Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 902054, 9 pages
http://dx.doi.org/10.1155/2014/902054

Research Article

Existence and Uniqueness Results for Hadamard-Type Fractional Differential Equations with Nonlocal Fractional Integral Boundary Conditions

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Received 1 April 2014; Accepted 10 June 2014; Published 26 June 2014

Academic Editor: Bashir Ahmad

We study the existence and uniqueness of solutions for a fractional boundary value problem involving Hadamard-type fractional differential equations and nonlocal fractional integral boundary conditions. Our results are based on some classical fixed point theorems. Some illustrative examples are also included.

1. Introduction

In this paper, we investigate the following Hadamard boundary value problem:

$$D^q x(t) = f(t, x(t)), \quad 1 < q \le 2, \ t \in (1, e), \tag{1}$$

$$x(1) = 0, \qquad \sum_{i=1}^{m} \lambda_i J^{\alpha_i} x(\eta_i) = \sum_{j=1}^{n} \mu_j \left(J^{\beta_j} x(e) - J^{\beta_j} x(\xi_j) \right), \tag{2}$$

where D^q denotes the Hadamard fractional derivative of order q, $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\eta_i, \xi_j \in (1, e)$, $\lambda_i, \mu_j \in \mathbb{R}$, for all $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $\eta_1 < \eta_2 < \cdots < \eta_m$, $\xi_1 < \xi_2 < \cdots < \xi_n$, and J^ϕ is the Hadamard fractional integral of order $\phi > 0$ ($\phi = \alpha_i, \beta_j, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$).

We mention that integral boundary conditions are encountered in various applications such as population dynamics, blood flow models, chemical engineering, cellular systems, heat transmission, plasma physics, and thermoelasticity.

Condition (2) is a general form of the integral boundary conditions considered in [1] and covers many special cases. For example, if $\alpha_i = \beta_j = 1$, for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, then condition (2) reduces to

$$\begin{gathered} x(1) = 0, \\ \lambda_1 \int_1^{\eta_1} x(s) \frac{ds}{s} + \cdots + \lambda_m \int_1^{\eta_m} x(s) \frac{ds}{s} \\ = \mu_1 \int_{\xi_1}^{e} x(s) \frac{ds}{s} + \cdots + \mu_n \int_{\xi_n}^{e} x(s) \frac{ds}{s}. \end{gathered} \tag{3}$$

Fractional differential equations provide appropriate models for describing real world problems, which cannot be described using classical integer order differential equations. The theory of fractional differential equations has received much attention over the past years and has become an important field of investigation due to its extensive applications in numerous branches of physics, economics, and engineering sciences [2–5]. Some recent contributions to the subject can be seen in [1, 6–20] and references cited therein.

It has been noticed that most of the work on this topic is based on Riemann-Liouville and Caputo type fractional differential equations. Another kind of fractional derivatives that appears side by side to Riemann-Liouville and Caputo derivatives in the literature is the fractional derivative due to Hadamard introduced in 1892 [21], which differs from the preceding ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains logarithmic function of arbitrary exponent. Details and properties of Hadamard fractional derivative and integral can be found in [2, 22–26]. For some recent results on Hadamard boundary value problem we refer to [27, 28].

We establish a variety of results for the problem (1)-(2) by using classical fixed point theorems. The first result, Theorem 4, relies on Banach contraction mapping principle and concerns an existence and uniqueness result for the solutions of the problem (1)-(2). A second existence and uniqueness result is proved in Theorem 7, via nonlinear contractions and a fixed point theorem due to Boyd and Wong. Existence results are proved in the third result, Theorem 9, by using Krasnoselskii fixed point theorem, and in the fourth result, Theorem 12, by using nonlinear alternative of Leray-Schauder type.

The paper is organized as follows. In Section 2, we recall some preliminary concepts that we need in the sequel and prove a preliminary lemma. Section 3 contains the main results for the problem (1)-(2). In Section 4, some illustrative examples are discussed.

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [2] and present preliminary results needed in our proofs later.

Definition 1. The Hadamard derivative of fractional order q for a function $f:[1,\infty)\to\mathbb{R}$ is defined as

$$D^q f(t)=\frac{1}{\Gamma(n-q)}\left(t\frac{d}{dt}\right)^n\int_1^t\left(\log\frac{t}{s}\right)^{n-q-1}\frac{f(s)}{s}ds,\quad n-1<q<n,\ n=[q]+1, \tag{4}$$

where $[q]$ denotes the integer part of the real number q, $\log(\cdot)=\log_e(\cdot)$, and Γ is the Gamma function.

Definition 2. The Hadamard fractional integral of order q for a function $f:[1,\infty)\to\mathbb{R}$ is defined by

$$J^q f(t)=\frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{f(s)}{s}ds,\quad q>0, \tag{5}$$

provided the integral exists.

