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Research Article

Sampled-Data Consensus for High-Order Multiagent Systems under Fixed and Randomly Switching Topology

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This paper studies the sampled-data based consensus of multiagent system with general linear time-invariant dynamics. It focuses on looking for a maximum allowable sampling period bound such that as long as the sampling period is less than this bound, there always exist linear consensus protocols solving the consensus problem. Both fixed and randomly switching topologies are considered. For systems under fixed topology, a necessary and sufficient sampling period bound is obtained for single-input multiagent systems, and a sufficient allowable bound is proposed for multi-input systems by solving the H_∞ optimal control problem of certain system with uncertainty. For systems under randomly switching topologies, tree-type and complete broadcasting network with Bernoulli packet losses are discussed, and explicit allowable sampling period bounds are, respectively, given based on the unstable eigenvalues of agent's system matrix and packet loss probability. Numerical examples are given to illustrate the results.

1. Introduction

In recent years, coordination of distributed dynamic systems operating over relative sensing/communication networks has attracted the attention of researchers in system theory, biology, and statistical physics, and so forth [1–5]. Consensus is an important problem in coordination of multiagent systems.

The insertion of the communication network among agents makes the analysis and design of multiagent systems more complex. One of the challenging problems associated with the multiagent systems is the influence of the sampling period. There are many researchers studying sampled-data consensus problem for first and second order multiagent systems. For first-order systems, Olfati-Saber and Murray [3] pointed out that the maximum allowable sampling period is the reciprocal of the maximum out-degree. Xie et al. [6] studied the first-order system via delayed sampled control and gave consensus conditions on delay and sampling period. Liu et al. [7] studied first-order consensus problem with logarithmic quantization and gave sufficient conditions on the sampling interval to ensure the β-asymptotic average consensus. For second-order integrator agents, Cao and Ren [8] proposed a necessary and sufficient condition on the sampling period, the control gain, and the communication graph. Gao and Wang [9] proposed an allowable bound of sampling period for a given protocol through solving a set of LMIs. Yu et al. [10] gave a necessary and sufficient consensus condition based on the sampling period, the coupling gains, and the spectra of the Laplacian matrix. Xiao and Chen [11] studied the state consensus of multiple double integrators in a sampled-data setting with the assumption that the position-like states were the only detectable information transmitted over the network and gave a necessary and sufficient condition and a sufficient condition for the uniform and nonuniform data-sampling cases, respectively. All the aforementioned literature concentrate on first or second order integrator agents. Nevertheless, the consensus problem of high-order agents is important and has been recently considered by many researchers [12–17]. Zhang and Tian [14] pointed out that for high-order multiagent systems the sampling period should be bounded and proposed an allowable bound based on given consensus protocol. In [15], Gao et al. studied the consensusability of sampled-data multiagent systems with general linear dynamics and gave some sufficient and necessary conditions for consensusability in the case of state feedback.

The other challenging problem associated with multiagent systems is the influence of packet losses. Consensus of discrete-time first and second-order systems in random networks has been studied in some works [18–21]. They show that for low-order integrator agents the solvability of the consensus problem just depends on the connectivity of the mean topology while it is independent of link weights and link existence probabilities. Zhang and Tian [22] studied the influence of packet loss probabilities on the consensus ability of discrete-time high-order multiagent systems and provided a maximum allowable sampling period bound. Although many efforts have been made on studying the consensus of multiagent systems with switching topologies or sampled-data, there is limited work investigating both the influences of packet losses and sampling period for high-order multiagent systems. Zhang and Tian [14] studied this problem, but the proposed allowable sampling period bound was not explicit and was based on given protocol gain.

This paper studies the consensus problem for high-order multiagent systems with sampled data and packet losses and focuses on looking for a maximum allowable sampling period such that as long as the sampling period is less than this period bound, there always exists a state feedback protocol solving the consensus problem. Firstly, fixed topology is discussed. For single input system, by using algebraic graph theory and nonsingular matrix transformation, the consensus problem is converted to the simultaneous stabilizing problem of subsystems, and a necessary and sufficient sampling period bound is obtained. For multi-input system, by solving the H_∞ optimal control problem of certain system with uncertainty, an allowable bound of sampling period is proposed. Both bounds depend on the eigenvalues of agents' system matrix and the Laplacian matrix and are easily computed. Secondly, randomly switching topology case is studied. When the topology is a rooted directed spanning tree with Bernoulli link losses, a sampling period bound is given by studying the spectral radius of expected value of multiagent system matrix. When the network is a complete topology with broadcasting schemes and Bernoulli packet losses, a sampling period bound is given by applying Lyapunov functional analysis. Both bounds are explicitly composed of unstable eigenvalues of agents' system matrix and packet loss probability.

Notations. I_n denotes the identity matrix with n dimensions. $\rho(\cdot)$ represents the spectral radius of a matrix. $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ represent the real part and imaginary part of a number, respectively. $\otimes$ denotes Kronecker product of matrixes. $\Pr(\cdot)$ and $E(\cdot)$ denote the probability and expected value of a random process, respectively. $\|x\|_P$ is the Euclidian norm $(x^T Px)^{1/2}$.

2. Preliminaries

Let $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ be an undirected graph of order n with the set of nodes $\mathscr{V} = \{1, 2, \dots, n\}$, edge set $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$, and adjacency matrix $\mathscr{A} = [a_{ij}]_{n\times n}$ which describes the linkages of nodes. If the edge $(i, j) \in \mathscr{E}$, that is, vehicle j can obtain information from vehicle i, then $a_{ji} > 0$, otherwise $a_{ji} = 0$. Suppose that each node has no self-edge; that is, $a_{ii} = 0$ for all i. For the edge (i, j), i is the parent node and j is the child node. A directed path is a sequence of edges in a directed graph of the form $(i, j_1), (j_1, j_2), \dots \in \mathscr{E}$. A graph is connected if there is a directed path from every node to every other node. A complete graph is a simple graph in which every pair of distinct vertices is connected by a unique edge; that is, $a_{ij} > 0$ for all $i \neq j$. A rooted directed spanning tree is a directed graph in which every node has exactly one parent except for one node, called the root, which has no parent and which has a directed path to every other node. If for any i and j, $a_{ij} = a_{ji}$, we say the graph is undirected. The Laplacian matrix of graph $\mathscr{G}$ is denoted by $L = [l_{ij}]_{n\times n}$ with $l_{ii} = \sum_{j=1}^{n} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$.

