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Research Article

Umbral Calculus and the Frobenius-Euler Polynomials

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We study some properties of umbral calculus related to the Appell sequence. From those properties, we derive new and interesting identities of the Frobenius-Euler polynomials.

1. Introduction

Let $\mathbf{C}$ be the complex number field. For $\lambda \in \mathbf{C}$ with $\lambda \neq 1$, the Frobenius-Euler polynomials are defined by the generating function to be

$$\frac{1-\lambda}{e^t-\lambda}e^{xt} = e^{H(x|\lambda)t} = \sum_{n=0}^{\infty} H_n(x \mid \lambda)\frac{t^n}{n!}, \qquad (1)$$

(see [1–5]) with the usual convention about replacing $H^n(x \mid \lambda)$ by $H_n(x \mid \lambda)$.

In the special case, $x = 0$, $H_n(0 \mid \lambda) = H_n(\lambda)$ are called the nth Frobenius-Euler numbers. By (1), we get

$$H_n(x \mid \lambda) = \sum_{l=0}^{n}\binom{n}{l} H_{n-l}(\lambda)x^l = (H(\lambda)+x)^n, \qquad (2)$$

(see [6–9]) with the usual convention about replacing $H^n(\lambda)$ by $H_n(\lambda)$.

Thus, from (1) and (2), we note that

$$(H(\lambda)+1)^n - \lambda H_n(\lambda) = (1-\lambda)\delta_{0,n}, \qquad (3)$$

where $\delta_{n,k}$ is the kronecker symbol (see [1, 10, 11]).

For $r \in \mathbf{Z}_+$, the Frobenius-Euler polynomials of order r are defined by the generating function to be

$$\begin{aligned}\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} &= \underbrace{\left(\frac{1-\lambda}{e^t-\lambda}\right)\times\cdots\times\left(\frac{1-\lambda}{e^t-\lambda}\right)}_{r\text{-times}} e^{xt} \\ &= \sum_{n=0}^{\infty} H_n^{(r)}(x \mid \lambda)\frac{t^n}{n!}.\end{aligned} \qquad (4)$$

In the special case, $x = 0$, $H_n^{(r)}(0 \mid \lambda) = H_n^{(r)}(\lambda)$ are called the nth Frobenius-Euler numbers of order r (see [1, 10]).

From (4), we can derive the following equation:

$$\begin{aligned}H_n^{(r)}(x \mid \lambda) &= \sum_{l=0}^{n}\binom{n}{l} H_{n-l}^{(r)}(\lambda)x^l, \\ H_n^{(r)}(\lambda) &= \sum_{l_1+\cdots+l_r=n}\binom{n}{l_1,\ldots,l_r} H_{l_1}(\lambda)\cdots H_{l_r}(\lambda).\end{aligned} \qquad (5)$$

By (5), we see that $H_n^{(r)}(x \mid \lambda)$ is a monic polynomial of degree n with coefficients in $\mathbf{Q}(\lambda)$.

Let $\mathbb{P}$ be the algebra of polynomials in the single variable x over $\mathbf{C}$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. As is known, $\langle L \mid p(x)\rangle$ denotes the action of the linear functional L on a polynomial $p(x)$ and we remind that the addition and scalar multiplication on $\mathbb{P}^*$ are, respectively, defined by

$$\langle L+M \mid p(x)\rangle = \langle L \mid p(x)\rangle + \langle M \mid p(x)\rangle, \qquad \langle cL \mid p(x)\rangle = c\langle L \mid p(x)\rangle, \tag{6}$$

where c is a complex constant (see [3, 12]).