For convenience, we set

$$\Lambda=\sum_{i=1}^m\lambda_i\frac{\Gamma(q)}{\Gamma(q+\alpha_i)}(\log\eta_i)^{q+\alpha_i-1}-\sum_{j=1}^n\mu_j\frac{\Gamma(q)}{\Gamma(q+\beta_j)}\left(1-(\log\xi_j)^{q+\beta_j-1}\right). \tag{6}$$

Lemma 3. *Let $\Lambda\neq0$, $1<q\le2$, $\alpha_i,\beta_j>0$, and $\eta_i,\xi_j\in(1,e)$ for $i=1,2,\ldots,m$, $j=1,2,\ldots,n$, and $h\in C([1,e],\mathbb{R})$. The unique solution of the following fractional differential equation,*

$$D^q x(t)=h(t),\quad t\in(1,e), \tag{7}$$

subject to the boundary condition,

$$x(1)=0,\qquad \sum_{i=1}^m\lambda_iJ^{\alpha_i}x(\eta_i)=\sum_{j=1}^n\mu_j\left(J^{\beta_j}x(e)-J^{\beta_j}x(\xi_j)\right), \tag{8}$$

is given by the integral equation

$$x(t)=\frac{(\log t)^{q-1}}{\Lambda}\sum_{j=1}^n\mu_j\left(J^{q+\beta_j}h(e)-J^{q+\beta_j}h(\xi_j)\right)-\frac{(\log t)^{q-1}}{\Lambda}\sum_{i=1}^m\lambda_iJ^{q+\alpha_i}h(\eta_i)+J^qh(t). \tag{9}$$

Proof. Applying the Hadamard fractional integral of order q to both sides of (7), we have

$$x(t)=z_1(\log t)^{q-1}+z_2(\log t)^{q-2}+J^qh(t), \tag{10}$$

where $z_1,z_2\in\mathbb{R}$.

The condition of $x(1)=0$ implies $z_2=0$. Therefore,

$$x(t)=z_1(\log t)^{q-1}+J^qh(t). \tag{11}$$

For any $p>0$, by Definition 2, it follows that

$$J^px(t)=z_1\frac{\Gamma(q)}{\Gamma(q+p)}(\log t)^{q+p-1}+J^{q+p}h(t). \tag{12}$$

The second condition of (8) with (12) leads to

$$z_1=\frac{1}{\Lambda}\sum_{j=1}^n\mu_j\left(J^{q+\beta_j}h(e)-J^{q+\beta_j}h(\xi_j)\right)-\frac{1}{\Lambda}\sum_{i=1}^m\lambda_iJ^{q+\alpha_i}h(\eta_i). \tag{13}$$

Substituting the value of a constant z_1 into (11), we obtain (9) as required. The proof is completed. □

3. Main Results

Let $\mathscr{C}=C([1,e],\mathbb{R})$ denote the Banach space of all continuous functions from $[1,e]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\|=\sup_{t\in[1,e]}|x(t)|$. As in Lemma 3, we define an operator $\mathscr{F}:\mathscr{C}\to\mathscr{C}$ by

$$(\mathscr{F}x)(t)=J^qf(s,x(s))(t)-\frac{(\log t)^{q-1}}{\Lambda}\sum_{i=1}^m\lambda_iJ^{\alpha_i+q}f(s,x(s))(\eta_i)+\frac{(\log t)^{q-1}}{\Lambda}\sum_{j=1}^n\mu_j\left(J^{\beta_j+q}f(s,x(s))(e)-J^{\beta_j+q}f(s,x(s))(\xi_j)\right), \tag{14}$$

with $\Lambda\neq0$. It should be noticed that problem (1)-(2) has solutions if and only if the operator $\mathscr{F}$ has fixed points.

For the sake of convenience, we put

$$\Phi = \frac{1}{\Gamma(q+1)} + \frac{1}{|\Lambda|}\sum_{i=1}^{m}|\lambda_i|\frac{(\log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} + \frac{1}{|\Lambda|}\sum_{j=1}^{n}|\mu_j|\frac{1+(\log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)}. \tag{15}$$

The first existence and uniqueness result is based on the Banach contraction mapping principle.

Theorem 4. *Let $f:[1,e]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the assumption that*

(H_1) *there exists a constant $L>0$ such that $|f(t,x)-f(t,y)|\le L|x-y|$, for each $t\in[1,e]$ and $x,y\in\mathbb{R}$.*

If

$$L\Phi < 1, \tag{16}$$

where Φ is given by (15)*, then the boundary value problem* (1)-(2) *has a unique solution on* $[1,e]$.

Proof. We transform the problem (1)-(2) into a fixed point problem, $x=\mathcal{F}x$, where the operator $\mathcal{F}$ is defined by (14). By using Banach's contraction mapping principle, we will show that $\mathcal{F}$ has a fixed point which is a unique solution of problem (1)-(2).

