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Research Article

Asymptotic Formula for Oscillatory Solutions of Some Singular Nonlinear Differential Equation

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Singular differential equation $(p(t)u')' = p(t)f(u)$ is investigated. Here f is Lipschitz continuous on $\mathbb{R}$ and has at least two zeros 0 and $L > 0$. The function p is continuous on $[0,\infty)$ and has a positive continuous derivative on $(0,\infty)$ and $p(0) = 0$. An asymptotic formula for oscillatory solutions is derived.

1. Introduction

In this paper, we investigate the equation

$$(p(t)u')' = p(t)f(u), \quad t \in (0,\infty), \tag{1.1}$$

where f satisfies

$$f \in Lip_{\text{loc}}(\mathbb{R}), \quad f(0) = f(L) = 0, \ f(x) < 0, \ x \in (0,L), \tag{1.2}$$

$$\exists \overline{B} \in (-\infty,0)\colon f(x) > 0, \quad x \in \left[\overline{B},0\right), \tag{1.3}$$

$$F\left(\overline{B}\right) = F(L), \quad \text{where } F(x) = -\int_0^x f(z)\mathrm{d}z, \ x \in \mathbb{R}, \tag{1.4}$$

and p fulfils

$$p \in C[0,\infty) \cap C^1(0,\infty), \quad p(0) = 0, \tag{1.5}$$

$$p'(t) > 0, \quad t \in (0,\infty), \quad \lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0. \tag{1.6}$$

Equation (1.1) is a generalization of the equation

$$u'' + \frac{k-1}{t}u' = f(u), \quad t \in (0,\infty), \tag{1.7}$$

which arises for $k > 1$ and special forms of f in many areas, for example: in the study of phase transitions of Van der Waals fluids [1–3], in population genetics, where it serves as a model for the spatial distribution of the genetic composition of a population [4, 5], in the homogeneous nucleation theory [6], in the relativistic cosmology for the description of particles which can be treated as domains in the universe [7], in the nonlinear field theory, in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces [8]. Numerical simulations of solutions of (1.1), where f is a polynomial with three zeros have been presented in [9–11]. Close problems about the existence of positive solutions can be found in [12–14].

Due to $p(0) = 0$, (1.1) has a singularity at $t = 0$.

Definition 1.1. A function $u \in C^1[0,\infty) \cap C^2(0,\infty)$ which satisfies (1.1) for all $t \in (0,\infty)$ is called a *solution* of (1.1).

Definition 1.2. Let u be a solution of (1.1) and let L be of (1.2). Denote $u_{\sup} = \sup\{u(t): t \in [0,\infty)\}$. If $u_{\sup} < L$ ($u_{\sup} = L$ or $u_{\sup} > L$), then u is called a *damped* solution (a *bounding homoclinic* solution or an *escape* solution).

These three types of solutions have been investigated in [15–19]. In particular, the existence of damped oscillatory solutions which converge to 0 has been proved in [19].

The main result of this paper is contained in Section 3 in Theorem 3.1, where we provide an asymptotic formula for damped oscillatory solutions of (1.1).

2. Existence of Oscillatory Solutions

Here, we will study solutions of (1.1) satisfying the initial conditions

$$u(0) = B, \qquad u'(0) = 0, \tag{2.1}$$

with a parameter $B \le L$. Reason is that we focus our attention on damped solutions of (1.1) and that each solution u of (1.1) must fulfil $u'(0) = 0$ (see [19]).

First, we bring two theorems about the existence of damped and oscillatory solutions.

