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*Research Article*

# Bounds of the Spectral Radius and the Nordhaus-Gaddum Type of the Graphs

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The Laplacian spectra are the eigenvalues of Laplacian matrix $L(G) = D(G) - A(G)$, where $D(G)$ and $A(G)$ are the diagonal matrix of vertex degrees and the adjacency matrix of a graph $G$, respectively, and the spectral radius of a graph $G$ is the largest eigenvalue of $A(G)$. The spectra of the graph and corresponding eigenvalues are closely linked to the molecular stability and related chemical properties. In quantum chemistry, spectral radius of a graph is the maximum energy level of molecules. Therefore, good upper bounds for the spectral radius are conducive to evaluate the energy of molecules. In this paper, we first give several sharp upper bounds on the adjacency spectral radius in terms of some invariants of graphs, such as the vertex degree, the average 2-degree, and the number of the triangles. Then, we give some numerical examples which indicate that the results are better than the mentioned upper bounds in some sense. Finally, an upper bound of the Nordhaus-Gaddum type is obtained for the sum of Laplacian spectral radius of a connected graph and its complement. Moreover, some examples are applied to illustrate that our result is valuable.

## 1. Introduction

The graphs in this paper are simple and undirected. Let $G$ be a simple graph with $n$ vertices and $m$ edges. For $v \in V$, denote by $d_v$, $m_v$, and $N_v$ the degree of $v$, the average 2-degree of $v$, and the set of neighbors of $v$, respectively. Then $d_v m_v$ is the 2-degree of $v$. Let $\Delta$, $\Delta'$, $\delta$, and $\delta'$ denote the maximum degree, second largest degree, minimum degree, and second smallest degree of vertices of $G$, respectively. Obviously, we have $\Delta' < \Delta$ and $\delta' > \delta$. A graph is $d$-regular if $\Delta = \delta = d$.

The complement graph $G^c$ of $G$ is the graph with the same set of vertices as $G$, where two distinct vertices are adjacent if and only if they are independent in $G$. The line graph $L_G$ of $G$ is defined by $V(L_G) = E(G)$, where any two vertices in $L_G$ are adjacent if and only if they are adjacent as edges of $G$.

Let $X$ be a nonnegative square matrix. The spectral radius $\rho(X)$ of $X$ is the maximum eigenvalue of $X$. Denote by $B$ the adjacency matrix of $L_G$, then $\rho(B)$ is the spectral radius of $B$. Let $D(G)$ and $A(G)$ denote the diagonal matrix of vertex degrees and the adjacency matrix of $G$, respectively. Then the matrix $L(G) = D(G) - A(G)$ is called the Laplacian matrix of a graph $G$. Obviously, it is symmetric and positive semidefinite. Similarly, the quasi-Laplacian matrix is defined as $Q(G) = D(G) + A(G)$, which is a nonnegative irreducible matrix. The largest eigenvalue of the Laplacian matrix, denoted by $\mu(G)$, is called the Laplacian spectral radius. The Laplacian eigenvalues of a graph are important in graph theory, because they have close relations to many graph invariants, including connectivity, isoperimetric number, diameter, and maximum cut. Particularly, good upper bounds for $\mu(G)$ are applied in many fields. For instance, it is used in theoretical chemistry, within the Heilbronner model, to determine the first ionization potential of alkanes, in combinatorial optimization to provide an upper bound on the size of the maximum cut in graph, in communication networks to provide a lower bound on the edge-forwarding index, and so forth. To learn more information on the applications of Laplacian spectral radius and other Laplacian eigenvalues of a graph, see references [1–4].

In the recent thirty years, the researchers obtained many good upper bounds for $\mu(G)$ [5–8]. These upper bounds improved the previous results constantly. In this paper, we focus on the bounds for the spectral radius of a graph, and the bound of Nordhaus-Gaddum type is also considered, which is the sum of Laplacian spectral radius of a connected graph $G$ and its complement $G^c$.

At the end of this section, we introduce some lemmas which will be used later on.