Lemma 1 (see [1, 3]). *For an undirected graph, L is symmetric and semipositive definite. Moreover, 0 is a simple eigenvalue of L, if and only if the undirected graph is connected. For a directed graph, 0 is a simple eigenvalue of L if the directed graph contains rooted directed spanning trees.*

3. Problem Formulation

Consider there are n agents in the communication network. Each agent in the network has identical continuous-time linear dynamics:

$$\dot{x}_i = Ax_i + Bu_i, \quad i = 1, 2, \dots, n, \tag{1}$$

where $x_i \in R^p$ is the state of agent i, $u_i \in R^q$ is the consensus protocol, and (A, B) are constant matrices with appropriate dimensions and are completely controllable.

By consensus, we mean a scenario where all states of agents in the network agree on a particular value; that is, $x_i = x_j$, for all i, j. Consensus is an important even fundamental problem in multiple agent coordination. It is of interest in studying flocking, swarming, and attitude alignment.

Since state information can be used, here we apply a state feedback consensus protocol:

$$u_i(t) = K\sum_{j=1}^{n} a_{ij}(t)\left(x_j(t) - x_i(t)\right), \tag{2}$$

where K is the protocol parameter to be designed. In practical applications, the topology is often varying due to communication packet losses and communication range constraints, and thus a_{ij} is varying.

Suppose all agents are clock synchronized and each agent transfers its state information periodically. Let T denote the sampling period, then at each transfer instant kT, $k = 0, 1, 2, \dots$, each agent sends its sampled information $x_i(kT)$ by network. If agent i receives j's information $x_j(kT)$ at the kth sampling period, then $a_{ij}(kT) > 0$ and agent i updates its control input u_i. ZOH is applied in the actuator of agents. Thus for $t \in [kT(k+1)T)$,

$$u_i(t) = K\sum_{j=1}^{n} a_{ij}(kT)\left(x_j(kT) - x_i(kT)\right). \quad (3)$$

And then

$$\dot{x}_i = Ax_i + BK\sum_{j=1}^{n} a_{ij}(kT)\left(x_j(kT) - x_i(kT)\right). \quad (4)$$

Denote $L = [l_{ij}]$ as the Laplacian matrix of the topology; define $x = \begin{bmatrix} x_1^T & \cdots & x_n^T \end{bmatrix}^T$; then for $t \in [kT(k+1)T)$

$$\dot{x} = \left(I_n \otimes A\right)x(t) - \left(L(kT) \otimes BK\right)x(kT). \quad (5)$$

Discretizing the above system we obtain that

$$x((k+1)T) = \left(I_n \otimes e^{AT} - L(kT) \otimes \int_0^T e^{A\tau}d\tau BK\right)x(kT). \quad (6)$$

This paper studies the consensus problem defined as follows.

Definition 2. The multiagent system (1), (3) converges to consensus, if $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$ holds for all $i \neq j$.

Zhang and Tian [14] have revealed that for high-order multiagent systems, the sampling period should be bounded. In the following sections, we investigate the maximum allowable sampling periods bound (MASPB) for systems under fixed topology and randomly switching topologies such that as long as the sampling period is less than this period bound, there exists a state feedback protocol solving the consensus problem.

4. MASPB of Systems under Fixed Undirected Topology

This section will look for a MASPB for multiagent systems under fixed undirected topology. To study the system, firstly perform some system transformations.

For Laplacian matrix L, there exists a nonsingular matrix $U = \begin{bmatrix} 1_n & U_1 \end{bmatrix}$ such that $U^{-1}LU = \begin{bmatrix} 0 & 0 \\ 0 & \overline{L} \end{bmatrix}$, $\overline{L}$ is a diagonal matrix with diagonal elements λ_i, $i = 1, 2, \ldots, n-1$. From Lemma 1, if the topology is connected, we have $\lambda_i > 0$. Applying nonsingular transformation $U^{-1} \otimes I$ to both sides of (6) we have that system (6) achieves consensus, if and only if $\rho(I_{n-1} \otimes e^{AT} - \overline{L} \otimes \int_0^T e^{A\tau}d\tau BK) < 1$. $I_{n-1} \otimes e^{AT} - \overline{L} \otimes \int_0^T e^{A\tau}d\tau BK$ is a block diagonal matrix. Thus the following lemma can be obtained.

Lemma 3. *The sampled-data multiagent system* (1), (3) *under fixed and undirected connected topology converges to consensus, if and only if*

$$\rho\left(e^{AT} - \lambda_i\int_0^T e^{A\tau}d\tau BK\right) < 1 \quad (7)$$

holds for i, where λ_i are the eigenvalues of L except the zero eigenvalue corresponding to the eigenvector 1_n, $i = 1, 2, \ldots, n-1$.

4.1. MASPB for Single Input Systems. To provide the maximum allowable sampling period for single input sampled-data multiagent systems, an important lemma is given firstly.

Lemma 4 (see [16]). *Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ be given. There exists a common control gain $K \in R^{1\times p}$ such that $\rho(G - \lambda_i HK) < 1$ hold for all i, if and only if*

(a) *(G, H) is stabilizable;*

(b) *the degree of instability of G is strictly upper bounded as follows:*

$$\prod_j \left|\lambda_j^u\right|^2 < \frac{\lambda_N + \lambda_1}{\lambda_N - \lambda_1}, \quad (8)$$

where λ_j^u are the unstable eigenvalues of G, $|\lambda_j^u| \geq 1$.

From Lemma 4 we can obtain the following theorem. Here we ignore the case when the discretized system $(e^{AT}, \int_0^T e^{A\tau}d\tau B)$ is uncontrollable.

Theorem 5. *For single input sampled-data multiagent systems* (1) *in fixed and connected undirected communication topology, there exists a linear consensus protocol asymptotically solving the consensus problem of the multiagent system, if and only if*

$$T < \frac{\ln\left(\lambda_{n-1} + \lambda_1\right) - \ln\left(\lambda_{n-1} - \lambda_1\right)}{2\sum_j \operatorname{Re}\left(\lambda_j^u\right)}, \quad (9)$$

where $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$ are the nonzero eigenvalues of Laplacian matrix and λ_j^u are the unstable eigenvalues of continuous-time system matrix A satisfying $\operatorname{Re}(\lambda_j^u) \geq 0$.

Proof. From Lemma 3, consensus is reached if and only if $\rho(\overline{A} - \lambda_i\overline{B}K) < 1$, $i = 1, 2, \ldots, n-1$.

Here we ignore the case when the discretized system is uncontrollable. Since (A, B) is completely controllable, the discretized system $(\overline{A}, \overline{B})$ is also completely controllable.