Let $\mathbf{F}$ denote the algebra of formal power series:

$$\mathbf{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \mid a_k \in \mathbf{C} \right\} \tag{7}$$

(see [3, 12]). The formal power series define a linear functional on $\mathbb{P}$ by setting

$$\left\langle f(t) \mid x^n \right\rangle = a_n, \quad \forall n \geq 0. \tag{8}$$

Indeed, by (7) and (8), we get

$$\langle t^k \mid x^n \rangle = n!\delta_{n,k} \quad (n, k \geq 0) \tag{9}$$

(see [3, 12]). This kind of algebra is called an umbral algebra.

The order $O(f(t))$ of a nonzero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. A series $f(t)$ for which $O(f(t)) = 1$ is said to be an invertible series (see [2, 12]). For $f(t), g(t) \in \mathbf{F}$, and $p(x) \in \mathbb{P}$, we have

$$\begin{aligned} \left\langle f(t) g(t) \mid p(x) \right\rangle &= \left\langle f(t) \mid g(t) p(x) \right\rangle \\ &= \left\langle g(t) \mid f(t) p(x) \right\rangle \end{aligned} \tag{10}$$

(see [12]). One should keep in mind that each $f(t) \in \mathbf{F}$ plays three roles in the umbral calculus: a formal power series, a linear functional, and a linear operator. To illustrate this, let $p(x) \in \mathbb{P}$ and $f(t) = e^{yt} \in \mathbf{F}$. As a linear functional, e^{yt} satisfies $\langle e^{yt} \mid p(x)\rangle = p(y)$. As a linear operator, e^{yt} satisfies $e^{yt}p(x) = p(x+y)$ (see [12]). Let $s_n(x)$ denote a polynomial in x with degree n. Let us assume that $f(t)$ is a delta series and $g(t)$ is an invertible series. Then there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)f(t)^k \mid s_n(x)\rangle = n!\delta_{n,k}$ for all $n, k \geq 0$ (see [3, 12]). This sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$. If $s_n(x) \sim (1, f(t))$, then $s_n(x)$ is called the associated sequence for $f(t)$. If $s_n(x) \sim (g(t), t)$, then $s_n(x)$ is called the Appell sequence.

Let $s_n(x) \sim (g(t), f(t))$. Then we see that

$$\begin{gathered} h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) \mid s_k(x)\rangle}{k!} g(t) f(t)^k, \quad h(t) \in \mathbf{F}, \\ p(x) = \sum_{k=0}^{\infty} \frac{\left\langle g(t) f(t)^k \mid p(x) \right\rangle}{k!} s_k(x), \quad p(x) \in \mathbb{P}, \\ f(t) s_n(x) = n s_{n-1}(x), \\ \left\langle f(t) \mid p(\alpha x) \right\rangle = \left\langle f(\alpha t \mid p(x) \right\rangle, \\ \frac{1}{g\left(\overline{f}(t)\right)} e^{y\overline{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k, \quad \forall y \in \mathbf{C}, \end{gathered} \tag{11, 12}$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$ (see [3]). In this paper, we study some properties of umbral calculus related to the Appell sequence. For those properties, we derive new and interesting identities of the Frobenius-Euler polynomials.

2. The Frobenius-Euler Polynomials and Umbral Calculus

By (4) and (12), we see that

$$H_n^{(r)}(x \mid \lambda) \sim \left(\left(\frac{e^t - \lambda}{1-\lambda} \right)^r, t \right). \tag{13}$$

Thus, by (13), we get

$$\left\langle \left(\frac{e^t - \lambda}{1-\lambda} \right)^r t^k \mid H_n^{(r)}(x \mid \lambda) \right\rangle = n!\delta_{n,k}. \tag{14}$$

Let

$$\mathbb{P}_n(\lambda) = \left\{ p(x) \in \mathbf{Q}(\lambda)[x] \mid \deg p(x) \leq n \right\}. \tag{15}$$

Then it is an $(n+1)$-dimensional vector space over $\mathbf{Q}(\lambda)$.