We set $\sup_{t\in[1,e]}|f(t,0)|=M<\infty$ and choose

$$r\ge\frac{M\Phi}{1-L\Phi}. \tag{17}$$

Now, we show that $\mathcal{F}B_r\subset B_r$, where $B_r=\{x\in\mathcal{C}:\|x\|\le r\}$. For any $x\in B_r$, we have

$$\begin{aligned}
&\|\mathcal{F}x\| \\
&\le \sup_{t\in[1,e]}\Big\{J^q|f(s,x(s))|(t) \\
&\quad+\frac{(\log t)^{q-1}}{|\Lambda|}\sum_{i=1}^{m}|\lambda_i|J^{\alpha_i+q}|f(s,x(s))|(\eta_i) \\
&\quad+\frac{(\log t)^{q-1}}{|\Lambda|}\sum_{j=1}^{n}|\mu_j|\left(J^{\beta_j+q}|f(s,x(s))|(e)\right. \\
&\quad\left.+J^{\beta_j+q}|f(s,x(s))|(\xi_j)\right)\Big\} \\
&\le J^q\left(|f(s,x(s))-f(s,0)|+|f(s,0)|\right)(e) \\
&\quad+\frac{1}{|\Lambda|}\sum_{i=1}^{m}|\lambda_i|J^{\alpha_i+q}\left(|f(s,x(s))-f(s,0)|+|f(s,0)|\right)(\eta_i) \\
&\quad+\frac{1}{|\Lambda|}\sum_{j=1}^{n}|\mu_j|\left(J^{\beta_j+q}\left(|f(s,x(s))-f(s,0)|+|f(s,0)|\right)(e)\right. \\
&\quad+J^{\beta_j+q}\left(|f(s,x(s))-f(s,0)|\right. \\
&\quad\left.\left.+|f(s,0)|\right)(\xi_j)\right) \\
&\le(Lr+M)\left\{\frac{1}{\Gamma(q+1)}+\frac{1}{|\Lambda|}\sum_{i=1}^{m}|\lambda_i|\frac{(\log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\right. \\
&\quad\left.+\frac{1}{|\Lambda|}\sum_{j=1}^{n}|\mu_j|\frac{1+(\log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right\} \\
&=(Lr+M)\Phi\le r.
\end{aligned} \tag{18}$$

It follows that $\mathcal{F}B_r\subset B_r$.

For $x,y\in\mathcal{C}$ and for each $t\in[1,e]$, we have

$$\begin{aligned}
&|\mathcal{F}x(t)-\mathcal{F}y(t)| \\
&\le J^q\left(|f(s,x(s))-f(s,y(s))|\right)(t) \\
&\quad+\frac{(\log t)^{q-1}}{|\Lambda|}\sum_{i=1}^{m}|\lambda_i|J^{\alpha_i+q}\left(|f(s,x(s))-f(s,y(s))|\right)(\eta_i) \\
&\quad+\frac{(\log t)^{q-1}}{|\Lambda|}\sum_{j=1}^{n}|\mu_j|\left(J^{\beta_j+q}\left(|f(s,x(s))-f(s,y(s))|\right)(e)\right. \\
&\quad+J^{\beta_j+q}\left(|f(s,x(s))-f(s,y(s))|\right) \\
&\quad\left.\times(\xi_j)\right) \\
&\le L\|x-y\|\left\{\frac{1}{\Gamma(q+1)}+\frac{1}{|\Lambda|}\sum_{i=1}^{m}|\lambda_i|\frac{(\log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\right. \\
&\quad\left.\times\frac{1}{|\Lambda|}\sum_{j=1}^{n}|\mu_j|\frac{1+(\log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right\} \\
&=L\Phi\|x-y\|.
\end{aligned} \tag{19}$$

The above result implies that $\|\mathcal{F}x-\mathcal{F}y\|\le L\Phi\|x-y\|$. As $L\Phi<1$, therefore $\mathcal{F}$ is a contraction. Hence, by the Banach contraction mapping principle, we deduce that $\mathcal{F}$ has a fixed point which is the unique solution of the problem (1)-(2). □

Next, we give the second existence and uniqueness result by using nonlinear contractions.

Definition 5. Let E be a Banach space and let $F:E\to E$ be a mapping. F is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\theta) < \theta$ for all $\theta > 0$ with the property

$$\|Fx - Fy\| \le \Psi(\|x - y\|), \quad \forall x, y \in E. \tag{20}$$

Lemma 6 (see [29]). *Let E be a Banach space and let $F : E \to E$ be a nonlinear contraction. Then F has a unique fixed point in E.*

Theorem 7. *Let $f: [1,e] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption*

(H_2) $|f(t,x) - f(t,y)| \le h(t)(|x-y|/(H^* + |x-y|))$, $t \in [1,e]$, $x, y \ge 0$, *where $h : [1,e] \to \mathbb{R}^+$ is continuous and a constant H^* is defined by*

$$H^* = J^q h(e) + \frac{1}{|\Lambda|}\sum_{i=1}^{m} |\lambda_i| J^{\alpha_i + q} h(\eta_i) + \frac{1}{|\Lambda|}\sum_{j=1}^{n} |\mu_j| \left(J^{\beta_j+q} h(e) + J^{\beta_j + q} h(\xi_j)\right). \tag{21}$$

Then the boundary value problem (1)-(2) *has a unique solution.*

Proof. We define the operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ as (14) and a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\Psi(\theta) = \frac{H^*\theta}{H^* + \theta}, \quad \forall \theta \ge 0. \tag{22}$$

Note that the function Ψ satisfies $\Psi(0) = 0$ and $\Psi(\theta) < \theta$ for all $\theta > 0$.