The degree of instability of $\overline{A}$ is $\prod_j|\overline{\lambda}_j^u|^2$, where $\overline{\lambda}_i^u$ are unstable eigenvalues of $\overline{A}$; that is, $|\overline{\lambda}_i^u| \geq 1$. Moreover, it is easy to obtain that $\overline{\lambda}_i^u = e^{\lambda_j^u T}$; then

$$\prod_j\left|\overline{\lambda}_j^u\right|^2 = \prod_j\left|e^{\lambda_j^u T}\right|^2 = \exp\left(2\sum_j \operatorname{Re}\left(\lambda_j^u\right)T\right). \quad (10)$$

Obviously, let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}$; then $\prod_j|\overline{\lambda}_j^u|^2 < (\lambda_{n-1} + \lambda_1)/(\lambda_{n-1} - \lambda_1)$ is equivalent to that $T < (\ln(\lambda_{n-1} + \lambda_1) - \ln(\lambda_{n-1} - \lambda_1))/2\sum_j \operatorname{Re}(\lambda_j^u)$. From Lemma 4, Theorem 5 has been proved. □

Remark 6. Theorem 5 shows that for first-order, second-order, even high-order integrator multiagent systems, the allowable sampling period can be arbitrarily large but bounded. It conforms to the results in [8].

4.2. MASPB for Multi-Input Systems. For multi-input systems, the degree of instability of the system matrix cannot be presented by the unstable eigenvalues directly. Here, we look for the allowable sampling period bound by studying the robust control of uncertain system.

From Lemma 3, the consensus problem is simplified to the simultaneous stabilization of $n-1$ specific systems like

$$z_i(k+1) = \left(\overline{A} - \lambda_i \overline{B} K\right) z_i(k) \quad i = 1, 2, \ldots, n-1, \tag{11}$$

where $\overline{A} = e^{AT}$, $\overline{B} = \int_0^T e^{A\tau} d\tau B$.

Define

$$\begin{aligned} \mu_{\min} &= \min_i \{|\lambda_i|\}, \qquad \mu_{\max} = \max_i \{|\lambda_i|\}, \\ \mu^* &= \frac{\mu_{\min} + \mu_{\max}}{2}, \qquad \delta_i = \frac{\mu^* - \lambda_i}{\mu^*}; \end{aligned} \tag{12}$$

then it can be easily obtained that $\lambda_i = \mu^*(1 - \delta_i)$, and $|\delta_i| \le \delta_m$, where $\delta_m = \max_i \{|(\mu^* - \lambda_i)/\mu^*|\}$.

Define $\overline{B}_\mu = \mu^* \overline{B}$; then the simultaneous stabilization problem of systems (11) can be solved by studying the robust control of the following system with uncertain parameters

$$z(k+1) = \left(\overline{A} - (1 - \Delta) \overline{B}_\mu K\right) z(k), \tag{13}$$

where the uncertainty Δ satisfies that $|\Delta| \le \delta_m$.

Denote system (13) by a linear model as follows:

$$\begin{aligned} z(k+1) &= \left(\overline{A} - \overline{B}_\mu K\right) z(k) + \overline{B}_\mu \omega(k), \\ y(k) &= K z(k), \\ \omega(k) &= \Delta y(k), \end{aligned} \tag{14}$$

where Δ is the uncertain parameter in the model and satisfies that $|\Delta| \le \delta_m$.

The uncertain system (14) is quadratic stable, if and only if

$$\|T(z)\|_\infty < \delta_m^{-1}, \tag{15}$$

where $T(z)$ is the transfer function of system (14) when $\Delta = 0$; that is,

$$T(z) = K\left(zI - \overline{A} + \overline{B}_\mu K\right)^{-1} \overline{B}_\mu. \tag{16}$$

Next, we study the condition of sampling period by solving the optimal H_∞ function problem.

Theorem 7. *For multi-input sampled-data multiagent systems* (1) *in a fixed communication topology, the topology is undirected and connected. If* $T < T^*$, *where*

$$\begin{aligned} T^* &= \max T \\ s.t. \quad & \gamma^*(T) < \delta_m^{-1}, \end{aligned} \tag{17}$$

where $\gamma^*(T)$ *is obtained by solving the following optimal problem*

$$\gamma^*(T) = \min \gamma \tag{18}$$

subject to the following inequality

$$\begin{bmatrix} Q & \left(\overline{A}Q - \overline{B}_\mu Y\right)^T & Y^T \\ \left(\overline{A}Q - \overline{B}_\mu Y\right) & Q - \gamma^{-2} \overline{B}_\mu \overline{B}_\mu^T & 0 \\ Y & 0 & I_q \end{bmatrix} > 0; \tag{19}$$

then there exists a linear consensus protocol asymptotically solving the consensus problem of the multiagent system.

Proof. From Bounded Real Lemma, $\|T(z)\|_\infty < \gamma$ if and only if there exists symmetric positive definite matrix $P > 0$ such that

$$\begin{aligned} &\left(\overline{A} - \overline{B}_\mu K\right)^T \left(P^{-1} - \gamma^{-2} \overline{B}_\mu \overline{B}_\mu^T\right)^{-1} \left(\overline{A} - \overline{B}_\mu K\right) \\ &\quad + K^T K - P < 0. \end{aligned} \tag{20}$$

Left- and right-multiplying the both sides of by P^{-1}, we obtain that

$$\begin{aligned} &P^{-1}\left(\overline{A} - \overline{B}_\mu K\right)^T \left(P^{-1} - \gamma^{-2} \overline{B}_\mu \overline{B}_\mu^T\right)^{-1} \left(\overline{A} - \overline{B}_\mu K\right) P^{-1} \\ &\quad + P^{-1} K^T K P^{-1} - P^{-1} < 0. \end{aligned} \tag{21}$$

Define $Q = P^{-1}$ and $KP^{-1} = Y$; then by applying Schur Complement Lemma, the above inequality is equivalent to (19). Solving the optimal H_∞ control problem, we obtain the minimum H_∞ spectrum γ^* and optimal control gain K. Therefore, if $\gamma^* < \delta_m^{-1}$, the optimal control gain K can always simultaneously stabilize systems (11).