So we see that $\{H_0^{(r)}(x \mid \lambda), H_1^{(r)}(x \mid \lambda), \ldots, H_n^{(r)}(x \mid \lambda)\}$ is a basis for $\mathbb{P}_n(\lambda)$. For $p(x) \in \mathbb{P}_n(\lambda)$, let

$$p(x) = \sum_{k=0}^{n} C_k H_k^{(r)}(x \mid \lambda), \quad (n \geq 0). \tag{16}$$

Then, by (13), (14), and (16), we get

$$\begin{aligned} &\left\langle \left(\frac{e^t - \lambda}{1-\lambda} \right)^r t^k \mid p(x) \right\rangle \\ &\quad = \sum_{l=0}^{n} C_l \left\langle \left(\frac{e^t - \lambda}{1-\lambda} \right)^r t^k \mid H_l^{(r)}(x \mid \lambda) \right\rangle \\ &\quad = \sum_{l=0}^{n} C_l l! \delta_{l,k} = k! C_k. \end{aligned} \tag{17}$$

From (17), we have

$$\begin{aligned} C_k &= \frac{1}{k!} \left\langle \left(\frac{e^t - \lambda}{1-\lambda} \right)^r t^k \mid p(x) \right\rangle \\ &= \frac{1}{k!} \left\langle \left(\frac{e^t - \lambda}{1-\lambda} \right)^r \mid D^k p(x) \right\rangle \\ &= \frac{1}{k!(1-\lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} \left\langle e^{jt} \mid D^k p(x) \right\rangle \\ &= \frac{1}{k!(1-\lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} \left\langle t^0 \mid e^{jt} D^k p(x) \right\rangle \\ &= \frac{1}{k!(1-\lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} \left\langle t^0 \mid D^k p(x+j) \right\rangle. \end{aligned} \tag{18}$$

Therefore, by (16) and (18), we obtain the following theorem.

Theorem 1. *For $p(x) \in \mathbb{P}_n(\lambda)$, let*

$$p(x) = \sum_{k=0}^{n} C_k H_k^{(r)}(x). \tag{19}$$

Then one has

$$C_k = \frac{1}{k!(1-\lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} D^k p(j), \tag{20}$$

where $Dp(x) = dp(x)/dx$.

From Theorem 1, we note that

$$p(x) = \frac{1}{(1-\lambda)^r} \cdot \sum_{k=0}^{n} \left\{ \sum_{j=0}^{r} \frac{1}{k!} \binom{r}{j} (-\lambda)^{r-j} D^k p(j) \right\} H_k^{(r)}(x \mid \lambda). \tag{21}$$

Let us consider the operator $\widetilde{\Delta}_\lambda$ with $\widetilde{\Delta}_\lambda f(x) = f(x+1) - \lambda f(x)$ and let $J_\lambda = (1/(1-\lambda))\widetilde{\Delta}_\lambda$. Then we have

$$J_\lambda(f)(x) = \frac{1}{1-\lambda}\{f(x+1) - \lambda f(x)\}. \tag{22}$$

Thus, by (22), we get

$$J_\lambda\left(H_n^{(r)}(x \mid \lambda)\right) = \frac{1}{1-\lambda}\left\{H_n^{(r)}(x+1 \mid \lambda) - \lambda H_n^{(r)}(x \mid \lambda)\right\}. \tag{23}$$

From (4), we can derive

$$\begin{aligned}
\sum_{n=0}^{\infty} &\left\{H_n^{(r)}(x+1 \mid \lambda) - \lambda H_n^{(r)}(x \mid \lambda)\right\} \frac{t^n}{n!} \\
&= \left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{(x+1)t} - \lambda \left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} \\
&= \left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt}\left(e^t-\lambda\right) = (1-\lambda)\left(\frac{1-\lambda}{e^t-\lambda}\right)^{r-1} e^{xt} \\
&= (1-\lambda)\sum_{n=0}^{\infty} H_n^{(r-1)}(x \mid \lambda)\frac{t^n}{n!}.
\end{aligned} \tag{24}$$