For any $x, y \in \mathcal{C}$ and for each $t \in [1,e]$, we have

$$\begin{aligned}
&|\mathcal{F}x(t) - \mathcal{F}y(t)| \\
&\le J^q\left(|f(s,x(s)) - f(s,y(s))|\right)(t) \\
&\quad + \frac{(\log t)^{q-1}}{|\Lambda|}\sum_{i=1}^{m}|\lambda_i| J^{\alpha_i+q}\left(|f(s,x(s)) - f(s,y(s))|\right)(\eta_i) \\
&\quad + \frac{(\log t)^{q-1}}{|\Lambda|}\sum_{j=1}^{n}|\mu_j| \left(J^{\beta_j+q}\left(|f(s,x(s)) - f(s,y(s))|\right)(e) \right. \\
&\qquad\qquad \left. + J^{\beta_j+q}\left(|f(s,x(s)) - f(s,y(s))|\right) \times (\xi_j)\right) \\
&\le J^q\left(h(s)\frac{|x(s)-y(s)|}{H^* + |x(s)-y(s)|}\right)(e) \\
&\quad + \frac{1}{|\Lambda|}\sum_{i=1}^{m}|\lambda_i| J^{\alpha_i+q}\left(h(s)\frac{|x(s)-y(s)|}{H^* + |x(s)-y(s)|}\right)(\eta_i) \\
&\quad + \frac{1}{|\Lambda|}\sum_{j=1}^{n}|\mu_j| \left\{J^{\beta_j+q}\left(h(s)\frac{|x(s)-y(s)|}{H^* + |x(s)-y(s)|}\right)(e) \right. \\
&\qquad\qquad \left. + J^{\beta_j+q}\left(h(s)\frac{|x(s)-y(s)|}{H^* + |x(s)-y(s)|}\right)(\xi_j)\right\} \\
&\le \frac{\Psi(\|x-y\|)}{H^*}\left(J^q h(e) + \frac{1}{|\Lambda|}\sum_{i=1}^{m}|\lambda_i| J^{\alpha_i+q}h(\eta_i) \right. \\
&\qquad\qquad \left. + \frac{1}{|\Lambda|}\sum_{j=1}^{n}|\mu_j|\left(J^{\beta_j+q}h(e) + J^{\beta_j+q}h(\xi_j)\right)\right) \\
&= \Psi(\|x-y\|).
\end{aligned} \tag{23}$$

This implies that $\|\mathcal{F}x - \mathcal{F}y\| \le \Psi(\|x-y\|)$. Therefore $\mathcal{F}$ is a nonlinear contraction. Hence, by Lemma 6 the operator $\mathcal{F}$ has a fixed point which is the unique solution of the problem (1)-(2). □

Next, we give an existence result by using Krasnoselskii's fixed point theorem.

Lemma 8 (Krasnoselskii's fixed point theorem [30]). *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that (a) $Ax + By \in M$, whenever $x, y \in M$; (b) A is compact and continuous; (c) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 9. *Assume that $f: [1,e] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying assumption (H_1). In addition we suppose that*

(H_3) $|f(t,x)| \le \kappa(t)$, $\forall (t,x) \in [1,e] \times \mathbb{R}$ *and* $\kappa \in C([1,e], \mathbb{R}^+)$.

If

$$\frac{L}{\Gamma(q+1)} < 1, \tag{24}$$

then the boundary value problem (1)-(2) *has at least one solution on* $[1,e]$.

Proof. We define $\sup_{t\in[1,e]}|\kappa(t)| = \|\kappa\|$ and choose a suitable constant $\bar{r}$ as

$$\bar{r} \ge \|\kappa\| \Phi, \tag{25}$$

where Φ is defined by (15). Furthermore, we define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\bar{r}} = \{x \in \mathcal{C} : \|x\| \le \bar{r}\}$ as

$$\begin{aligned}
(\mathcal{P}x)(t) &= \frac{(\log t)^{q-1}}{\Lambda}\sum_{j=1}^{n}\mu_j\left(J^{\beta_j+q}f(s,x(s))(e) - J^{\beta_j+q}f(s,x(s))(\xi_j)\right) \\
&\quad - \frac{(\log t)^{q-1}}{\Lambda}\sum_{i=1}^{m}\lambda_i J^{\alpha_i+q}f(s,x(s))(\eta_i), \quad t \in [1,e], \\
(\mathcal{Q}x)(t) &= J^q f(s,x(s))(t), \quad t \in [1,e].
\end{aligned} \tag{26}$$