For a given communication topology, δ_m is fixed. If a sampling period T guarantees that the obtained minimum H_∞ spectrum $\gamma^*(T)$ for this sampled-data system matrix $(\overline{A}, \overline{B})$ is less than δ_m^{-1}, then under this sampling period there exists control gain K simultaneously stabilizing systems (11). We look for the maximum allowable sampling period by searching the maximum T under which there holds $\gamma^*(T) < \delta_m^{-1}$. So, as long as the sampling period $T < T^*$, there exists a common control gain K simultaneously stabilizing systems (11), and hence from Lemma 1 there exists a linear consensus protocol solving the consensus problem of the sampled-data multiagent system. This theorem has been proved. □

Remark 8. Zhang and Tian [14] studied the consensus of general linear dynamical multiagent systems and gave allowable sampling period bounds based on given protocol gain. Comparing with their results, the bound in Theorem 7 is less conservative since the bound is obtained by finding optimal protocol gain.

5. MASPB of Systems under Randomly Switching Topologies

This section focuses on looking for MASPBs for multiagent systems under randomly switching topologies. Two types of topology are discussed.

5.1. Rooted Directed Spanning Tree with Bernoulli Link Losses. Firstly we consider systems under tree-type topology. Due to communication constraints, the links between agents are time-varying and driven by a Bernoulli process. Assume all links in the network are independent. For an edge $(i, j) \in \mathcal{E}$ in the tree, $a_{ij}(kT)$ is varying between 0 and 1. Define r $(0 < r < 1)$ as the packet loss probability of the network; then $\Pr(a_{ij}(kT) = 0) = r$.

For a tree-type graph, number the agents such that each agent's parent node in the graph is lower numbered than itself. Then the Laplacian matrix is a lower triangular matrix with diagonal elements $l_{11}(kT) \equiv 0$ and $l_{ii}(kT)$ $(i > 1)$ is switching among 0 and 1 with probability $\Pr(l_{ii}(kT) = 0) = r$. $L(kT)$ can be denoted as $\left[\begin{smallmatrix} 0 & 0 \\ * & \overline{L}(kT)\end{smallmatrix}\right]$, where $\overline{L}(kT)$ is a $(n-1)$-dimensional lower triangular matrix.

For multiagent system (1)–(3), let $e_i(t) = x_i(t) - x_1(t)$, $i = 2, \ldots, n$, $e = [e_2^T, \ldots, e_n^T]^T$; then the system achieves consensus in mean square sense, if and only if $\lim_{t \to \infty} E(\|e(t)\|^2) = 0$. From system (4) and discretization, we can obtain that

$$e((k+1)T) = \left(I_n \otimes \overline{A} - \overline{L}(kT) \otimes \overline{B}K\right) e(kT). \tag{22}$$

To obtain the result, an important lemma from [23–25] is firstly given.

Lemma 9. *Consider a controllable system with packet loss*

$$x(k+1) = Gx(k) + Hu(k), \qquad u(k) = -\alpha(k) Kx(k), \tag{23}$$

where $\alpha(k) \in \{0, 1\}$ denotes packet loss process and is driven by an i.i.d. process with loss probability r. Then the closed-loop system $x(k+1) = (G - \alpha(k)HK)x(k)$ is mean square stable, if and only if $\rho(E((G - \alpha(k)HK) \otimes (G - \alpha(k)HK))) < 1$. And it is stabilizable in mean square sense, if the packet loss probability r satisfies $r < 1/\prod_j |\lambda_j^u|^2$, where λ_j^u are the unstable eigenvalues of G. Moreover, If the packet loss probability $r < 1/\prod_j |\lambda_j^u|^2$, then there exist $Q > 0$ and K such that

$$(1-r)(G-HK)^T Q (G-HK) + rG^T QG < Q. \tag{24}$$

Theorem 10. *For multiagent systems* (1)–(3) *under tree-type lossy network, there exists a linear consensus protocol solving the mean square consensus problem, if*

$$T < \frac{-\ln r}{2\sum_j \operatorname{Re}\left(\lambda_j^u\right)}, \tag{25}$$

where λ_j^u are the unstable eigenvalues of A.

Proof. System (25) is a Bernoulli switching system. It is mean square stable, if and only if $\rho(E(\Gamma(kT))) < 1$, where $\Gamma(kT) = (I_n \otimes \overline{A} - \overline{L}(kT) \otimes \overline{B}K) \otimes (I_n \otimes \overline{A} - \overline{L}(kT) \otimes \overline{B}K)$.

Obviously, $I_n \otimes \overline{A} - \overline{L}(kT) \otimes \overline{B}K$ is a lower block triangular matrix with diagonal blocks $\overline{A} - l_{ii}(k)\overline{B}K$. Then $\Gamma(kT)$ is also a lower block triangular matrix with diagonal blocks $(\overline{A} - l_{ii}(k)\overline{B}K) \otimes (\overline{A} - l_{jj}(k)\overline{B}K)$. Thus by applying the approach used in [22], $\rho(E(\Gamma(kT))) < 1$ if and only if for $i, j \in \{2, \ldots, n\}$, $\rho(E((\overline{A} - l_{ii}(k)\overline{B}K) \otimes (\overline{A} - l_{jj}(k)\overline{B}K))) < 1$. Since the edges of network are independent, there have that the mean square consensus condition is equivalent to $\rho(E((\overline{A} - l_{ii}(k)\overline{B}K) \otimes (\overline{A} - l_{ii}(k)\overline{B}K))) < 1$. Since $\Pr(l_{ii}(kT) = 0) = r$, then

$$\begin{aligned} \Phi &= E\left(\left(\overline{A} - l_{ii}(k)\overline{B}K\right) \otimes \left(\overline{A} - l_{ii}(k)\overline{B}K\right)\right) \\ &= r\overline{A} \otimes \overline{A} + (1-r)\left(\overline{A} - \overline{B}K\right) \otimes \left(\overline{A} - \overline{B}K\right). \end{aligned} \tag{26}$$

By applying Lemma 9, there exists K mean square stabilizing Φ, if $r < 1/\prod_j |\overline{\lambda}_j^u|^2$, where $\overline{\lambda}_j^u$ are the unstable eigenvalues of e^{AT}. Denote λ_j^u as the unstable eigenvalue of continuous-time system matrix A, then $\prod_j |\overline{\lambda}_j^u|^2 = \prod_j |e^{\lambda_j^u T}|^2 = \exp(2\sum_j \operatorname{Re}(\lambda_j^u) T)$. Thus if $T < (-\ln r)/2\sum_j \operatorname{Re}(\lambda_j^u)$, there exists K mean square stabilizing $\Gamma(kT)$, and there exists a linear consensus protocol solving the mean square consensus problem. Theorem 10 has been proved. □

5.2. Complete Network with Broadcasting Schemes and Bernoulli Link Losses. This subsection considers a network of agents with complete graph and broadcasting schemes. At each sampling instant, all agents compete for the communication channel with the same opportunity. Just one of the agents can succeed and broadcast its information to all other agents. Due to communication constraints, when the agent broadcasts its information, the links between itself and other agents may be lost. The process is driven by a Bernoulli process. Assume all links in the network are independent. Thus at each sampling instant kT, agent i broadcasts its information to other $n-1$ agents with probability $1/n$. When agent i sends its information to agent j, the edge (i, j) may be lost due to packet losses. Then $a_{ji}(kT)$ is varying between 0 and 1. Define r $(0 < r < 1)$ as the packet loss probability of the network, then $\Pr(a_{ji}(kT) = 0) = r$.