Theorem 2.1 (see [19]). *Assume that* (1.2)–(1.6) *hold. Then for each* $B \in [\overline{B}, L)$ *problem* (1.1), (2.1) *has a unique solution. This solution is damped.*

Theorem 2.2. *Assume that* (1.2)–(1.6) *hold. Further, let there exists* $k_0 \in (0,\infty)$ *such that*

$$p \in C^2(0,\infty), \qquad \limsup_{t\to\infty}\left|\frac{p''(t)}{p'(t)}\right| < \infty, \qquad \liminf_{t\to\infty}\frac{p(t)}{t^{k_0}} \in (0,\infty], \tag{2.2}$$

$$\lim_{x\to 0+}\frac{f(x)}{x} < 0, \qquad \lim_{x\to 0-}\frac{f(x)}{x} < 0. \tag{2.3}$$

Then for each $B \in [\overline{B}, L)$ problem (1.1), (2.1) *has a unique solution u. If $B \neq 0$, then the solution u is damped and oscillatory with decreasing amplitudes and*

$$\lim_{t\to\infty} u(t) = 0. \tag{2.4}$$

Proof. The assertion follows from Theorems 2.3, 2.10 and 3.1 in [19]. □

Example 2.3. The functions

(i) $p(t) = t^k$, $p(t) = t^k \ln(t^\ell + 1)$, $k, \ell \in (0, \infty)$,

(ii) $p(t) = t + \alpha \sin t$, $\alpha \in (-1, 1)$,

(iii) $p(t) = t^k/(1 + t^\ell)$, $k, \ell \in (0, \infty)$, $\ell < k$

satisfy (1.5), (1.6), and (2.2).

The functions

(i) $p(t) = \ln(t + 1)$, $p(t) = \arctan t$, $p(t) = t^k/(1 + t^k)$, $k \in (0, \infty)$

satisfy (1.5), (1.6), but not (2.2) (the third condition).

The function

(i) $p(t) = t^k + \alpha \sin t^k$, $\alpha \in (-1, 1)$, $k \in (1, \infty)$,

satisfy (1.5), (1.6) but not (2.2) (the second and third conditions).

Example 2.4. Let $k \in (0, \infty)$.

(i) The function

$$f(x) = \begin{cases} -kx, & \text{for } x \le 0, \\ x(x-1), & \text{for } x > 0, \end{cases} \tag{2.5}$$

satisfies (1.2) with $L = 1$, (1.3), (1.4) with $\overline{B} = -(3k)^{-1/2}$ and (2.3).

(ii) The function

$$f(x) = \begin{cases} kx^2, & \text{for } x \le 0, \\ x(x-1), & \text{for } x > 0, \end{cases} \tag{2.6}$$

satisfies (1.2) with $L = 1$, (1.3), (1.4) with $\overline{B} = -(2k)^{-1/3}$ but not (2.3) (the second condition).

In the next section, the generalized Matell's theorem which can be found as Theorem 6.5 in the monograph by Kiguradze will be useful. For our purpose, we provide its following special case.

Consider an interval $J \subset \mathbb{R}$. We write $AC(J)$ for the set of functions absolutely continuous on J and $AC_{\text{loc}}(J)$ for the set of functions belonging to $AC(I)$ for each compact interval $I \subset J$. Choose $t_0 > 0$ and a function matrix $A(t) = (a_{i,j}(t))_{i,j\le 2}$ which is defined on (t_0,∞). Denote by $\lambda(t)$ and $\mu(t)$ eigenvalues of $A(t),\ t \in (t_0,\infty)$. Further, suppose

$$\lambda = \lim_{t\to\infty}\lambda(t), \qquad \mu = \lim_{t\to\infty}\mu(t) \tag{2.7}$$

be different eigenvalues of the matrix $A = \lim_{t\to\infty}A(t)$, and let $\mathbf{l}$ and $\mathbf{m}$ be eigenvectors of A corresponding to λ and μ, respectively.

Theorem 2.5 (see [20]). *Assume that*

$$a_{i,j} \in AC_{\mathrm{loc}}(t_0,\infty), \qquad \left|\int_{t_0}^{\infty} a'_{i,j}(t)dt\right| < \infty,\ i,j = 1,2, \tag{2.8}$$

and that there exists $c_0 > 0$ such that

$$\int_s^t \operatorname{Re}\big(\lambda(\tau)-\mu(\tau)\big)\mathrm{d}\tau \le c_0, \quad t_0 \le s < t, \tag{2.9}$$

or

$$\int_{t_0}^{\infty} \operatorname{Re}\big(\lambda(\tau)-\mu(\tau)\big)\mathrm{d}\tau = \infty, \qquad \int_s^t \operatorname{Re}\big(\lambda(\tau)-\mu(\tau)\big)\mathrm{d}\tau \ge -c_0, \quad t_0 \le s < t. \tag{2.10}$$