**Lemma 1** (see [9]). *Let $M = (m_{ij})_{n\times n}$ be an irreducible nonnegative matrix with spectral radius $\rho(M)$, and let $R_i(M)$ be the ith row sum of M; that is, $R_i(M) = \sum_j m_{ij}$. Then*

$$\min_{1\le i\le n} R_i(M) \le \rho(M) \le \max_{1\le i\le n} R_i(M). \quad (1)$$

*Moreover, if the row sums of M are not all equal, then both inequalities are strict.*

**Lemma 2** (see [10]). *Let $G = [V, E]$ be a connected graph with n vertices; then*

$$\rho(G) \le \frac{1}{2}\rho(L_G) + 1. \quad (2)$$

*The equality holds if and only if G is a regular graph.*

This lemma gives a relation between the spectral radius of a graph and its line graph. Therefore, we can estimate the spectral radius of the adjacency matrix of graph by estimating that of its line graph.

**Lemma 3** (see [11]). *Let B be a real symmetric $n\times n$ matrix, and let $\rho(B)$ be the largest eigenvalue of B. If $P(\lambda)$ is a polynomial on $\lambda$, then*

$$\min_{v\in V} R_v(P(B)) \le P(\rho(B)) \le \max_{v\in V} R_v(P(B)). \quad (3)$$

*Here $R_v(P(B))$ is the vth row sum of matrix $P(B)$. Moreover, if the row sums of $P(B)$ are not all equal, then both inequalities are strict.*

**Lemma 4** (see [11]). *Let G be a simple connected graph with n vertices and let $\rho(Q)$ be the largest eigenvalue of the quasi-Laplacian matrix of graph G. Then*

$$\mu(G) \le \rho(Q), \quad (4)$$

*with equality holds if and only if G is a bipartite graph.*

By these lemmas, we will give some improved upper bounds for the spectral radius and determine the corresponding extremal graphs.

This paper is organized as follows. In Section 2, we will give several sharp upper and lower bounds for the spectral radius of graphs and determine the extremal graphs which achieve these bounds. In Section 3, some bounds of Nordhaus-Gaddum type will be given. Furthermore, in Sections 2 and 3, we present some examples to illustrate that our results are better than all of the mentioned upper bounds in this paper, in some sense.

## 2. Bounds on the Spectral Radius

*2.1. Previous Results.* The eigenvalues of adjacency matrix of the graph have wide applications in many fields. For instance, it can be used to present the energy level of specific electrons. Specially, the spectral radius of a graph is the maximum energy level of molecules. Hence, good upper bound for the spectral radius helps to estimate the energy level of molecules [12–15]. Recently, there are some classic upper bounds for the spectral radius of graphs.

In the early time Cao [16] gave a bound as follows:

$$\rho(G) \le \sqrt{2m - \delta(n-1) + \Delta(\delta - 1)}. \quad (5)$$

The equality holds if and only if $G$ is regular graph or a star plus of $K_2$, or a complete graph plus a regular graph with smaller degree of vertices.

Hu [17] obtained an upper bound with simple form as follows:

$$\rho(G) \le \sqrt{2m - n - \delta + 2}. \quad (6)$$

The equality holds if and only if $G$ is $n-2$ regular graph.

In 2005, Xu [18] proved that

$$\rho(G) \le \sqrt{2m - n + 1 - (\delta - 1)(n - 1 - \Delta)}. \quad (7)$$

The equality holds if and only if $G$ is regular graph or a star graph.

Using the average 2-degree of the vertices, the researchers got more upper bounds.

Cao's [16] another upper bound:

$$\rho(G) \le \max_{u\in V(G)} \sqrt{d_u m_u}. \quad (8)$$

The equality holds if and only if $G$ is a regular graph or a semiregular bipartite graph.

Similarly, Abrham and Zhang [19] proved that

$$\rho(G) \le \max_{uv\in E(G)} \sqrt{d_u d_v}. \quad (9)$$

The equality holds if and only if $G$ is a regular graph or a semiregular bipartite graph.