Denote $\mathcal{A}_i(kT) = [a_{js}^i(kT)]_{n \times n}$, $L_i(kT)$ as the adjacency and Laplacian matrix of the topology when agent i broadcasts its information, then for $j \neq i$, $a_{ji}^i(kT) \in \{0, 1\}$, for $s \neq i$, $a_{js}^i(kT) \equiv 0$. In this network, $L(kT) = \sum_{i=1}^n p_i(kT) L_i(kT)$, where $p_i(kT) \in \{0, 1\}$, $\sum_{i=1}^n p_i(kT) = 1$, and $\Pr(p_i(kT) = 1) = 1/n$. Then the system can be given as

$$x((k+1)T) = \left(I_n \otimes \overline{A} - \sum_{i=1}^n p_i(kT) L_i(kT) \otimes \overline{B}K\right) x(kT). \tag{27}$$

Theorem 11. *For multiagent systems* (1)–(3) *under complete and broadcasting lossy network, there exists a linear consensus protocol solving the mean square consensus problem, if*

$$T < \frac{-\ln(\alpha+\beta)^{\mu-1}\beta}{2\sum_{i=1}^{\mu} \operatorname{Re}(\lambda_i)}, \tag{28}$$

where λ_j^u are the unstable eigenvalues of A, μ is the number of unstable eigenvalues of A, $\alpha = ((1-r)/n)(n + (n-2)r)$, $\beta = (r/n)(2n - 2 - (n-2)r)$.

Proof. Firstly, we will show that for agents' system matrices A, just unstable parts should be considered. For A and B, there always exists a nonsingular matrix Q such that $QAQ^{-1} = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}$, $QB = [B_1 \ \ B_2]$, and then $Q\overline{A}Q^{-1} = \begin{bmatrix} \overline{A}_s & 0 \\ 0 & \overline{A}_u \end{bmatrix}$, $Q\overline{B} = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix}$, where A_s, A_u are Hurwitz stable and unstable matrices, respectively, and $\overline{A}_s$, $\overline{A}_u$ are Schur stable and unstable matrices, respectively. Define $KQ^{-1} = [0 \ \ K_u]$, then system matrix in (6) is similar to $\begin{bmatrix} I_n \otimes \overline{A}_s & -L(kT)\otimes \overline{B}_1 K_u \\ 0 & I_n \otimes \overline{A}_u - L(kT)\otimes \overline{B}_2 K_u \end{bmatrix}$. Obviously, the system achieves consensus if and only if the part $\tilde{x}((k+1)T) = (I_n \otimes \overline{A}_u - L(kT) \otimes \overline{B}_2 K_u)\tilde{x}(kT)$ achieves consensus. Therefore, just unstable part of A should be controlled. For depiction simplicity, in the following we just consider the unstable matrix A.

For edge (i, j), define $e_{ij}(kT) = x_i(kT) - x_j(kT)$, $V_{ij}(kT) = e_{ij}^T(kT) P e_{ij}(kT)$, $V(kT) = \sum_{i,j=1}^{n} E(V_{ij}(kT))$. For $V_{ij}(kT)$, there are 8 cases.

Case 1. Agent i broadcasts and j successfully receives its packet. Then $x_j((k+1)T) = \overline{A}x_j(kT) + \overline{B}K(x_i(kT) - x_j(kT))$ and $V_{ij}((k+1)T) = e_{ij}^T(kT)(\overline{A} - \overline{B}K)^T P(\overline{A} - \overline{B}K)e_{ij}(kT)$.

Case 2. Agent i broadcasts and j doesn't receive its packet $V_{ij}((k+1)T) = e_{ij}^T(kT)\overline{A}^T P\overline{A}e_{ij}(kT)$.

Case 3. Agent j broadcasts and i successfully receives its packet. Then $V_{ij}((k+1)T) = e_{ji}^T(kT)(\overline{A} - \overline{B}K)^T P(\overline{A} - \overline{B}K)e_{ji}(kT)$.

Case 4. Agent j broadcasts and i doesn't receive its packet. $V_{ij}((k+1)T) = e_{ji}^T(kT)\overline{A}^T P\overline{A}e_{ji}(kT)$.

Case 5. Agent s $(s \neq i, j)$ broadcasts and both i and j receive the packet. Then $x_j((k+1)T) = \overline{A}x_j(kT) + \overline{B}K(x_s(kT) - x_j(kT))$, $x_i((k+1)T) = \overline{A}x_i(kT) + \overline{B}K(x_s(kT) - x_i(kT))$, and $V_{ij}((k+1)T) = e_{ij}^T(kT)(\overline{A} - \overline{B}K)^T P(\overline{A} - \overline{B}K)e_{ij}(kT)$.

Case 6. Agent s $(s \neq i, j)$ broadcasts and i successfully receives its packet while j doesn't receive the packet. Then $x_i((k+1)T) = \overline{A}x_i(kT) + \overline{B}K(x_s(kT) - x_i(kT))$, $x_j((k+1)T) = \overline{A}x_j(kT)$, and

$$\begin{aligned}
V_{ij}((k+1)T) &= \left\| x_i((k+1)T) - x_j((k+1)T) \right\|_P^2 \\
&\leq \Big(\left\| x_i((k+1)T) - x_s((k+1)T) \right\|_P \\
&\qquad + \left\| x_s((k+1)T) - x_j((k+1)T) \right\|_P \Big)^2 \\
&\leq 2\left\| e_{si}((k+1)T) \right\|_P^2 + 2\left\| e_{sj}((k+1)T) \right\|_P^2 \\
&= 2e_{si}^T(kT)\left(\overline{A} - \overline{B}K\right)^T P \left(\overline{A} - \overline{B}K\right) e_{si}(kT) \\
&\quad + 2e_{sj}^T(kT)\,\overline{A}^T P\overline{A} e_{sj}(kT).
\end{aligned} \tag{29}$$

Case 7. Agent s $(s \neq i)$ broadcasts and j successfully receives its packet while i doesn't receive the packet. Then

$$\begin{aligned}
V_{ij}((k+1)T) &= \left\| x_i((k+1)T) - x_j((k+1)T) \right\|_P^2 \\
&\leq 2\left\| e_{si}((k+1)T) \right\|_P^2 + 2\left\| e_{sj}((k+1)T) \right\|_P^2 \\
&= 2e_{sj}^T(kT)\left(\overline{A} - \overline{B}K\right)^T P \left(\overline{A} - \overline{B}K\right) e_{sj}(kT) \\
&\quad + 2e_{si}^T(kT)\,\overline{A}^T P\overline{A} e_{si}^T(kT).
\end{aligned} \tag{30}$$

Case 8. Agent s $(s \neq i)$ broadcasts and neither i nor j receives the packet. Then $V_{ij}((k+1)T) = e_{ij}^T(kT)\overline{A}^T P\overline{A}e_{ij}^T(kT)$.