Then the differential system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) \tag{2.11}$$

has a fundamental system of solutions $x(t),\ y(t)$ such that

$$\lim_{t\to\infty}\mathbf{x}(t)\mathrm{e}^{-\int_{t_0}^t \lambda(\tau)\mathrm{d}\tau} = \mathbf{l}, \qquad \lim_{t\to\infty}\mathbf{y}(t)\mathrm{e}^{-\int_{t_0}^t \mu(\tau)\mathrm{d}\tau} = \mathbf{m}. \tag{2.12}$$

3. Asymptotic Formula

In order to derive an asymptotic formula for a damped oscillatory solution u of problem (1.1), (2.1), we need a little stronger assumption than (2.3). In particular, the function $f(x)/x$ should have a negative derivative at $x = 0$.

Theorem 3.1. *Assume that* (1.2)–(1.6), *and* (2.2) *hold. Assume, moreover, that there exist $\eta > 0$ and $c > 0$ such that*

$$\frac{f(x)}{x} \in AC[-\eta,\eta], \qquad \lim_{x\to 0}\frac{f(x)}{x} = -c. \tag{3.1}$$

Then for each $B \in [\overline{B}, L)$ *problem* (1.1), (2.1) *has a unique solution* u*. If* $B \neq 0$*, then the solution* u *is damped and oscillatory with decreasing amplitudes such that*

$$\limsup_{t\to\infty} \sqrt{p(t)}|u(t)| < \infty. \tag{3.2}$$

Proof. We have the following steps:

Step 1 (construction of an auxiliary linear differential system). Choose $B \in [\overline{B}, L)$, $B \neq 0$. By Theorem 2.2, problem (1.1), (2.1) has a unique oscillatory solution u with decreasing amplitudes and satisfying (2.4). Having this solution u, define a linear differential equation

$$v'' + \frac{p'(t)}{p(t)}v' = \frac{f(u(t))}{u(t)}v, \tag{3.3}$$

and the corresponding linear differential system

$$x_1' = x_2, \qquad x_2' = \frac{f(u(t))}{u(t)}x_1 - \frac{p'(t)}{p(t)}x_2. \tag{3.4}$$

Denote

$$A(t) = \left(a_{i,j}(t)\right)_{i,j\le 2} = \begin{pmatrix} 0 & 1 \\ \dfrac{f(u(t))}{u(t)} & -\dfrac{p'(t)}{p(t)} \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -c & 0 \end{pmatrix}. \tag{3.5}$$

By (1.6), (2.4), and (3.1),

$$A = \lim_{t\to\infty} A(t). \tag{3.6}$$

Eigenvalues of A are numbers $\lambda = i\sqrt{c}$ and $\mu = -i\sqrt{c}$, and eigenvectors of A are $\mathbf{l} = (1, i\sqrt{c})$ and $\mathbf{m} = (1, -i\sqrt{c})$, respectively. Denote

$$D(t) = \left(\frac{p'(t)}{2p(t)}\right)^2 + \frac{f(u(t))}{u(t)}, \quad t \in (0,\infty). \tag{3.7}$$

Then eigenvalues of $A(t)$ have the form

$$\lambda(t) = -\frac{p'(t)}{2p(t)} + \sqrt{D(t)}, \qquad \mu(t) = -\frac{p'(t)}{2p(t)} - \sqrt{D(t)}, \quad t \in (0,\infty). \tag{3.8}$$

We see that

$$\lim_{t\to\infty} \lambda(t) = \lambda, \qquad \lim_{t\to\infty} \mu(t) = \mu. \tag{3.9}$$

Step 2 (verification of the assumptions of Theorem 2.5). Due to (1.6), (2.4), and (3.1), we can find $t_0 > 0$ such that

$$u(t_0) \neq 0, \qquad |u(t)| \leq \eta, \qquad D(t) < 0, \quad t \in (t_0, \infty). \tag{3.10}$$