By (23) and (24), we get

$$J_\lambda\left(H_n^{(r)}(x \mid \lambda)\right) = H_n^{(r-1)}(x \mid \lambda). \tag{25}$$

From (25), we have

$$\begin{aligned}
J_\lambda^r\left(H_n^{(r)}(x \mid \lambda)\right) &= J_\lambda^{r-1}\left(H_n^{(r-1)}(x \mid \lambda)\right) \\
&= \cdots = H_n^{(0)}(x \mid \lambda) = x^n, \\
J_\lambda^r(x^n) = J_\lambda^r H_n^{(0)}(x \mid \lambda) &= H_n^{(-r)}(x \mid \lambda) = J_\lambda^{2r} H_n^{(r)}(x \mid \lambda).
\end{aligned} \tag{26}$$

For $s \in \mathbf{Z}_+$, from (25), we have

$$J_\lambda^s\left(H_n^{(r)}(x \mid \lambda)\right) = H_n^{(r-s)}(x \mid \lambda). \tag{27}$$

On the other hand, by (12), (13), and (25),

$$\begin{aligned}
J_\lambda^s\left(H_n^{(r)}(x \mid \lambda)\right) &= \left(\frac{e^t-\lambda}{1-\lambda}\right)^s \left(H_n^{(r)}(x \mid \lambda)\right) \\
&= \frac{1}{(1-\lambda)^s}\left((1-\lambda) + \sum_{k=1}^{\infty}\frac{t^k}{k!}\right)^s \\
&\quad \cdot \left(H_n^{(r)}(x \mid \lambda)\right).
\end{aligned} \tag{28}$$

Thus, by (28), we get

$$\begin{aligned}
&J_\lambda^s\left(H_n^{(r)}(x \mid \lambda)\right) \\
&= \sum_{m=0}^{s} \frac{\binom{s}{m}}{(1-\lambda)^m} \sum_{l=m}^{\infty} \left(\sum_{\substack{k_1+\cdots+k_m=l \\ k_j \geq 1}} \frac{1}{k_1!\cdots k_m!} \right) t^l \left(H_n^{(r)}(x \mid \lambda)\right) \\
&= \sum_{m=0}^{s} \frac{\binom{s}{m}}{(1-\lambda)^m} \sum_{l=m}^{\infty} \frac{1}{l!} \left(\sum_{\substack{k_1+\cdots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1,\ldots,k_m} D^l \right) \\
&\quad \cdot H_n^{(r)}(x \mid \lambda) \\
&= \sum_{m=0}^{\min\{s,n\}} \frac{\binom{s}{m}}{(1-\lambda)^m} \sum_{l=m}^{n} \binom{n}{l} \sum_{\substack{k_1+\cdots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1,\ldots,k_m} H_{n-l}^{(r)}(x \mid \lambda) \\
&= \sum_{l=0}^{\min\{s,n\}} \left\{ \binom{n}{l} \sum_{m=0}^{l} \frac{\binom{s}{m}}{(1-\lambda)^m} \right. \\
&\qquad \left. \cdot \sum_{\substack{k_1+\cdots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1,\ldots,k_m} \right\} H_{n-l}^{(r)}(x \mid \lambda) \\
&\quad + \sum_{l=\min\{s,n\}+1}^{n} \left\{ \binom{n}{l} \sum_{m=0}^{\min\{s,n\}} \frac{\binom{s}{m}}{(1-\lambda)^m} \right. \\
&\qquad \left. \cdot \sum_{\substack{k_1+\cdots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1,\ldots,k_m} \right\} H_{n-l}^{(r)}(x \mid \lambda).
\end{aligned} \tag{29}$$

Therefore, by (27) and (29), we obtain the following theorem.

