\alpha}u(t)+f_{\sigma(t)}(t,u(t))+g_{\sigma(t)}(t,u(t))=0,\\ t\in J=[0,1],\\ u(0)=u''(0)=0,\qquad u(1)=\int_0^1 u(s)\,ds,\end{gathered}\tag{3}$$

where ${}^{c}D_{0+}^{\alpha}$ is the Caputo fractional derivative with $2 < \alpha \le 3$, $\sigma(t) : J \to \{1, 2, \ldots, N\}$ is a piecewise constant function depending on t, and $\mathbb{R}^+ = [0, +\infty)$, $f_i, g_i \in C[J \times \mathbb{R}^+, \mathbb{R}^+]$, $i = 1, 2, \ldots, N$.

The paper is organized as follows. In Section 2, we present some background materials and preliminaries. Section 3 deals with some existence results. In Section 4, two examples are given to illustrate the results.

2. Background Materials and Preliminaries

Definition 1 (see [3, 4]). The fractional integral of order α with the lower limit t_0 for a function f is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s)\, ds, \qquad t > t_0,\ \alpha > 0, \quad (4)$$

where Γ is the gamma function.

Definition 2 (see [3, 4]). For a function $f : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order is defined as

$$ {}^{c}D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \qquad \alpha > 0, \quad n = [\alpha] + 1. \quad (5)$$

In the following, let us recall some basic information on cone (see more from [19, 20]). Let E be a real Banach space and let P be a cone in E which defined a partial ordering in E by $x \le y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$. P is called solid if its interior $\mathring{P}$ is nonempty. If $x \le y$ and $x \ne y$, we write $x < y$. We say that an operator A is increasing if $x \le y$ implies $Ax \le Ay$.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \le y \le \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \ge \theta$ and $h \ne \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$.

Definition 3. Let $D = P$ or $D = \mathring{P}$ and let γ be a real number with $0 \le \gamma < 1$. An operator $A : P \to P$ is said to be γ-concave if it satisfies

$$A(tx) \ge t^{\gamma} Ax, \qquad \forall t \in (0,1),\ x \in D. \quad (6)$$

Definition 4. An operator $A : E \to E$ is said to be homogeneous if it satisfies

$$A(tx) = tAx, \qquad \forall t > 0,\ x \in E. \quad (7)$$

An operator $A : P \to P$ is said to be subhomogeneous if it satisfies

$$A(tx) \ge tAx, \qquad \forall t \in (0,1),\ x \in P. \quad (8)$$

From [2], we have the following result.

Lemma 5. *Assume that $2 < \alpha \le 3$ and $f_i, g_i \in C[J \times \mathbb{R}^+, \mathbb{R}^+]$, $i = 1, 2, \ldots, N$. Then the problem* (3) *has a solution if and only if u is a solution of the integral equation*

$$u(t) = \int_0^1 G(t,s) \left(f_{\sigma(s)}(s, u(s)) + g_{\sigma(s)}(s, u(s)) \right) ds, \quad (9)$$

where

$$G(t,s) = \begin{cases} \dfrac{2t(1-s)^{\alpha-1}(\alpha-1+s) - \alpha(t-s)^{\alpha-1}}{\Gamma(\alpha+1)}, & 0 \le s \le t \le 1, \\ \dfrac{2t(1-s)^{\alpha-1}(\alpha-1+s)}{\Gamma(\alpha+1)}, & 0 \le t \le s \le 1. \end{cases} \quad (10)$$

Lemma 6. *$G(t,s)$ in Lemma 5 has the following property:*

(i) $G(t,s) > 0$ *for all* $t, s \in (0,1)$.

(ii) $(1/\Gamma(\alpha+1))h(t)(1-s)^{\alpha-1}(\alpha-2+2s) \le G(t,s) \le (2/\Gamma(\alpha+1))h(t)(1-s)^{\alpha-1}(\alpha-1+s)$, $t, s \in [0,1]$, $2 < \alpha \le 3$, $h(t) = t$.