Therefore, by (3.1),

$$a_{21}(t) = \frac{f(u(t))}{u(t)} \in AC_{\mathrm{loc}}(t_0, \infty), \tag{3.11}$$

and so

$$\left| \int_{t_0}^{\infty} \left(\frac{f(u(t))}{u(t)} \right)' \mathrm{d}t \right| = \left| \lim_{t \to \infty} \frac{f(u(t))}{u(t)} - \frac{f(u(t_0))}{u(t_0)} \right| = \left| -c - \frac{f(u(t_0))}{u(t_0)} \right| < \infty. \tag{3.12}$$

Further, by (2.2), $a_{22}(t) = -p'(t)/p(t) \in C^1(t_0, \infty)$. Hence, due to (1.6),

$$\left| \int_{t_0}^{\infty} \left(\frac{p'(t)}{p(t)} \right) \mathrm{d}t \right| = \left| \lim_{t \to \infty} \frac{p'(t)}{p(t)} - \frac{p'(t_0)}{p(t_0)} \right| = \frac{p'(t_0)}{p(t_0)} < \infty. \tag{3.13}$$

Since $a_{11}(t) \equiv 0$ and $a_{12}(t) \equiv 1$, we see that (2.8) is satisfied. Using (3.8) we get $\mathrm{Re}(\lambda(t) - \mu(t)) \equiv 0$. This yields

$$\int_s^t \mathrm{Re}(\lambda(\tau) - \mu(\tau)) \mathrm{d}\tau = 0 < c_0, \quad t_0 \leq s < t, \tag{3.14}$$

for any positive constant c_0. Consequently (2.9) is valid.

Step 3 (application of Theorem 2.5). By Theorem 2.5 there exists a fundamental system $\mathbf{x}(t) = (x_1(t), x_2(t))$, $\mathbf{y}(t) = (y_1(t), y_2(t))$ of solutions of (3.4) such that (2.12) is valid. Hence

$$\lim_{t \to \infty} x_1(t) \mathrm{e}^{-\int_{t_0}^t \lambda(\tau)\mathrm{d}\tau} = 1, \qquad \lim_{t \to \infty} y_1(t) \mathrm{e}^{-\int_{t_0}^t \mu(\tau)\mathrm{d}\tau} = 1. \tag{3.15}$$

Using (3.8) and (3.10), we get

$$\begin{aligned} \exp\left(-\int_{t_0}^t \lambda(\tau)\mathrm{d}\tau \right) &= \exp\left(\int_{t_0}^t \left(\frac{p'(\tau)}{2p(\tau)} - \sqrt{D(\tau)} \right) \mathrm{d}\tau \right) \\ &= \exp\left(\frac{1}{2} \ln \frac{p(t)}{p(t_0)} \right) \exp\left(-i \int_{t_0}^t \sqrt{|D(\tau)|} \mathrm{d}\tau \right), \end{aligned} \tag{3.16}$$

and, hence,

$$\left| e^{-\int_{t_0}^{t} \lambda(\tau)d\tau} \right| = \sqrt{\frac{p(t)}{p(t_0)}}, \quad t \in (t_0, \infty). \tag{3.17}$$

Similarly

$$\left| e^{-\int_{t_0}^{t} \mu(\tau)d\tau} \right| = \sqrt{\frac{p(t)}{p(t_0)}}, \quad t \in (t_0, \infty). \tag{3.18}$$

Therefore, (3.15) implies

$$\begin{aligned} 1 &= \lim_{t\to\infty} \left| x_1(t) e^{-\int_{t_0}^{t} \lambda(\tau)d\tau} \right| = \lim_{t\to\infty} |x_1(t)| \sqrt{\frac{p(t)}{p(t_0)}}, \\ 1 &= \lim_{t\to\infty} \left| y_1(t) e^{-\int_{t_0}^{t} \mu(\tau)d\tau} \right| = \lim_{t\to\infty} |y_1(t)| \sqrt{\frac{p(t)}{p(t_0)}}. \end{aligned} \tag{3.19}$$