In recent years, Feng et al. [10] give some upper bounds for the spectral radius as follows:

$$\rho(G) \le \max_{u\in V(G)} \sqrt{\frac{d_u^2 + d_u m_u}{2}}. \quad (10)$$

The equality holds if and only if $G$ is regular graph.

$$\rho(G) \le \max_{uv\in E(G)} \sqrt{\frac{d_u(d_u + m_u) + d_v(d_v + m_v)}{2}}. \quad (11)$$

The equality holds if and only if $G$ is regular graph.

$$\rho(G) \le \max_{u\in V(G)} \frac{d_u + \sqrt{d_u m_u}}{2}. \quad (12)$$

The equality holds if and only if $G$ is regular graph.

$$\rho(G) \le \max_{uv\in E(G)} \frac{d_u + d_v + \sqrt{(d_u - d_v)^2 + 4m_u m_v}}{4}. \quad (13)$$

The equality holds if and only if $G$ is regular graph.

*2.2. Main Results.* All of these upper bounds mentioned in Section 2.1 are characterized by the degree and the average 2-degree of the vertices. Actually, we can also use other invariants of the graph to estimate the spectral radius. In the following, such an invariant will be introduced.

In a graph, a circle with length 3 is called a triangle. If $u$ is a triangle's vertex in a graph, then $u$ is incident with this triangle. Denote by $T_u$ the number of the triangles associated with the vertex $u$. For example, in Figure 1, we have $T_u = 3$ and $T_v = T_w = 0$.

Let $N_u \cap N_v$ be the set of the common adjacent points of vertex $u$ and $v$; then $|N_u \cap N_v|$ present the cardinality of $N_u \cap N_v$.

Now, some new and sharp upper and lower bounds for the spectral radius will be given.

**Theorem 5.** *Let G be a simple connected graph with n vertices. Then*

$$\rho(G) \le \max_{uv\in E(G)} \frac{d_u^2 m_u + d_v^2 m_v - 2(T_u + T_v)}{2(d_u d_v - |N_u \cap N_v|)}; \tag{14}$$

*the equality holds if and only if G is a regular graph.*

*Proof.* Let $K = \operatorname{diag}(d_u d_v - |N_u \cap N_v| : uv \in E(G))$ is a diagonal matrix and $B$ is the adjacency matrix of the line graph. Denote $N = K^{-1}BK$, then $N$ and $B$ have the same eigenvalues. Since $G$ is a simple connected graph, it is easy to obtain that $N$ is nonnegative and irreducible matrix. The $(uv, pq)$th entry of $N$ is equal to

$$\begin{cases} \dfrac{d_p d_q - |N_p \cap N_q|}{d_u d_v - |N_u \cap N_v|}, & pq \sim uv, \\ 0, & \text{else,} \end{cases} \tag{15}$$

here $pq \sim uv$ implies that $pq$ and $uv$ are adjacent in graph. Hence, the $uv$th row sum $R_{uv}(N)$ of $N$ is

$$\begin{aligned} &\sum_{pq\sim uv} \frac{d_p d_q - |N_p \cap N_q|}{d_u d_v - |N_u \cap N_v|} \\ &\quad = \frac{\sum_{q\sim u} d_u d_q + \sum_{p\sim v} d_p d_v - 2 d_u d_v}{d_u d_v - |N_u \cap N_v|} \\ &\quad\quad - \frac{\sum_{q\sim u} |N_u \cap N_q| + \sum_{p\sim v} |N_p \cap N_v| - 2|N_u \cap N_v|}{d_u d_v - |N_u \cap N_v|} \\ &\quad = \frac{d_u^2 m_u + d_v^2 m_v - 2 d_u d_v - 2(T_u + T_v) + 2|N_u \cap N_v|}{d_u d_v - |N_u \cap N_v|} \\ &\quad = \frac{d_u^2 m_u + d_v^2 m_v - 2(T_u + T_v)}{d_u d_v - |N_u \cap N_v|} - 2. \end{aligned} \tag{16}$$

From Lemmas 1 and 2, we have

$$\begin{aligned} \rho(G) &\le \frac{1}{2}\rho(B) + 1 \\ &\le \max\left\{\frac{1}{2}R_{uv}(N) + 1 : uv \in V(H)\right\}. \end{aligned} \tag{17}$$

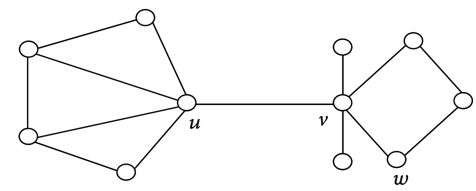


Figure 1: Graph with triangles.