For $x, y \in B_{\bar{r}}$, we have

$$\|\mathscr{P}x + \mathscr{Q}y\| \le \|\kappa\| \left(\frac{1}{\Gamma(q+1)} + \frac{1}{|\Lambda|}\sum_{i=1}^{m} |\lambda_i| \frac{(\log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} + \frac{1}{|\Lambda|}\sum_{j=1}^{n} |\mu_j| \frac{1+(\log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \right) = \|\kappa\|\,\Phi \le \bar{r}. \tag{27}$$

This shows that $\mathscr{P}x + \mathscr{Q}y \in B_{\bar{r}}$. It follows from assumption (H_1) together with (24) that $\mathscr{Q}$ is a contraction mapping. Since the function f is continuous, we have that the operator $\mathscr{P}$ is continuous. It is easy to verify that

$$\|\mathscr{P}x\| \le \|\kappa\| \left(\frac{1}{|\Lambda|}\sum_{i=1}^{m} |\lambda_i| \frac{(\log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} + \frac{1}{|\Lambda|}\sum_{j=1}^{n} |\mu_j| \frac{1+(\log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \right). \tag{28}$$

Therefore, $\mathscr{P}$ is uniformly bounded on $B_{\bar{r}}$.

Next, we prove the compactness of the operator $\mathscr{P}$. Let us set $\sup_{(t,x)\in[1,e]\times B_{\bar{r}}}|f(t,x)| = \bar{f} < \infty$; consequently we get

$$\begin{aligned}
&\left|(\mathscr{P}x)(t_1) - (\mathscr{P}x)(t_2)\right| \\
&= \Bigg| \frac{(\log t_1)^{q-1}}{\Lambda}\sum_{j=1}^{n}\mu_j \left(J^{\beta_j+q}f(s,x(s))(e) - J^{\beta_j+q}f(s,x(s))(\xi_j)\right) \\
&\quad - \frac{(\log t_1)^{q-1}}{\Lambda}\sum_{i=1}^{m}\lambda_i J^{\alpha_i+q}f(s,x(s))(\eta_i) \\
&\quad - \frac{(\log t_2)^{q-1}}{\Lambda}\sum_{j=1}^{n}\mu_j \left(J^{\beta_j+q}f(s,x(s))(e) - J^{\beta_j+q}f(s,x(s))(\xi_j)\right) \\
&\quad + \frac{(\log t_2)^{q-1}}{\Lambda}\sum_{i=1}^{m}\lambda_i J^{\alpha_i+q}f(s,x(s))(\eta_i) \Bigg| \\
&\le \bar{f}\frac{\left|(\log t_2)^{q-1} - (\log t_1)^{q-1}\right|}{|\Lambda|}\sum_{i=1}^{m}|\lambda_i|\frac{(\log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
&\quad + \bar{f}\frac{\left|(\log t_2)^{q-1} - (\log t_1)^{q-1}\right|}{|\Lambda|}\sum_{j=1}^{n}|\mu_j|\frac{1-(\log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)},
\end{aligned} \tag{29}$$

which is independent of x and tends to zero as $t_2 \to t_1$. Thus, $\mathscr{P}$ is equicontinuous. So $\mathscr{P}$ is relatively compact on $B_{\bar{r}}$. Hence, by the Arzelá-Ascoli theorem, $\mathscr{P}$ is compact on $B_{\bar{r}}$. Thus, all the assumptions of Lemma 8 are satisfied. So the boundary value problem (1)-(2) has at least one solution on $[1, e]$. The proof is completed. □

Remark 10. In the above theorem we can interchange the roles of the operators $\mathscr{P}$ and $\mathscr{Q}$ to obtain a second result replacing (24) by the following condition:

$$\frac{L}{|\Lambda|}\sum_{i=1}^{m}|\lambda_i|\frac{(\log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} + \frac{L}{|\Lambda|}\sum_{j=1}^{n}|\mu_j|\frac{1+(\log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)} < 1. \tag{30}$$

Now, our last existence result is based on Leray-Schauder's nonlinear alternative.

Theorem 11 (nonlinear alternative for single-valued maps [31]). *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C, and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (i.e., $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$ or*

(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$, with $u = \lambda F(u)$.*

Theorem 12. *Assume that $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. In addition we suppose that*

(H_4) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([1, e], \mathbb{R}^+)$ such that*

$$|f(t,x)| \le p(t)\,\psi(|x|) \quad \text{for each } (t,x) \in [1,e]\times\mathbb{R}; \tag{31}$$

(H_5) *there exists a constant $N > 0$ such that*

$$\frac{N}{\|p\|\,\psi(N)\,\Phi} > 1, \tag{32}$$

where Φ is defined by (15).

Then the boundary value problem (1)-(2) *has at least one solution on* $[1, e]$.