Proof. From [2], we know that (i) is obvious. For $0 \le s \le t \le 1$, $2 < \alpha \le 3$, we have

$$\begin{aligned} &2t(1-s)^{\alpha-1}(\alpha-1+s) - \alpha(t-s)^{\alpha-1} \\ &\quad = 2t(1-s)^{\alpha-1}(\alpha-1+s) - \alpha t^{\alpha-1}\left(1 - \frac{s}{t}\right)^{\alpha-1} \\ &\quad \ge 2t(1-s)^{\alpha-1}(\alpha-1+s) - \alpha t(1-s)^{\alpha-1} \\ &\quad = t(1-s)^{\alpha-1}(\alpha-2+2s). \end{aligned} \quad (11)$$

This means that (ii) holds. □

Theorem 7 (see [19]). *Let P be a normal cone in a real Banach space E, $A : P \to P$ an increasing γ-concave operator, and $B : P \to P$ an increasing subhomogeneous operator. Assume that*

(i) *there is $h > \theta$ such that $Ah \in P_h$ and $Bh \in P_h$;*

(ii) *there exists a constant $\delta_0 > 0$ such that $Ax \ge \delta_0 Bx$, $\forall x \in P$.*

Then the operator equation $Ax + Bx = x$ has a unique solution x^* in P_h. Moreover, constructing successively the sequence $y_n = Ay_{n-1} + By_{n-1}$, $n = 1, 2, \ldots$, for any initial value $y_0 \in P_h$, we have $y_n \to x^*$ as $n \to \infty$.

3. Main Results

In this section, we will deal with the existence and uniqueness of positive solutions for problem (3). Let

$$G_1(s,s) = \frac{2}{\Gamma(\alpha+1)}(1-s)^{\alpha-1}(\alpha-1+s). \quad (12)$$

It is obvious that

$$G(t,s) \leq G_1(s,s), \quad t,s \in [0,1]. \tag{13}$$

We consider the Banach space $E = C[[0,1],\mathbb{R}]$ endowed with the norm defined by $\| u \| = \sup_{0\leq t\leq 1}|u(t)|$. Letting $P = \{u \in E \mid u(t) \geq 0\}$, then P is a cone in E. Define an operator $\mathcal{F} : E \to E$ as

$$(\mathcal{F}u)(t) = \int_0^1 G(t,s)\left(f_{\sigma(s)}(s,u(s)) + g_{\sigma(s)}(s,u(s))\right) ds. \tag{14}$$

Then $\mathcal{F}$ has a solution if and only if the operator $\mathcal{F}$ has a fixed point.

Theorem 8. *Let $f_i, g_i \in C[J \times \mathbb{R}^+, \mathbb{R}^+]$, $i = 1,2,\ldots,N$. Suppose that the following conditions are satisfied:*

$$\begin{gathered}\left|f_i(t,u(t)) - f_i(t,v(t))\right| \leq l_i(t)|u(t)-v(t)|, \\ \left|g_i(t,u(t)) - g_i(t,v(t))\right| \leq \bar{l}_i(t)|u(t)-v(t)|, \\ 0 < \int_0^1 G_1(s,s)\left(l_i(s)+\bar{l}_i(s)\right) ds < 1,\end{gathered} \tag{15}$$

where

$$l_i, \bar{l}_i \in C[J,\mathbb{R}^+], \quad i = 1,2,\ldots,N. \tag{16}$$

Then the problem (3) *has a unique solution on* $[0,1]$.

Proof. It follows from Lemma 6 that $\mathcal{F} : P \to P$. For $t \in J$, $i = 1,2,\ldots,N$, we set $\max_{i=1,2,\ldots,N}\sup_{t\in J}|f_i(t,0)| = M$, $\max_{i=1,2,\ldots,N}\sup_{t\in J}|g_i(t,0)| = \overline{M}$, and $B_r = \{u \in C[J,\mathbb{R}^+] : \|u\| \leq r\}$, where

$$r \geq \frac{\left(M+\overline{M}\right)\int_0^1 G_1(s,s)\,ds}{1-\max_{i=1,2,\ldots,N}\int_0^1 G_1(s,s)\left(l_i(s)+\bar{l}_i(s)\right)ds}. \tag{17}$$

Step 1. We show that $\mathcal{F}(B_r) \subset B_r$.