It means that (14) holds and the equality in (14) holds if and only if $G$ is a regular graph. □

In a graph, let $\alpha$ and $\beta$ represent the number of vertices with the maximum degree and minimum degree, respectively. Then, we get the following results.

**Theorem 6.** *Let G be a simple connected graph with n vertices. If* $\Delta \le \min\{n-1-\beta, n-1-\alpha\}$, *then*

$$\rho(G) \le \sqrt{2m + \Delta(\delta' - 1) - \beta\delta - (n-1-\beta)\delta'}, \tag{18}$$

$$\rho(G) \ge \sqrt{2m + (\Delta' - 1)\delta - \alpha\Delta - (n-1-\alpha)\Delta'}; \tag{19}$$

*the equality holds if and only if G is a regular graph.*

*Proof.* Since $R_v(A^2)$ is exactly the number of walks of length 2 in $G$ with a starting point $v$, thus

$$R_v(A^2) = \sum_{u\sim v} d_u = 2m - d_v - \sum_{u\nsim v} d_u. \tag{20}$$

Therefore, from Lemmas 1 and 3, if $\Delta \le n-1-\beta$, we have $d_v \le n-1-\beta$ for any $v \in V(G)$. Then

$$\begin{aligned} \rho(A^2) &\le \max_{v\in V(G)} \left(2m - d_v - \sum_{u\nsim v} d_u\right) \\ &\le \max_{v\in V(G)} \left(2m - d_v - \left(\beta\delta + (n - d_v - 1 - \beta)\delta'\right)\right) \\ &= \max_{v\in V(G)} \left(2m + (\delta' - 1) d_v - \beta\delta - (n-1-\beta)\delta'\right) \\ &\le 2m + \Delta(\delta' - 1) - \beta\delta - (n-1-\beta)\delta'. \end{aligned} \tag{21}$$

Hence, it is easy to obtain that (18) holds.

If equality in (18) holds, then all equalities in the above argument must hold. Thus, for all $v \in V(G)$

$$\sum_{u\nsim v} d_u = \beta\delta + (n - d_v - 1 - \beta)\delta'. \tag{22}$$

It means that $d_v = n-1$ and $\delta' = \delta$, or $d_u = \delta = \delta'$; this shows that the graph $G$ is regular. Conversely, if $G$ is $k$-regular, it is not difficult to check that $\rho(G)$ attains the upper bound by direct calculation.

Similarly for the lower bound, if $\Delta \leq n-1-\alpha$, we have

$$\begin{aligned}\rho\left(A^2\right) &\geq \min_{v\in V(G)}\left(2m-d_v-\sum_{u\neq v} d_u\right)\\ &\geq \min_{v\in V(G)}\left(2m-d_v-\left(\alpha\Delta+\left(n-d_v-1-\alpha\right)\Delta'\right)\right)\\ &= \min_{v\in V(G)}\left(2m+\left(\Delta'-1\right)d_v-\alpha\Delta-(n-1-\alpha)\Delta'\right)\\ &\geq 2m+\left(\Delta'-1\right)\delta-\alpha\Delta-(n-1-\alpha)\Delta'.\end{aligned} \tag{23}$$

It means that (19) holds and the equality in (19) holds if and only if $G$ is a regular graph by similar discussion. □

**Theorem 7.** *Let G be a simple connected graph with n vertices. If $\Delta \leq n-1-\beta$; then*

$$\rho(G) \leq \frac{\delta'-1+\sqrt{(\delta'+1)^2+8m-4\beta(\delta-\delta')-4n\delta'}}{2}; \tag{24}$$

*the equality holds if and only if G is a regular graph.*

*Proof.* According to the proof of Theorem 6, we have

$$\begin{aligned}R_v\left(A^2\right) &= 2m-d_v-\sum_{u\neq v} d_u\\ &\leq 2m+\left(\delta'-1\right)d_v-\beta\delta-(n-1-\beta)\delta'.\end{aligned} \tag{25}$$

Thus

$$R_v\left(A^2-\left(\delta'-1\right)A\right) \leq 2m-\beta\delta-(n-1-\beta)\delta'. \tag{26}$$

From Lemma 3, we have

$$\rho^2(A)-\left(\delta'-1\right)\rho(A)-2m+\beta\delta+(n-1-\beta)\delta' \leq 0. \tag{27}$$

Solving this quadratic inequality, we obtain that upper bound (24) holds.