Since agent i broadcasts its information with probability $1/n$, and the packet loss probability is r, there is

$$\begin{aligned}
&E\left(V_{ij}((k+1)T)\right) \\
&\leq \left(\frac{1}{n}(1-r) + \frac{n-1}{n}(1-r)^2\right) \\
&\quad \times E\left(e_{ij}^T(kT)\left(\overline{A} - \overline{B}K\right)^T P\left(\overline{A} - \overline{B}K\right) e_{ij}(kT)\right) \\
&\quad + \left(\frac{1}{n}r + \frac{n-1}{n}r^2\right) E\left(e_{ij}^T(kT)\,\overline{A}^T P\overline{A} e_{ij}(kT)\right) \\
&\quad + \frac{1}{n}(1-r)\, E\left(e_{ji}^T(kT)\left(\overline{A} - \overline{B}K\right)^T P\left(\overline{A} - \overline{B}K\right) e_{ji}(kT)\right) \\
&\quad + \frac{1}{n} r E\left(e_{ji}^T(kT)\,\overline{A}^T P\overline{A} e_{ji}(kT)\right) \\
&\quad + 2\sum_{s\neq i,j} \frac{1}{n} r(1-r)\, E\left(e_{si}^T(kT)\right) \\
&\quad \times \left(\left(\overline{A} - \overline{B}K\right)^T P\left(\overline{A} - \overline{B}K\right) + \overline{A}^T P\overline{A}\right) e_{si}(kT) \\
&\quad + 2\sum_{s\neq i,j} \frac{1}{n} r(1-r)\, E\left(e_{sj}^T(kT)\right) \\
&\quad \times \left(\left(\overline{A} - \overline{B}K\right)^T P\left(\overline{A} - \overline{B}K\right) + \overline{A}^T P\overline{A}\right) e_{sj}(kT),
\end{aligned} \tag{31}$$

and then

$$\begin{aligned}
V((k+1)T) &= \sum_{i,j=1}^{n} E\left(V_{ij}((k+1)T)\right) \\
&\leq \sum_{i,j=1}^{n} E\Big(e_{ij}^T(kT)\Big(\alpha\left(\overline{A} - \overline{B}K\right)^T P\left(\overline{A} - \overline{B}K\right) \\
&\qquad + \beta\overline{A}^T P\overline{A}\Big) e_{ij}(kT)\Big),
\end{aligned} \tag{32}$$

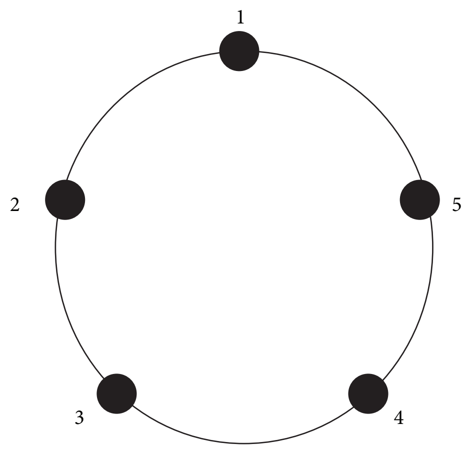

Figure 1: Communication topology among agents.

where

$$\alpha = \frac{1}{n}(1-r) + \frac{1}{n}(1-r) + \frac{n-2}{n}(1-r)^2 + 2\frac{n-2}{n}r(1-r) = \frac{1-r}{n}(n + (n-2)r), \tag{33}$$

$$\beta = \frac{1}{n}r + \frac{1}{n}r + \frac{n-2}{n}r^2 + 2\frac{n-2}{n}r(1-r) = \frac{r}{n}(2n - 2 - (n-2)r). \tag{34}$$

If $\alpha(\overline{A} - \overline{B}K)^T P(\overline{A} - \overline{B}K) + \beta\overline{A}^T P\overline{A} < P$, then $V((k+1)T) < V(kT)$, and mean square consensus is achieved. The inequality is converted to

$$\frac{\alpha}{\alpha+\beta}\left(\sqrt{\alpha+\beta}\overline{A} - \sqrt{\alpha+\beta}\overline{B}K\right)^T P\left(\sqrt{\alpha+\beta}\overline{A} - \sqrt{\alpha+\beta}\overline{B}K\right) + \frac{\beta}{\alpha+\beta}\sqrt{\alpha+\beta}\overline{A}^T P\sqrt{\alpha+\beta}\overline{A} < P \tag{35}$$

By applying Lemma 9 we have that if $\beta/(\alpha+\beta) < 1/\prod_j|\lambda_j^u(\sqrt{\alpha+\beta}A)|^2$, then the above LMI is feasible, $\lambda_j^u(\sqrt{\alpha+\beta}A)$ are the unstable eigenvalues of $\sqrt{\alpha+\beta}A$. From the form of $\overline{A}$, the above inequalities can be simplified as $(\alpha+\beta)^{\mu-1}\beta e^{2\sum_{i=1}^{\mu}\mathrm{Re}(\lambda_i)T} < 1$, where the eigenvalue λ_i of A satisfies $\mathrm{Re}(\lambda_i) \geq 0$, μ is the number of unstable eigenvalues. Thus as long as $T < (-\ln(\alpha+\beta)^{\mu-1}\beta)/2\sum_{i=1}^{\mu}\mathrm{Re}(\lambda_i)$, there exists a linear consensus protocol solving the mean square consensus problem. Theorem 11 has been proved. □

6. Simulation Examples

6.1. Fixed Topology Cases. Firstly consider a network of 5 agents. The topology is undirected and cyclic as given in Figure 1. Then the eigenvalues of Laplacian matrix are $\lambda_4 = 3.618$, $\lambda_1 = 1.382$.