Theorem 2. *For any $r, s \geq 0$, one has*

$$\begin{aligned}
&H_n^{(r-s)}(x \mid \lambda) \\
&= \sum_{l=0}^{\min\{s,n\}} \left\{ \binom{n}{l} \sum_{m=0}^{l} \frac{\binom{s}{m}}{(1-\lambda)^m} \sum_{\substack{k_1+\cdots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \ldots, k_m} \right\} \\
&\quad \cdot H_{n-l}^{(r)}(x \mid \lambda) \\
&\quad + \sum_{l=\min\{s,n\}+1}^{n} \left\{ \binom{n}{l} \sum_{m=0}^{\min\{s,n\}} \frac{\binom{s}{m}}{(1-\lambda)^m} \right. \\
&\qquad \left. \cdot \sum_{\substack{k_1+\cdots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \ldots, k_m} \right\} H_{n-l}^{(r)}(x \mid \lambda).
\end{aligned} \tag{30}$$

Let us take $s = r-1$ $(r \geq 1)$ in Theorem 2. Then we obtain the following corollary.

Corollary 3. *For $n \geq 0$, $r \geq 1$, one has*

$$\begin{aligned}
&H_n(x \mid \lambda) \\
&= \sum_{l=0}^{\min\{r-1,n\}} \left\{ \binom{n}{l} \sum_{m=0}^{l} \frac{\binom{r-1}{m}}{(1-\lambda)^m} \sum_{\substack{k_1+\cdots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \ldots, k_m} \right\} \\
&\quad \cdot H_{n-l}^{(r)}(x \mid \lambda) \\
&\quad + \sum_{l=\min\{r-1,n\}+1}^{n} \left\{ \binom{n}{l} \sum_{m=0}^{\min\{r-1,n\}} \frac{\binom{r-1}{m}}{(1-\lambda)^m} \right. \\
&\qquad \left. \cdot \sum_{\substack{k_1+\cdots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \ldots, k_m} \right\} H_{n-l}^{(r)}(x \mid \lambda).
\end{aligned} \tag{31}$$

Let us take $s = r$ $(r \geq 1)$ in Theorem 2. Then we obtain the following corollary.

Corollary 4. *For $n \geq 0, r \geq 1$, one has*

$$\begin{aligned}
x^n &= \sum_{l=0}^{\min\{r,n\}} \left\{ \binom{n}{l} \sum_{m=0}^{l} \frac{\binom{r}{m}}{(1-\lambda)^m} \sum_{\substack{k_1+\cdots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \ldots, k_m} \right\} \\
&\quad \cdot H_{n-l}^{(r)}(x \mid \lambda) \\
&\quad + \sum_{l=\min\{r,n\}+1}^{n} \left\{ \binom{n}{l} \sum_{m=0}^{\min\{r,n\}} \frac{\binom{r}{m}}{(1-\lambda)^m} \right. \\
&\qquad \left. \cdot \sum_{\substack{k_1+\cdots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \ldots, k_m} \right\} H_{n-l}^{(r)}(x \mid \lambda).
\end{aligned} \tag{32}$$

Now, we define the analogue of Stirling numbers of the second kind as follows:

$$S_\lambda(n,k) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-\lambda)^{k-j} j^n, \quad (n, k \geq 0). \tag{33}$$

Note that $S_1(n,k) = S(n,k)$ is the Stirling number of the second kind.

From the definition of $\widetilde{\Delta}_\lambda$, we have

$$\widetilde{\Delta}_\lambda^n f(0) = \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} f(k). \tag{34}$$

By (33) and (34), we get

$$S_\lambda(n,k) = \frac{1}{k!} \widetilde{\Delta}_\lambda^k 0^n, \quad (n, k \geq 0). \tag{35}$$