If equality in (24) holds, then all equalities in the argument must hold. By the similar discussion of Theorem 6, the equality holds if and only if $G$ is a regular graph. □

*2.3. Numerical Examples.* In this section, we will present two graphs to illustrate that our some new bounds are better than other bounds in some sense. Let Figures 2 and 3 be graphs of orders 7 and 8.

The estimated value of each upper bound is listed in Table 1. Obviously, from Table 1, bound (24) is the best in all known upper bounds for Figure 2 and bound (14) is the best for Figure 3. Furthermore, bound (18) is the best except (13) and (24) for Figure 2. Hence, commonly, these upper bounds are incomparable.

## 3. Bounds of the Nordhaus-Gaddum Type

*3.1. Previous Results.* In this part, we mainly discuss the upper bounds on the sum of Laplacian spectral radius of a connected graph $G$ and its complement $G^c$, which is called the upper bound of the Nordhaus-Gaddum type. For convenience, let

$$\sigma(G) = \mu(G)+\mu\left(G^c\right). \tag{28}$$

The following are some classic upper bounds of Nordhaus-Gaddum type. The coarse bound $\mu(G) \leq 2\Delta$ easily implies the simplest upper bound on $\sigma(G)$:

$$\sigma(G) \leq 2(n-1)+2(\Delta-\delta). \tag{29}$$

In particular, if both $G$ and $G^c$ are connected and irregular, Shi [20] gave a better upper bound as follows:

$$\sigma(G) \leq 2\left(n-1-\frac{2}{2n^2-n}\right)+2(\Delta-\delta). \tag{30}$$

Liu et al. [21] proved that

$$\sigma(G) \leq n-2+\left\{(\Delta-\omega)^2+n^2+4(\Delta-\delta)(n-1)\right\}^{1/2}, \tag{31}$$

where $\omega = n-\delta-1$.

Shi [20] gives another upper bound

$$\sigma(G) \leq 2\left\{(n-1)(2\omega-\delta)+(\Delta+\delta)^2-\Delta+\delta\right\}^{1/2}. \tag{32}$$

To learn other bounds of the Nordhaus-Gaddum type, see references [22, 23]. In order to state the main result of this section, we first give an upper bound for the Laplacian spectral radius.

*3.2. Laplacian Spectral Radius.* Here we give a new upper bound for the Laplacian spectral radius. For convenience, let

$$f(m,\Delta,\delta) = \left(\left(\Delta-\frac{\delta}{2}-1\right)^2+16m-2\delta(4n-\delta-2)\right)^{1/2}. \tag{33}$$

**Theorem 8.** *Let G be a simple connected graph of order n with $\Delta$ and $\delta$; then*

$$\mu(G) \leq \frac{\Delta+(3/2)\delta-1+f(m,\Delta,\delta)}{2}, \tag{34}$$

*with equality holds if and only if G is bipartite regular.*

*Proof.* Let $K = Q - \delta E$; then $R_v(K) = 2d_v - \delta$, it means that $2d_v = R_v(K) + \delta$. Considering the $v$th row sum of matrix $K^2$, we have

$$\begin{aligned} R_v\left(K^2\right) &= R_v\left(Q^2\right) - 2\delta R_v(Q) + \delta^2 \\ &= 2d_v^2 + 2\sum_{u\sim v} d_u - 4\delta d_v + \delta^2 \\ &= 2d_v^2 + 2\left(2m - d_v - \sum_{u \nsim v, u\neq v} d_u\right) - 4\delta d_v + \delta^2 \\ &\leq 2d_v^2 + 2\left(2m - d_v - (n - d_v - 1)\delta\right) - 4\delta d_v + \delta^2 \\ &= 2d_v^2 - 2d_v - 2\delta d_v + 4m - 2(n-1)\delta + \delta^2 \\ &= \left(2d_v - \delta\right) d_v - (2+\delta) d_v \\ &\quad + 4m - 2(n-1)\delta + \delta^2 \\ &\leq \Delta R_v(K) - (2+\delta)\frac{R_v(K) + \delta}{2} \\ &\quad + 4m - 2(n-1)\delta + \delta^2 \\ &= \left(\Delta - \frac{\delta}{2} - 1\right) R_v(K) + 4m - 2n\delta + \delta + \frac{\delta^2}{2}. \end{aligned} \tag{35}$$