Proof. Firstly, we will show that the operator $\mathscr{F}$, defined by (14), *maps bounded sets (balls) into bounded sets in* $\mathscr{C}$. For a positive number R, let $B_R = \{x \in \mathscr{C} : \|x\| \leq R\}$ be a bounded ball in $\mathscr{C}$. Then for $t \in [1, e]$, we have

$$\begin{aligned}
|\mathscr{F}x(t)| &\leq J^q |f(s, x(s))| (e) \\
&\quad + \frac{1}{|\Lambda|} \sum_{i=1}^{m} |\lambda_i| J^{\alpha_i+q} |f(s, x(s))| (\eta_i) \\
&\quad + \frac{1}{|\Lambda|} \sum_{j=1}^{n} |\mu_j| \left(J^{\beta_j+q} |f(s, x(s))| (e) \right. \\
&\qquad \left. + J^{\beta_j+q} |f(s, x(s))| (\xi_j) \right) \\
&\leq \|p\| \psi(\|x\|) \frac{1}{\Gamma(q+1)} \\
&\quad + \|p\| \psi(\|x\|) \frac{1}{|\Lambda|} \sum_{i=1}^{m} |\lambda_i| \frac{(\log \eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
&\quad + \|p\| \psi(\|x\|) \frac{1}{|\Lambda|} \sum_{j=1}^{n} |\mu_j| \frac{1 + (\log \xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \\
&\leq \|p\| \psi(R) \frac{1}{\Gamma(q+1)} \\
&\quad + \|p\| \psi(R) \frac{1}{|\Lambda|} \sum_{i=1}^{m} |\lambda_i| \frac{(\log \eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
&\quad + \|p\| \psi(R) \frac{1}{|\Lambda|} \sum_{j=1}^{n} |\mu_j| \frac{1 + (\log \xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \\
&:= K.
\end{aligned} \tag{33}$$

Therefore, we conclude that $\|\mathscr{F}x\| \leq K$.

Secondly, we show that $\mathscr{F}$ *maps bounded sets into equicontinuous sets of* $\mathscr{C}$. Let $\sup_{(t,x)\in[1,e]\times B_R} |f(t, x)| = f^* < \infty$, $\nu_1, \nu_2 \in [1, e]$ with $\nu_1 < \nu_2$ and $x \in B_R$. Then we have

$$\begin{aligned}
&\left| (\mathscr{F}x)(\nu_2) - (\mathscr{F}x)(\nu_1) \right| \\
&= \left| J^q f(s, x(s))(\nu_2) \right. \\
&\quad - \frac{(\log \nu_2)^{q-1}}{\Lambda} \sum_{i=1}^{m} \lambda_i J^{\alpha_i+q} f(s, x(s))(\eta_i) \\
&\quad + \frac{(\log \nu_2)^{q-1}}{\Lambda} \sum_{j=1}^{n} \mu_j \left(J^{\beta_j+q} f(s, x(s))(e) \right. \\
&\qquad \left. - J^{\beta_j+q} f(s, x(s))(\xi_j) \right) \\
&\quad - J^q f(s, x(s))(\nu_1) \\
&\quad + \frac{(\log \nu_1)^{q-1}}{\Lambda} \sum_{i=1}^{m} \lambda_i J^{\alpha_i+q} f(s, x(s))(\eta_i) \\
&\quad - \frac{(\log \nu_1)^{q-1}}{\Lambda} \sum_{j=1}^{n} \mu_j \left(J^{\beta_j+q} f(s, x(s))(e) \right. \\
&\qquad \left. \left. - J^{\beta_j+q} f(s, x(s))(\xi_j) \right) \right| \\
&\leq f^* \frac{\left| (\log \nu_2)^q - (\log \nu_1)^q \right|}{\Gamma(q+1)} \\
&\quad + f^* \frac{\left| (\log \nu_2)^{q-1} - (\log \nu_1)^{q-1} \right|}{|\Lambda|} \sum_{i=1}^{m} |\lambda_i| \frac{(\log \eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
&\quad + f^* \frac{\left| (\log \nu_2)^{q-1} - (\log \nu_1)^{q-1} \right|}{|\Lambda|} \sum_{j=1}^{n} |\mu_j| \frac{1 - (\log \xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)}.
\end{aligned} \tag{34}$$

Obviously, the right hand side of the above inequality tends to zero independently of $x \in B_R$ as $\nu_2 \to \nu_1$. Therefore it follows from the Arzelá-Ascoli theorem that $\mathscr{F} : \mathscr{C} \to \mathscr{C}$ is completely continuous.