Consider a network of agents with single input. The system matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{36}$$

From Theorem 5, the maximum allowable sampling period bound is 0.4024 s. Choose $T = 0.35$ s, then by solving LMIs $(\overline{A} - \lambda_i\overline{B}K)^T Q(\overline{A} - \lambda_i\overline{B}K) < Q$ we obtain a control gain $K = [0.2183 \ \ -0.2537 \ \ 1.5837]$. The trajectories of agents are given in Figure 2. Obviously, the system reaches consensus asymptotically.

If $T = 0.45$ s, then the LMIs $(\overline{A} - \lambda_i\overline{B}K)^T Q(\overline{A} - \lambda_i\overline{B}K) < Q$ are infeasible. For a control gain $K = [0 \ \ -0.6643 \ \ 1.2524]$, the trajectory of average consensus errors is given in Figure 3. Obviously, the system does not achieve consensus.

Next consider a network of agents with multi-input. The system matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}. \tag{37}$$

From Theorem 7, the maximum allowable sampling period bound is 0.85 s. Choose $T = 0.75$ s, then by solving LMI (19) we obtain a control gain $K = \begin{bmatrix} 0 & -0.9836 & 0.7337 \\ 0 & 0.4841 & 0.2499 \end{bmatrix}$. The trajectories of agents are given in Figure 4. Obviously, the system reaches consensus asymptotically.

6.2. Tree-Type Network with Bernoulli Link Losses. Consider a network of 5 agents with system matrix (36). The topology is a rooted spanning tree as shown in Figure 5. The packet loss probability is 0.1. Then by applying Theorem 10, the maximum allowable sampling period bound is 0.5756 s. Choose $T = 0.5$ s, then by solving LMI $(1-r)(\overline{A} - \overline{B}K)^T Q(\overline{A} - \overline{B}K) + r\overline{A}^T Q\overline{A} < Q$ we obtain a control gain $K = [0.3569 \ \ -0.9649 \ \ 3.1304]$. The trajectories of agents are given in Figure 6. Obviously, the system reaches consensus.

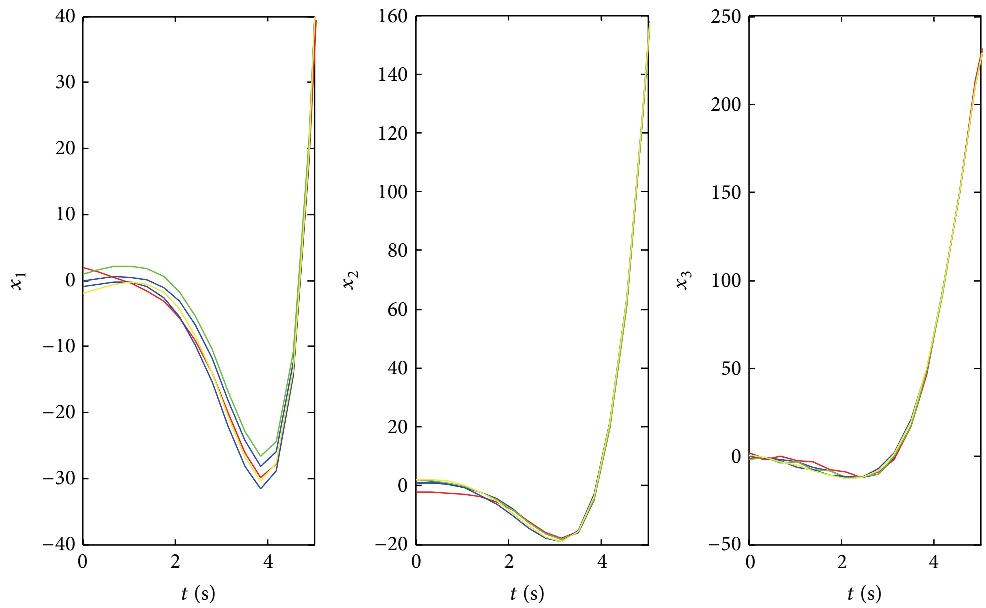

FIGURE 2: Trajectories of single input agents under fixed topology.

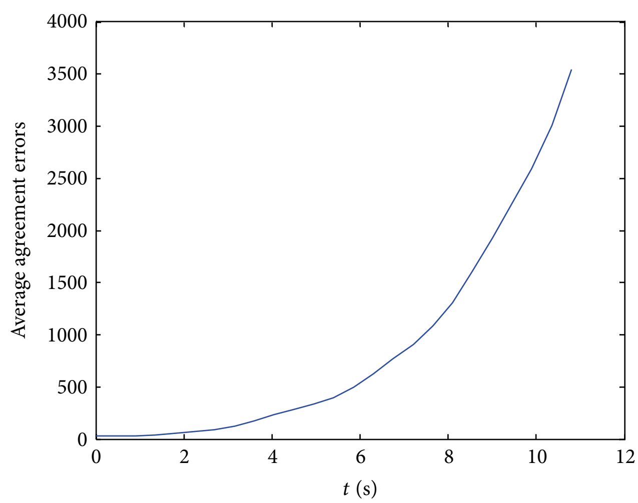

FIGURE 3: Average consensus errors with unallowable sampling period.

6.3. Complete Network with Broadcasting Schemes and Bernoulli Link Losses. Consider a network of 5 agents with system matrix (36). The network is a complete broadcasting graph. The packet loss probability is 0.1. Then by applying Theorem 11, the maximum allowable sampling period bound is 0.4164 s. Choose $T = 0.4$ s, then by solving LMIs $\alpha(\overline{A} - \overline{B}K)^T P(\overline{A} - \overline{B}K) + \beta\overline{A}^T P\overline{A} < P$ we obtain a control gain $K = [0.3999 \quad -0.6330 \quad 3.6687]$. The trajectories of agents are given in Figure 7. Obviously, the system reaches consensus.