Step 4 (asymptotic formula). In Step 1, we have assumed that u is a solution of (1.1), which means that

$$u''(t) + \frac{p'(t)}{p(t)} u'(t) = f(u(t)), \quad \text{for } t \in (0, \infty). \tag{3.20}$$

Consequently

$$u''(t) + \frac{p'(t)}{p(t)} u'(t) = \frac{f(u(t))}{u(t)} u(t), \quad \text{for } t \in (0, \infty), \tag{3.21}$$

and, hence, u is also a solution of (3.3). This yields that there are $c_1, c_2 \in \mathbb{R}$ such that $u(t) = c_1 x_1(t) + c_2 y_1(t),\ t \in (0, \infty)$. Therefore,

$$\limsup_{t\to\infty} \sqrt{p(t)} |u(t)| \le (|c_1| + |c_2|) \sqrt{p(t_0)} < \infty. \tag{3.22}$$

□

Remark 3.2. Due to (2.2) and (3.2), we have for a solution u of Theorem 3.1

$$u(t) = O\left(t^{-k_0/2}\right), \quad \text{for } t \longrightarrow \infty. \tag{3.23}$$

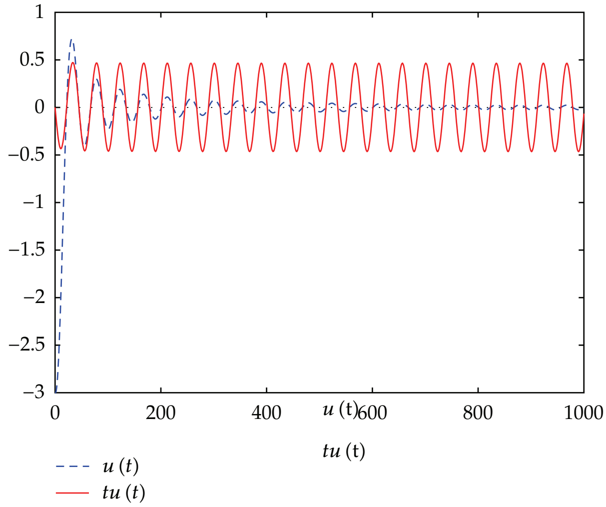

Figure 1

Example 3.3. Let $k \in (1, \infty)$.

(i) The functions $f(x) = x(x-1)$ and $f(x) = x(x-1)(x+2)$ satisfy all assumptions of Theorem 3.1.

(ii) The functions $f(x) = x^{2k-1}(x-1)$ and $f(x) = x^{2k-1}(x-1)(x+2)$

satisfy (1.2)–(1.4) but not (3.1) (the second condition).

Example 3.4. Consider the initial problem

$$\left(t^2 u'\right)' = t^2 u(u-5)(u+10), \qquad u(0) = -3,\ u'(0) = 0. \tag{3.24}$$

Here $L_0 = -10$, $L = 5$ and we can check that $\overline{B} < -3$. Further, all assumptions of Theorems 2.2 and 3.1 are fulfilled. Therefore, by Theorem 2.2, there exists a unique solution u of problem (3.24) which is damped and oscillatory and converges to 0. By Theorem 3.1, we have

$$\limsup_{t\to\infty} t|u(t)| < \infty, \quad \text{that is,}\ \ u(t) = O\left(\frac{1}{t}\right), \quad \text{for } t \longrightarrow \infty. \tag{3.25}$$

The behaviour of the solution $u(t)$ and of the function $tu(t)$ is presented on Figure 1.

Remark 3.5. Our further research of this topic will be focused on a deeper investigation of all types of solutions defined in Definition 1.2. For example, we have proved in [15, 19] that damped solutions of (1.1) can be either oscillatory or they have a finite number of zeros or no zero and converge to 0. A more precise characterization of behaviour of nonoscillatory solutions are including their asymptotic formulas in as open problem. The same can be said about homoclinic solutions. In [17] we have found some conditions which guarantee their existence, and we have shown that if u is a homoclinic solution of (1.1), then $\lim_{t\to\infty} u(t) = L$.

In order to discover other existence conditions for homoclinic solutions, we would like to estimate their convergence by proper asymptotic formulas.

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