This is equivalent to the following inequality:

$$R_v\left(K^2 - \left(\Delta - \frac{\delta}{2} - 1\right)K\right) \leq 4m - 2n\delta + \delta + \frac{\delta^2}{2}. \tag{36}$$

From Lemma 3, we obtain that

$$\rho^2(K) - \left(\Delta - \frac{\delta}{2} - 1\right)\rho(K) \leq 4m - 2n\delta + \delta + \frac{\delta^2}{2}. \tag{37}$$

By simple calculation, we get the upper bound of the spectral radius of matrix $K$ as follows:

$$\begin{aligned} \rho(K) \leq\ & \frac{\Delta - (\delta/2) - 1}{2} \\ & + \frac{\left((\Delta - (\delta/2) - 1)^2 + 16m - 2\delta(4n - \delta - 2)\right)^{1/2}}{2}. \end{aligned} \tag{38}$$

Since $\rho(K) = \rho(Q) - \delta$, therefore from Lemma 4 we obtain that the result (34) holds.

If the spectral radius $\mu(G)$ achieves the upper bound in (34), then each inequality in the above proof must be equal. This implies that $\Delta = \delta$ for all $v \in V(G)$, thus $G$ is regular graph. From Lemma 4 again, $G$ is regular bipartite graph.

Conversely, it is easy to verify that equality in (34) holds for regular bipartite graphs. □

*3.3. Bound of the Nordhaus-Gaddum Type.* In this part, based on Theorem 8, an upper bound of Nordhaus-Gaddum type of Laplacian matrix will be given.

**Theorem 9.** *Let $G$ be a simple graph of order $n$ with $\Delta$ and $\delta$; then*

$$\sigma(G) \leq \frac{5n - \Delta + \delta - 9 + \sqrt{2}\left\{2(2\Delta - \delta - 2)^2 + 8\delta(2+\delta) + (\omega - \Delta)(n + 3\Delta - 3\delta - 5) + 32n\omega - 8\pi(3n + \Delta - 1)\right\}^{1/2}}{4} \tag{39}$$

*here $\omega = n - \delta - 1$ and $\pi = n - \Delta - 1$. Moreover, if both $G$ and $G^c$ are connected, then the upper bound is strict.*

*Proof.* According to the relation of a graph $G$ and its complement, it is not difficult to obtain the invariants of $G^c$. Denote it by $\Delta(G^c) = n - \delta - 1$, $\delta(G^c) = n - \Delta - 1$, and $m(G^c) = C_n^2 - m$. From Theorem 8, we have

$$\begin{aligned} \mu(G^c) \leq\ & \frac{\Delta(G^c) + (3/2)\delta(G^c) - 1}{2} \\ & + \frac{f\left(m(G^c), \Delta(G^c), \delta(G^c)\right)}{2}. \end{aligned} \tag{40}$$

Let

$$g(m) = f(m, \Delta, \delta) + f\left(m(G^c), \Delta(G^c), \delta(G^c)\right). \tag{41}$$

Then the upper bound of the Nordhaus-Gaddum type of Laplacian matrix is

$$\sigma(G) = \mu(G) + \mu(G^c) \leq \frac{5n - \Delta + \delta - 9 + 2g(m)}{4} \tag{42}$$

since

$$g'(m) = \frac{8}{f(m, \Delta, \delta)} - \frac{8}{f\left(m(G^c), \Delta(G^c), \delta(G^c)\right)}. \tag{43}$$

Obviously, $g'(m) \geq 0$ holds if and only if the following inequality holds:

$$f(m, \Delta, \delta) \leq f\left(C_n^2 - m, n - \delta - 1, n - \Delta - 1\right). \tag{44}$$

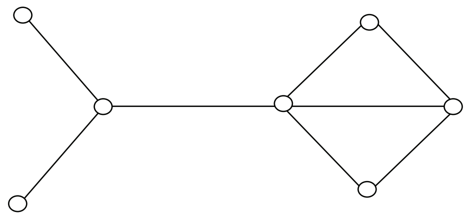

Figure 2: Graph of order 7.