Let x be a solution. Then, for $t \in [1, e]$, following the similar computations as in the first step, we have

$$\begin{aligned}
\|x\| &\leq \|p\| \psi(\|x\|) \frac{1}{\Gamma(q+1)} \\
&\quad + \|p\| \psi(\|x\|) \frac{1}{|\Lambda|} \sum_{i=1}^{m} |\lambda_i| \frac{(\log \eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
&\quad + \|p\| \psi(\|x\|) \frac{1}{|\Lambda|} \sum_{j=1}^{n} |\mu_j| \frac{1 + (\log \xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \\
&= \|p\| \psi(\|x\|) \Phi.
\end{aligned} \tag{35}$$

Consequently, we have

$$\frac{\|x\|}{\|p\| \psi(\|x\|) \Phi} \leq 1. \tag{36}$$

In view of (H_5), there exists N such that $\|x\| \neq N$. Let us set

$$U = \{x \in \mathscr{C} : \|x\| < N\}. \tag{37}$$

Note that the operator $\mathscr{F} : \overline{U} \to \mathscr{C}$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \theta \mathscr{F} x$ for some $\theta \in (0, 1)$. Consequently, by nonlinear alternative of Leray-Schauder type (Theorem 11) we deduce that $\mathscr{F}$ has a fixed point in $\overline{U}$, which is a solution of the boundary value problem (1)-(2). This completes the proof. □

4. Examples

Example 1. Consider the following boundary value problem for Hadamard fractional differential equation:

$$D^{3/2}x(t)=\frac{\log t^5}{e^t(t+2)^2}\frac{|x(t)|}{(3+|x(t)|)},\quad t\in J=[1,e],$$
$$x(1)=0,$$
$$2J^{1/4}x\left(\frac{5}{4}\right)+\frac{1}{5}J^{3/2}x\left(\frac{9}{5}\right)+3J^2x\left(\frac{15}{7}\right)$$
$$=J^{2/3}x(e)-J^{2/3}x\left(\frac{10}{7}\right)+5\left(J^{9/7}x(e)-J^{9/7}x(2)\right)$$
$$-2\left(J^{11/4}x(e)-J^{11/4}x\left(\frac{9}{4}\right)\right). \tag{38}$$

Here $q=3/2$, $\lambda_1=2$, $\lambda_2=1/5$, $\lambda_3=3$, $\alpha_1=1/4$, $\alpha_2=3/2$, $\alpha_3=2$, $\eta_1=5/4$, $\eta_2=9/5$, $\eta_3=15/7$, $\mu_1=1$, $\mu_2=5$, $\mu_3=-2$, $\beta_1=2/3$, $\beta_2=9/7$, $\beta_3=11/4$, $\xi_1=10/7$, $\xi_2=2$, $\xi_3=9/4$, and $f(t,x)=(\log t^5|x|)/(e^t(t+2)^2(3+|x|))$. Since

$$|f(t,x)-f(t,y)|\le\left(\frac{5}{27e}\right)|x-y|, \tag{39}$$

then (H_1) is satisfied with $L=5/27e$. We can show that

$$\Lambda=\sum_{i=1}^{m}\lambda_i\frac{\Gamma(q)}{\Gamma(q+\alpha_i)}(\log\eta_i)^{q+\alpha_i-1}-\sum_{j=1}^{n}\mu_j\frac{\Gamma(q)}{\Gamma(q+\beta_j)}\left(1-(\log\xi_j)^{q+\beta_j-1}\right)\approx-0.6895040549,$$

$$\Phi=\frac{1}{\Gamma(q+1)}+\frac{1}{|\Lambda|}\sum_{i=1}^{m}|\lambda_i|\frac{(\log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}+\frac{1}{|\Lambda|}\sum_{j=1}^{n}|\mu_j|\frac{1+(\log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\approx 3.975680952, \tag{40}$$

$$L\Phi=\frac{5}{27e}(3.975680952)\approx 0.2708465347<1.$$

Hence, by Theorem 4, the boundary value problem (38) has a unique solution on $[1,e]$.

Example 2. Consider the following boundary value problem for Hadamard fractional differential equation:

$$D^{7/4}x(t)=\frac{e^t}{(t+1)^2}\frac{|x(t)|}{(2+|x(t)|)},\quad t\in J=[1,e],$$
$$x(1)=0,$$
$$\frac{1}{4}J^{6/7}x\left(\frac{7}{3}\right)-\frac{2}{3}J^3x\left(\frac{7}{5}\right)-2J^{5/2}x(2)$$
$$=4\left(J^5x(e)-J^5x\left(\frac{11}{5}\right)\right)+\frac{11}{4}\left(J^{3/4}x(e)-J^{3/4}x\left(\frac{16}{13}\right)\right). \tag{41}$$