For $u \in B_r$ and $t \in J$, $i = 1,2,\ldots,N$,

$$\begin{aligned}&\int_0^1 G(t,s)\left|f_i(s,u(s)) + g_i(s,u(s))\right| ds \\ &\leq \int_0^1 G_1(s,s)\left(\left|f_i(s,u(s)) - f_i(s,0)\right| + \left|f_i(s,0)\right|\right) ds \\ &\quad + \int_0^1 G_1(s,s)\left(\left|g_i(s,u(s)) - g_i(s,0)\right| + \left|g_i(s,0)\right|\right) ds \\ &\leq r\max_{i=1,2,\ldots,N}\int_0^1 G_1(s,s)\left(l_i(s)+\bar{l}_i(s)\right) ds \\ &\quad + \left(M+\overline{M}\right)\int_0^1 G_1(s,s)\,ds \\ &\leq r,\end{aligned} \tag{18}$$

which implies that $|(\mathcal{F}u)(t)| \leq r$. Thus, $\|\mathcal{F}u\| \leq r$. Therefore,

$$\mathcal{F}(B_r) \subset B_r. \tag{19}$$

Step 2. We show that $\mathcal{F}$ is a contraction mapping.

For $u, v \in B_r$ and for each $t \in J$, $i = 1,2,\ldots,N$, we have

$$\begin{aligned}&\int_0^1 G(t,s)\left|f_i(s,u(s)) - f_i(s,v(s))\right| ds \\ &\quad + \int_0^1 G(t,s)\left|g_i(s,u(s)) - g_i(s,v(s))\right| ds \\ &\leq \int_0^1 G_1(s,s)\,l_i(s)|u(s)-v(s)|\,ds \\ &\quad + \int_0^1 G_1(s,s)\,\bar{l}_i(s)|u(s)-v(s)|\,ds \\ &\leq \int_0^1 G_1(s,s)\left(l_i(s)+\bar{l}_i(s)\right) ds\,\|u-v\|.\end{aligned} \tag{20}$$

This, together with $0 < \int_0^1 G_1(s,s)(l_i(s)+\bar{l}_i(s))ds < 1$, $i = 1,2,\ldots,N$, yields that

$$\left|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\right| \leq k\|u-v\|, \tag{21}$$

where

$$0 < k = \max_{i=1,2,\ldots,N}\int_0^1 G_1(s,s)\left(l_i(s)+\bar{l}_i(s)\right) ds < 1. \tag{22}$$

Thus,

$$\|\mathcal{F}u - \mathcal{F}v\| \leq k\|u-v\|. \tag{23}$$

This means that $\mathcal{F}$ is a contraction mapping.

It follows from Banach's contraction mapping that $\mathcal{F}$ has a unique fixed point in B_r. Therefore, the problem (3) has a unique solution. □

Corollary 9. *Let $f_i \in C[J\times\mathbb{R}^+,\mathbb{R}^+]$, $i = 1,2,\ldots,N$. Suppose that the following conditions are satisfied:*

$$\begin{gathered}\left|f_i(t,u(t)) - f_i(t,v(t))\right| \leq l_i(t)|u(t)-v(t)|, \\ 0 < \int_0^1 G_1(s,s)\,l_i(s)\,ds < 1,\end{gathered} \tag{24}$$

where

$$l_i \in C[J,\mathbb{R}^+], \quad i = 1,2,\ldots,N. \tag{25}$$

Then the following fractional switched system

$$\begin{gathered}{}^cD_{0+}^{\alpha}u(t) + f_{\sigma(t)}(t,u(t)) = 0, \quad t \in J = [0,1], \\ u(0) = u''(0) = 0, \qquad u(1) = \int_0^1 u(s)\,ds,\end{gathered} \tag{26}$$

has a unique solution on $[0,1]$.

Theorem 10. *Assume that;*

- (H_1) $f_i, g_i \in C[J\times\mathbb{R}^+,\mathbb{R}^+]$ *and* $f_i(t,x)$, $g_i(t,x)$ *are increasing in x for $x \in \mathbb{R}^+$, $g_i(t,0) \neq 0$, $i = 1,2,\ldots,N$;*
- (H_2) $g_i(t,\lambda x) \geq \lambda g_i(t,x)$ *for* $\lambda \in (0,1)$, $t \in J$, $x \in \mathbb{R}^+$, *and there exists a constant* $\gamma \in (0,1)$ *such that* $f_i(t,\lambda x) \geq \lambda^{\gamma} f_i(t,x)$, $\forall t \in J$, $\lambda \in (0,1)$, $x \in \mathbb{R}^+$, $i = 1,2,\ldots,N$;

(H_3) *there exists a constant* $\delta_0 > 0$ *such that* $f_i(t,x) \geq \delta_0 g_i(t,x)$, $t \in J$, $x \in \mathbb{R}^+$, $i = 1,2,\ldots,N$.