7. Conclusion

This paper focuses on looking for an allowable sampling period bound such that as long as the sampling period is less than this period bound, there exists a state feedback consensus algorithm solving the consensus problem. The allowable sampling period bounds for sampled-data multiagent systems under fixed topology and two specific Bernoulli lossy networks are provided. Comparing with existing results, the proposed MASPDs are explicitly related to unstable eigenvalues of agents' system matrix and packet loss probability and can be directly computed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

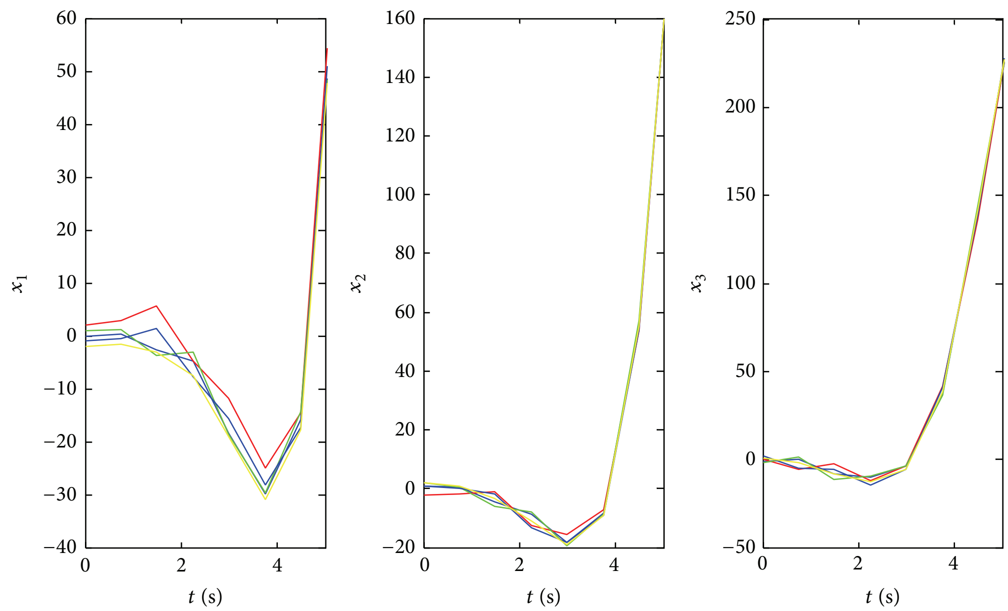

Figure 4: Trajectories of multi-input agents under fixed topology.

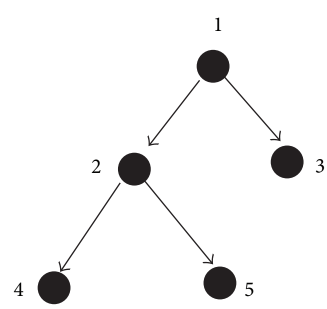

Figure 5: Tree-type topology among agents.

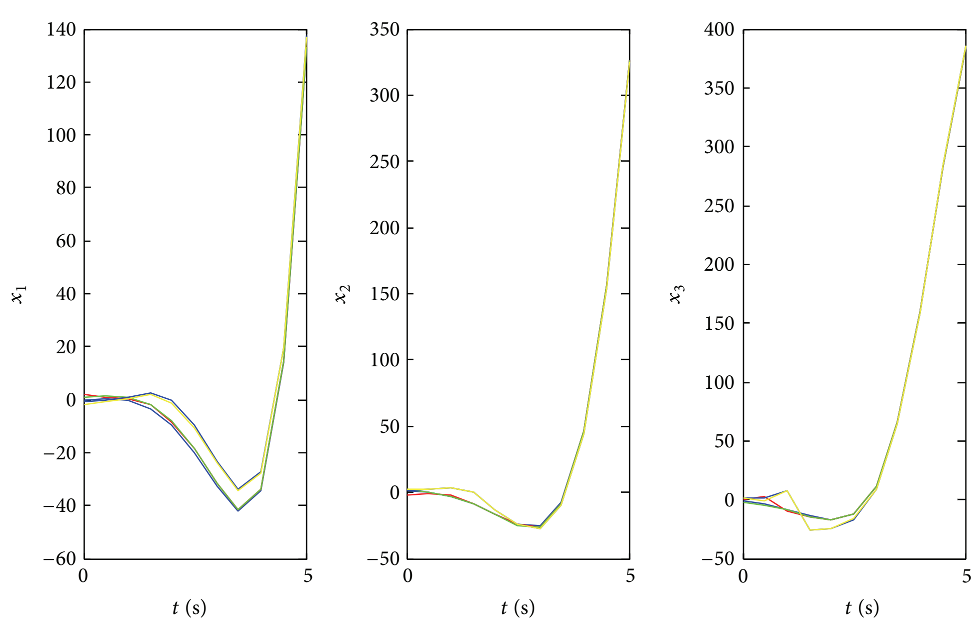

Figure 6: Trajectories of agents under tree-type lossy network.

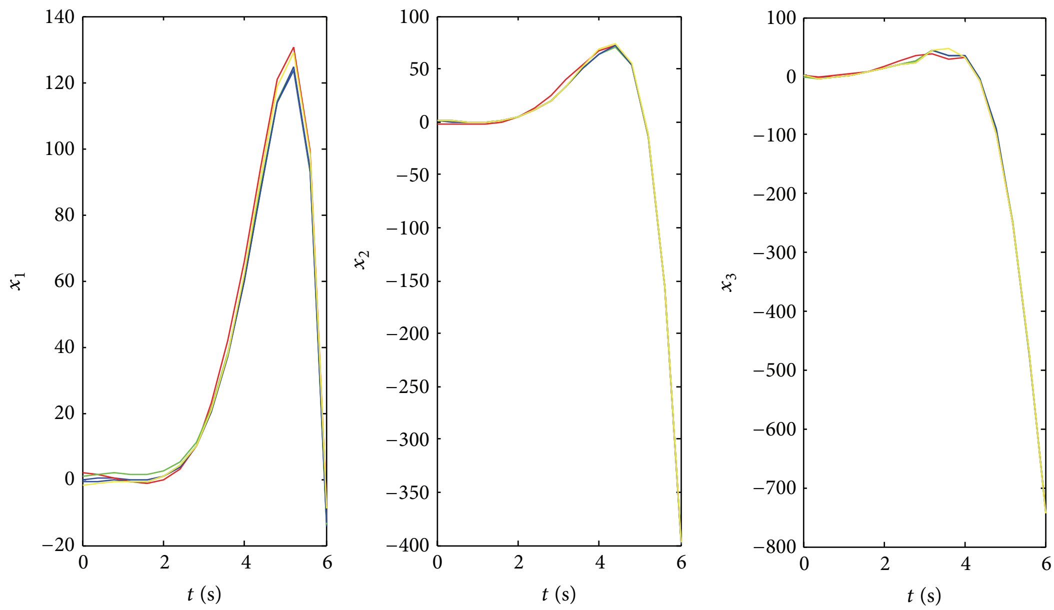

Figure 7: Trajectories of agents under complete broadcasting lossy network.

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