Here $q=7/4$, $\lambda_1=1/4$, $\lambda_2=-2/3$, $\lambda_3=-2$, $\alpha_1=6/7$, $\alpha_2=3$, $\alpha_3=5/2$, $\eta_1=7/3$, $\eta_2=7/5$, $\eta_3=2$, $\mu_1=4$, $\mu_2=11/4$, $\beta_1=5$, $\beta_2=3/4$, $\xi_1=11/5$, $\xi_2=16/13$, and $f(t,x)=(e^t|x|)/((t+1)^2(2+|x|))$. We choose $h(t)=e^t/4$ and that

$$\Lambda=\sum_{i=1}^{m}\lambda_i\frac{\Gamma(q)}{\Gamma(q+\alpha_i)}(\log\eta_i)^{q+\alpha_i-1}-\sum_{j=1}^{n}\mu_j\frac{\Gamma(q)}{\Gamma(q+\beta_j)}\left(1-(\log\xi_j)^{q+\beta_j-1}\right)\approx-1.672972140,$$
$$H^*=J^qh(e)+\frac{1}{|\Lambda|}\sum_{i=1}^{m}|\lambda_i|J^{\alpha_i+q}h(\eta_i)+\frac{1}{|\Lambda|}\sum_{j=1}^{n}|\mu_j|\left(J^{\beta_j+q}h(e)+J^{\beta_j+q}h(\xi_j)\right)\approx 1.295076743. \tag{42}$$

Clearly,

$$|f(t,x)-f(t,y)|=\frac{e^t}{(1+t)^2}\left(\frac{2|x|-2|y|}{4+2|x|+2|y|+|x||y|}\right)\le\frac{e^t}{4}\left(\frac{|x-y|}{1.295076743+|x-y|}\right). \tag{43}$$

Hence, by Theorem 7, the boundary value problem (41) has a unique solution on $[1,e]$.

Example 3. Consider the following boundary value problem for Hadamard fractional differential equation:

$$D^{6/5}x(t) = \frac{2\sin(x/4)}{5\pi + (e^x+1)^2} + \frac{2+\cos(\pi t)}{10\pi+3}, \quad t \in J = [1,e],$$
$$x(1) = 0,$$
$$J^4 x\left(\frac{3}{2}\right) - 3J^{9/4}x(2) - 10J^{1/5}x\left(\frac{7}{4}\right) + 6J^{7/2}x\left(\frac{5}{2}\right) + \frac{14}{3}J^5 x\left(\frac{11}{9}\right) = 3\left(J^{3/2}x(e) - J^{3/2}x\left(\frac{11}{7}\right)\right) - 7\left(J^3 x(e) - J^3 x\left(\frac{17}{13}\right)\right) + \frac{4}{3}\left(J^{5/3}x(e) - J^{5/3}x(2)\right). \tag{44}$$

Here $q = 6/5$, $\lambda_1 = 1$, $\lambda_2 = -3$, $\lambda_3 = -10$, $\lambda_4 = 6$, $\lambda_5 = 14/3$, $\alpha_1 = 4$, $\alpha_2 = 9/4$, $\alpha_3 = 1/5$, $\alpha_4 = 7/2$, $\alpha_5 = 5$, $\eta_1 = 3/2$, $\eta_2 = 2$, $\eta_3 = 7/4$, $\eta_4 = 5/2$, $\eta_5 = 11/9$, $\mu_1 = 3$, $\mu_2 = -7$, $\mu_3 = 4/3$, $\beta_1 = 3/2$, $\beta_2 = 3$, $\beta_3 = 5/3$, $\xi_1 = 11/7$, $\xi_2 = 17/13$, $\xi_3 = 2$, and $f(t,x) = (2\sin(x/4))/(5\pi + (e^x+1)^2) + (2+\cos(\pi t))/(10\pi+3)$. Clearly,

$$|f(t,x)| = \left|\frac{2\sin(x/4)}{5\pi + (e^x+1)^2} + \frac{2+\cos(\pi t)}{10\pi+3}\right| \le (2+\cos(\pi t))\left(\frac{|x|+1}{10\pi}\right). \tag{45}$$

Choosing $p(t) = 2+\cos(\pi t)$ and $\psi(|x|) = (|x|+1)/(10\pi)$, we can show that

$$\Lambda = \sum_{i=1}^{m}\lambda_i \frac{\Gamma(q)}{\Gamma(q+\alpha_i)}(\log\eta_i)^{q+\alpha_i-1} - \sum_{j=1}^{n}\mu_j \frac{\Gamma(q)}{\Gamma(q+\beta_j)}\left(1 - (\log\xi_j)^{q+\beta_j-1}\right) \approx -9.148087406,$$
$$\Phi = \frac{1}{\Gamma(q+1)} + \frac{1}{|\Lambda|}\sum_{i=1}^{m}|\lambda_i|\frac{(\log\eta_i)^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} + \frac{1}{|\Lambda|}\sum_{j=1}^{n}|\mu_j|\frac{1+(\log\xi_j)^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \approx 1.462649525, \tag{46}$$
$$\frac{N}{(3)((N+1)/10\pi)(1.462649525)} > 1,$$

which implies that $N > 0.1623483851$. Hence, by Theorem 12, the boundary value problem (44) has at least one solution on $[1,e]$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The research of Phollakrit Thiramanus is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand. The research of Jessada Tariboon is supported by King Mongkut's University of Technology North Bangkok, Thailand. Sotiris K. Ntouyas is a member of Nonlinear Analysis and Applied Mathematics (NAAM) Research Group at King Abdulaziz University, Jeddah, Saudi Arabia.

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