Let us take $s = 2r$. Then we have

$$\begin{aligned}
&J_\lambda^r x^n \\
&= H_n^{(-r)}(x \mid \lambda) \\
&= \sum_{l=0}^{\min\{2r,n\}} \left\{ \binom{n}{l} \sum_{m=0}^{l} \frac{\binom{2r}{m}}{(1-\lambda)^m} \sum_{\substack{k_1+\cdots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \ldots, k_m} \right\} \\
&\quad \cdot H_{n-l}^{(r)}(x \mid \lambda) \\
&\quad + \sum_{l=\min\{2r,n\}+1}^{n} \left\{ \binom{n}{l} \sum_{m=0}^{\min\{2r,n\}} \frac{\binom{2r}{m}}{(1-\lambda)^m} \right. \\
&\qquad \left. \cdot \sum_{\substack{k_1+\cdots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1, \ldots, k_m} \right\} H_{n-l}^{(r)}(x \mid \lambda), \\
&J_\lambda^r x^n = \left(\frac{1}{1-\lambda} \widetilde{\Delta}_\lambda \right)^r x^n \\
&\quad = \frac{1}{(1-\lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} (x+j)^n.
\end{aligned} \tag{36}$$

By (36), we get

$$\begin{aligned}
&\frac{1}{(1-\lambda)^r}\sum_{j=0}^{r}\binom{r}{j}(-\lambda)^{r-j}(x+j)^n\\
&=\frac{1}{(1-\lambda)^r}\widetilde{\Delta}_\lambda^r x^n\\
&=\sum_{l=0}^{\min\{2r,n\}}\left\{\binom{n}{l}\sum_{m=0}^{l}\frac{\binom{2r}{m}}{(1-\lambda)^m}\sum_{\substack{k_1+\cdots+k_m=l\\k_j\geq 1}}\binom{l}{k_1,\ldots,k_m}\right\}\\
&\quad\cdot H_{n-l}^{(r)}(x\mid\lambda)\\
&\quad+\sum_{l=\min\{2r,n\}+1}^{n}\left\{\binom{n}{l}\sum_{m=0}^{\min\{2r,n\}}\frac{\binom{2r}{m}}{(1-\lambda)^m}\right.\\
&\qquad\left.\cdot\sum_{\substack{k_1+\cdots+k_m=l\\k_j\geq 1}}\binom{l}{k_1,\ldots,k_m}\right\}H_{n-l}^{(r)}(x\mid\lambda).
\end{aligned}\tag{37}$$

Let us take $x = 0$ in (37). Then we obtain the following theorem.

Theorem 5. *We have*

$$\begin{aligned}
&\frac{r!}{(1-\lambda)^r}S_\lambda(n,r)\\
&=\frac{r!}{(1-\lambda)^r}\frac{\widetilde{\Delta}_\lambda^r 0^n}{r!}\\
&=\sum_{l=0}^{\min\{2r,n\}}\left\{\binom{n}{l}\sum_{m=0}^{l}\frac{\binom{2r}{m}}{(1-\lambda)^m}\sum_{\substack{k_1+\cdots+k_m=l\\k_j\geq 1}}\binom{l}{k_1,\ldots,k_m}\right\}\\
&\quad\cdot H_{n-l}^{(r)}(\lambda)\\
&\quad+\sum_{l=\min\{2r,n\}+1}^{n}\left\{\binom{n}{l}\sum_{m=0}^{\min\{2r,n\}}\frac{\binom{2r}{m}}{(1-\lambda)^m}\right.\\
&\qquad\left.\cdot\sum_{\substack{k_1+\cdots+k_m=l\\k_j\geq 1}}\binom{l}{k_1,\ldots,k_m}\right\}H_{n-l}^{(r)}(\lambda)\\
&=\sum_{m=0}^{\min\{r,n\}}\frac{\binom{r}{m}}{(1-\lambda)^m}\sum_{\substack{k_1+\cdots+k_m=n\\k_j\geq 1}}\binom{n}{k_1,\ldots,k_m}.
\end{aligned}\tag{38}$$