Then problem (3) *has a unique solution* u^* *in* P_h, *where* $h(t) = t$, $t \in J$. *Moreover, for any initial value* $u_0 \in P_h$, *constructing successively the sequence*

$$u_{n+1}(t) = \int_0^1 G(t,s)\left(f_{\sigma(s)}(s,u_n(s)) + g_{\sigma(s)}(s,u_n(s))\right)ds, \quad n = 0,1,2,\ldots, \tag{27}$$

we have $u_n(t) \to u^*(t)$ *as* $n \to \infty$.

Proof. Define the two operators

$$Au(t) = \int_0^1 G(t,s) f_{\sigma(s)}(s,u(s))\,ds,$$
$$Bu(t) = \int_0^1 G(t,s) g_{\sigma(s)}(s,u(s))\,ds. \tag{28}$$

From Lemma 6, we have $A : P \to P$ and $B : P \to P$. It is obvious that u is the solution of problem (3) if and only if $u = Au + Bu$. It follows from (H_1) that A and B are two increasing operators. Thus, for $u, v \in P, u \geq v$, we have $Au \geq Av$ and $Bu \geq Bv$.

Step 1. We show that A is a γ-concave operator and B is a subhomogeneous operator.

In fact, for $\lambda \in (0,1)$, $u \in P$, $t \in J$, $i = 1,2,\ldots,N$, from (H_2), we have

$$\int_0^1 G(t,s) f_i(s,\lambda u(s))\,ds \geq \lambda^\gamma \int_0^1 G(t,s) f_i(s,u(s))\,ds, \tag{29}$$

which yields that

$$A(\lambda u)(t) \geq \lambda^\gamma Au(t). \tag{30}$$

Thus, A is a γ-concave operator. By a closely similar way, we can see that B is a subhomogeneous operator.

Step 2. We show that $Ah \in P_h$ and $Bh \in P_h$.

From Lemma 6 and (H_1), we have, for $t \in J$, $i = 1,2,\ldots,N$,

$$\int_0^1 G(t,s) f_i(s,h(s))\,ds \leq \frac{2}{\Gamma(\alpha+1)} h(t) \times \int_0^1 (1-s)^{\alpha-1}(\alpha-1+s) f_i(s,1)\,ds,$$
$$\int_0^1 G(t,s) f_i(s,h(s))\,ds \geq \frac{1}{\Gamma(\alpha+1)} h(t) \times \int_0^1 (1-s)^{\alpha-1}(\alpha-2+2s) f_i(s,0)\,ds. \tag{31}$$

For $i = 1,2,\ldots,N$, let

$$m_i = \frac{1}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha-1}(\alpha-2+2s) f_i(s,0)\,ds,$$
$$\overline{m}_i = \frac{2}{\Gamma(\alpha+1)} \int_0^1 1-s^{\alpha-1}(\alpha-1+s) f_i(s,1)\,ds. \tag{32}$$

It follows from $g(t,0) \neq 0$ that $\int_0^1 f_i(s,1)\,ds \geq \int_0^1 f_i(s,0)\,ds \geq \delta_0 \int_0^1 g_i(s,0)\,ds > 0$.

Thus,

$$m_i > 0, \quad \overline{m}_i > 0, \quad i = 1,2,\ldots,N. \tag{33}$$

Letting $\widetilde{m} = \min\{m_i, i = 1,2,\ldots,N\}$ and $\widehat{m} = \max\{\overline{m}_i, i = 1,2,\ldots,N\}$, then $\widetilde{m} > 0$ and $\widehat{m} > 0$. Therefore,

$$\widetilde{m}h(t) \leq Ah(t) \leq \widehat{m}h(t), \tag{34}$$

which implies that

$$Ah \in P_h. \tag{35}$$

Similarly, we have $Bh \in P_h$.

Step 3. There exists a constant $\delta_0 > 0$ such that $Au \geq \delta_0 Bu$, $\forall u \in P$.