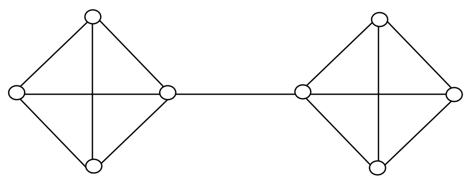

Figure 3: Graph of order 8.

Let $m$ be a variable; then solving this inequality, we have

$$m \leq \frac{(n-\delta-\Delta-1)(n-3\delta+3\Delta-5)+32n(n+\delta-1)}{128} - \frac{8\delta(\delta+2)-8(n-\Delta-1)(3n+\Delta-1)}{128} = m^{*}. \tag{45}$$

Here, the symbol $m^*$ represents the right hand of the above inequality. Then we can assert that $g(m)$ is an increasing function for $m \leq m^*$, and it implies that $g(m) \leq g(m^*)$. Therefore, we have

$$\sigma(G) \leq \frac{5n-\Delta+\delta-9+2g(m^*)}{4} = \frac{5n-\Delta+\delta-9+4f(m^*,\Delta,\delta)}{4}. \tag{46}$$

Simplifying this expression by direct calculation, we prove that the result (39) is correct.

If equality in (39) holds, then each inequality in the above proof must be equality. From Theorem 8, we obtain that both $G$ and $G^c$ are regular bipartite. But it is impossible for a connected graph, this implies that the Laplacian spectral radius of either $G$ or $G^c$ fails to achieve its upper bound and so does the sum. Hence the inequality in (39) is strict. □

*3.4. Numerical Examples.* In this section, we give some examples to illustrate that the new bound is better than other bounds for some graphs. Considering the graph of order 10 in Figure 4 and Figures 1–3, the estimated value of each upper bound of the Nordhaus-Gaddum type is given in Table 2.

Clearly, from Table 2, we can see that new bound (39) is the best in all known upper bounds for all figures mentioned in this paper.

## 4. Conclusion

From numerical examples of Sections 2 and 3, the estimated value of new upper bounds of the spectral radius and the Nordhaus-Gaddum type of graphs are the smallest in all known upper bounds for the graphs considered in these examples. It means that our results are better than the existing upper bounds in some sense.

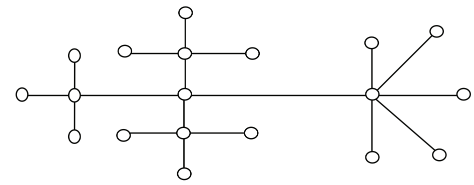

Figure 4: Graph of order 10.

Table 1: Estimated value of each upper bound.

| Upper bounds | Figure 2 | Figure 3 |
|---|---|---|
| Bound (5) | 3.1623 | 4.1231 |
| Bound (6) | 3.1623 | 3.6056 |
| Bound (7) | 3.1623 | 3.6056 |
| Bound (8) | 3.1623 | 4.0000 |
| Bound (9) | 3.4641 | 3.8079 |
| Bound (10) | 3.6056 | 3.8079 |
| Bound (11) | 3.2787 | 3.6056 |
| Bound (12) | 3.5811 | 3.8028 |
| Bound (13) | 3.0650 | 3.6250 |
| Bound (14) | 3.5000 | 3.5000 |
| Bound (18) | 3.1623 | 4.0000 |
| Bound (24) | 3.0000 | 4.0000 |
| Actual value | 2.7321 | 3.3028 |

Table 2: Estimated value of each upper bound.

| Upper bound | Figure 1 | Figure 2 | Figure 3 | Figure 4 |
|---|---|---|---|---|
| Bound (29) | 28 | 18 | 16 | 46 |
| Bound (30) | 27.98 | 17.96 | 15.96 | 45.99 |
| Bound (31) | 26.23 | 16.05 | 15.59 | 46.02 |
| Bound (32) | 28.43 | 17.44 | 18.22 | 50.52 |
| Bound (39) | 25.84 | 15.88 | 15.52 | 44.97 |

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