Let us consider $s = 2r - 1$ in the identity of Theorem 2. Then we have

$$\begin{aligned}
&J_\lambda^{r-1}x^n\\
&=H_n^{-(r-1)}(x\mid\lambda)\\
&=\sum_{l=0}^{\min\{2r-1,n\}}\left\{\binom{n}{l}\sum_{m=0}^{l}\frac{\binom{2r-1}{m}}{(1-\lambda)^m}\sum_{\substack{k_1+\cdots+k_m=l\\k_j\geq 1}}\binom{l}{k_1,\ldots,k_m}\right\}\\
&\quad\cdot H_{n-l}^{(r)}(x\mid\lambda)\\
&\quad+\sum_{l=\min\{2r-1,n\}+1}^{n}\left\{\binom{n}{l}\sum_{m=0}^{\min\{2r-1,n\}}\frac{\binom{2r-1}{m}}{(1-\lambda)^m}\right.\\
&\qquad\left.\cdot\sum_{\substack{k_1+\cdots+k_m=l\\k_j\geq 1}}\binom{l}{k_1,\ldots,k_m}\right\}H_{n-l}^{(r)}(x\mid\lambda)\\
&=\frac{1}{(1-\lambda)^{r-1}}\sum_{j=0}^{r-1}\binom{r-1}{j}(-\lambda)^{r-1-j}(x+j)^n\\
&=\frac{1}{(1-\lambda)^{r-1}}\widetilde{\Delta}_\lambda^{r-1}x^n.
\end{aligned}\tag{39}$$

Let us take $x = 0$ in (39). Then we obtain the following theorem.

Theorem 6. *For $n \geq 0$ and $r \geq 1$, one has*

$$\begin{aligned}
&\frac{(r-1)!}{(1-\lambda)^{r-1}}S_\lambda(n,r-1)\\
&=\frac{(r-1)!}{(1-\lambda)^{r-1}}\frac{\widetilde{\Delta}_\lambda^{r-1}0^n}{(r-1)!}\\
&=\sum_{l=0}^{\min\{2r-1,n\}}\left\{\binom{n}{l}\sum_{m=0}^{l}\frac{\binom{2r-1}{m}}{(1-\lambda)^m}\sum_{\substack{k_1+\cdots+k_m=l\\k_j\geq 1}}\binom{l}{k_1,\ldots,k_m}\right\}\\
&\quad\cdot H_{n-l}^{(r)}(\lambda)\\
&\quad+\sum_{l=\min\{2r-1,n\}+1}^{n}\left\{\binom{n}{l}\sum_{m=0}^{\min\{2r-1,n\}}\frac{\binom{2r-1}{m}}{(1-\lambda)^m}\right.\\
&\qquad\left.\cdot\sum_{\substack{k_1+\cdots+k_m=l\\k_j\geq 1}}\binom{l}{k_1,\ldots,k_m}\right\}H_{n-l}^{(r)}(\lambda).
\end{aligned}\tag{40}$$

Remark 7. Note that

$$\begin{aligned}
&\frac{(r-1)!}{(1-\lambda)^{r-1}} S_{\lambda}(n, r-1) \\
&\quad = \sum_{l=0}^{\min\{r,n\}} \left\{ \binom{n}{l} \sum_{m=0}^{l} \frac{\binom{r}{m}}{(1-\lambda)^{m}} \sum_{\substack{k_1+\cdots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1,\ldots,k_m} \right\} \\
&\qquad \cdot H_{n-l}(\lambda) \\
&\qquad + \sum_{l=\min\{r,n\}+1}^{n} \left\{ \binom{n}{l} \sum_{m=0}^{\min\{r,n\}} \frac{\binom{r}{m}}{(1-\lambda)^{m}} \right. \\
&\qquad\qquad \left. \cdot \sum_{\substack{k_1+\cdots+k_m=l \\ k_j \geq 1}} \binom{l}{k_1,\ldots,k_m} \right\} H_{n-l}(\lambda) .
\end{aligned} \tag{41}$$

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