For $u \in P$ and $t \in J$, $i = 1,2,\ldots,N$, by (H_3), we have

$$\int_0^1 G(t,s) f_i(s,u(s))\,ds \geq \delta_0 \int_0^1 G(t,s) g_i(s,u(s))\,ds. \tag{36}$$

This means that

$$Au \geq \delta_0 Bu, \quad u \in P. \tag{37}$$

Therefore, the conditions of Theorem 7 are satisfied. By means of Theorem 7, we obtain that the operator equation $Au + Bu = u$ has a unique solution u^* in P_h. Moreover, for any initial value $u_0 \in P_h$, constructing successively the sequence

$$u_{n+1}(t) = \int_0^1 G(t,s)\left(f_{\sigma(s)}(s,u_n(s)) + g_{\sigma(s)}(s,u_n(s))\right)ds, \quad n = 0,1,2,\ldots, \tag{38}$$

we have $u_n(t) \to u^*(t)$ as $n \to \infty$. □

In Theorem 10, if we let B be a null operator, we have the following conclusion.

Corollary 11. *Assume that;*

(H_4) $f_i \in C[J \times \mathbb{R}^+, \mathbb{R}^+]$ *and* $f_i(t,x)$ *is increasing in* x *for* $x \in \mathbb{R}^+$, $f_i(t,0) \neq 0$, $i = 1,2,\ldots,N$;

(H_5) *there exists a constant* $\gamma \in (0,1)$ *such that* $f_i(t,\lambda x) \geq \lambda^{\gamma} f_i(t,x), \forall t \in J,\ \lambda \in (0,1),\ x \in \mathbb{R}^+, i = 1,2,\ldots,N$.

Then the following fractional switched system

$$\begin{gathered} {}^{c}D_{0+}^{\alpha} u(t) + f_{\sigma(t)}(t,u(t)) = 0, \quad t \in J = [0,1], \\ u(0) = u''(0) = 0, \qquad u(1) = \int_0^1 u(s)\,ds, \end{gathered} \tag{39}$$

has a unique solution u^* *in* P_h, *where* $h(t) = t,\ t \in J$. *Moreover, for any initial value* $u_0 \in P_h$, *constructing successively the sequence*

$$u_{n+1}(t) = \int_0^1 G(t,s)\, f_{\sigma(s)}(s,u_n(s))\,ds, \quad n = 0,1,2,\ldots, \tag{40}$$

we have $u_n(t) \to u^*(t)$ *as* $n \to \infty$.

4. Examples

Example 1. Consider the following boundary value problem:

$$\begin{gathered} {}^{c}D_{0+}^{\alpha} u(t) + f_{\sigma(t)}(t,u(t)) + g_{\sigma(t)}(t,u(t)) = 0, \\ t \in J = [0,1], \\ u(0) = u''(0) = 0, \qquad u(1) = \int_0^1 u(s)\,ds, \end{gathered} \tag{41}$$

where $\alpha = 5/2, \sigma(t) : J \to M = \{1,2\}$ is a finite switching signal,

$$\begin{aligned} f_1(t,x) &= \frac{1}{4(t+2)^2}\frac{x}{1+x} + 1, \\ g_1(t,x) &= \frac{1}{16}\sin^2 x + t^2, \\ f_2(t,x) &= \frac{1}{8(t+2)^2}\frac{x}{1+x} + 1, \\ g_2(t,x) &= \frac{1}{32}\sin^2 x + t^2. \end{aligned} \tag{42}$$

Thus,

$$f_i, g_i \in C[J \times \mathbb{R}^+, \mathbb{R}^+], \quad i = 1,2. \tag{43}$$

By computation, we deduce that

$$\begin{aligned} |f_1(t,x_1) - f_1(t,x_2)| &\leq \frac{1}{16}|x_2 - x_1|, \\ |g_1(t,x_1) - g_1(t,x_2)| &\leq \frac{1}{16}|x_2 - x_1|, \\ |f_2(t,x_1) - f_2(t,x_2)| &\leq \frac{1}{32}|x_2 - x_1|, \\ |g_2(t,x_1) - g_2(t,x_2)| &\leq \frac{1}{32}|x_2 - x_1|. \end{aligned} \tag{44}$$

On the other hand,

$$\begin{aligned} &\int_0^1 G_1(s,s)\left(l_1(s) + \bar{l}_1(s)\right)ds \\ &\quad = \int_0^1 \frac{2(1-s)^{(5/2)-1}((5/2)-1+s)}{\Gamma((5/2)+1)}\left(\frac{1}{16} + \frac{1}{16}\right)ds \\ &\quad = \frac{1}{4\Gamma(7/2)}\int_0^1 (1-s)^{3/2}\left(\frac{3}{2} + s\right)ds \\ &\quad \leq \frac{1}{4\Gamma(7/2)}\int_0^1 (1-s)^{3/2}\left(\frac{3}{2} + 1\right)ds \\ &\quad = \frac{1}{3\sqrt{\pi}} \times \frac{2}{5} \\ &\quad < 1, \\ &\int_0^1 G_1(s,s)\left(l_1(s) + \bar{l}_1(s)\right)ds \\ &\quad = \int_0^1 \frac{2(1-s)^{(5/2)-1}((5/2)-1+s)}{\Gamma((5/2)+1)}\left(\frac{1}{32} + \frac{1}{32}\right)ds \\ &\quad = \frac{1}{3\sqrt{\pi}} \times \frac{1}{5} \\ &\quad < 1. \end{aligned} \tag{45}$$

Hence, by Theorem 8, BVP (41) has a unique positive solution on [0, 1].

Example 2. Consider the following boundary value problem:

$$\begin{gathered} cD_{0+}^{\alpha} u(t) + f_{\sigma(t)}(t,u(t)) + g_{\sigma(t)}(t,u(t)) = 0, \\ t \in J = [0,1], \\ u(0) = u''(0) = 0, \qquad u(1) = \int_0^1 u(s)\,ds, \end{gathered} \tag{46}$$

where $\alpha = 5/2, \sigma(t) : J \to \{1,2,3\}$ is a finite switching signal,

$$\begin{aligned} f_1(t,x) &= x^{1/3} + t^2 + c, \\ g_1(t,x) &= \frac{x}{(1+t^2)(1+x)} + b - c, \\ f_2(t,x) &= 2x^{1/3} + t^2 + 2c, \\ g_2(t,x) &= \frac{2x}{(1+t^2)(1+x)} + 2(b-c), \end{aligned}$$

$$f_3(t,x) = 3x^{1/3} + t^2 + 3c,$$
$$g_3(t,x) = \frac{3x}{(1+t^2)(1+x)} + 3(b-c). \tag{47}$$

Let $\gamma = 1/3$ and $0 < c < b$. It is obvious that $f_i, g_i \in C[J \times \mathbb{R}^+, \mathbb{R}^+]$ and are increasing with respect to the second argument, $g_i(t,0) = b - c > 0$, $i = 1,2,3$. On the other hand, for $\lambda \in (0,1)$, $t \in J$, $x \in [0,+\infty)$, $i = 1,2,3$, we have

$$\begin{aligned} g_i(t,\lambda x) &= \frac{i\lambda x}{(1+t^2)(1+\lambda x)} + i(b-c) \\ &\geq \frac{i\lambda x}{(1+t^2)(1+\lambda x)} + i\lambda(b-c) \\ &= \lambda g_i(t,x), \end{aligned} \tag{48}$$
$$\begin{aligned} f_i(t,\lambda x) &= i\lambda^{1/3}x^{1/3} + t^2 + ic \\ &\geq \lambda^{1/3}\left(ix^{1/3} + t^2 + ic\right) \\ &= \lambda^{\gamma} f_i(t,x). \end{aligned}$$

Moreover, for $t \in J$, $x \in \mathbb{R}^+$, $i = 1,2,3$, we have

$$\begin{aligned} f_i(t,x) &= ix^{1/3} + t^2 + ic \\ &\geq ic \geq \frac{c}{3+(b-c)}(i + i(b-c)) \\ &\geq \frac{c}{3+(b-c)}\left(\frac{ix}{(1+t^2)(1+x)} + i(b-c)\right) \\ &= \delta_0 g_i(t,x), \end{aligned} \tag{49}$$

where

$$\delta_0 = \frac{c}{3+(b-c)}. \tag{50}$$

Hence all the conditions of Theorem 10 are satisfied. Thus, BVP (46) has a unique positive solution in P_h, where $h(t) = t$, $t \in [0,1]$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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