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Research Article

τ-Complexity and Tilting Modules

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Let A be a finite dimensional algebra over an algebraic closed field k. In this note, we will show that if T is a separating and splitting tilting A-module, then τ-complexities of A and B are equal, where $B = \mathrm{End}_A(T)$.

1. Introduction

Background. Tilting theory plays an important role in the modern representation theory of algebras. Let A be a finite dimensional algebra over a field k and T a tilting A-module. It is well known that A and $\mathrm{End}_A(T)$ are derived equivalent. The endomorphism algebra of a tilting module preserves many significant invariants, for example, the center of an algebra, the number of nonisomorphic simple modules, the Hochschild cohomology groups, and Cartan determinants. In particular, if T is a separating and splitting tilting A-module (see the definition in Section 2), then $\mathrm{End}_A(T)$ preserves representation dimension [1].

On the other hand, τ-complexity (see the definition in Section 2) is an important invariant in the representation theory of algebras. With τ-complexity, Bergh and Oppermann described the classification of hereditary algebras and studied the classification of cluster tilted algebra [2].

However, the precise value of τ-complexity of a given algebra is not known in general, and it is hard to compute even for small examples. One possible way is to compare τ-complexities of "nicely" related algebras.

Question. Suppose B is the endomorphism algebra of a tilting module T over an algebra A. What is the relationship between τ-complexities of A and B?

Note that in general A and B do not have the same τ-complexities, since there are examples where A is representation finite while B is representation infinite. Our main result in this paper is the following theorem.

Theorem 1. *Let T be a tilting module over a finite dimensional k-algebra A, with $B = \mathrm{End}_A(T)$. If T is separating and splitting, then $c_{\tau_A} = c_{\tau_B}$, where c_{τ_A}, c_{τ_B} denote τ-complexity of A and B, respectively.*

Organization. This paper is organized as follows. In Section 2, we shall give the proof of our main result Theorem 1. In Section 3, we shall give two examples to illustrate our results.

2. Proof of the Main Theorem

Throughout this paper, k is an algebraically closed field, A is a finite dimensional k-algebra. Denote by $\mathrm{mod}\,A$ the category of finitely generated left A-modules, $\mathscr{P}(\mathrm{mod}\,A)$ the full subcategory of $\mathrm{mod}\,A$ consisting of all projective objects in $\mathrm{mod}\,A$, and $\mathrm{gl}\cdot\dim A$ the global dimension of A. $D := \mathrm{Hom}_k(-,k)$ denotes the standard duality functor between $\mathrm{mod}\,A$ and $\mathrm{mod}\,A^{\mathrm{op}}$. Given a left A-module M, $\mathrm{add}\,M$ denotes the full subcategory of $\mathrm{mod}\,A$ consisting of all direct summands of finite direct sums of copies of M.

Torsion Pair. A pair $(\mathscr{T},\mathscr{F})$ of full subcategories of $\mathrm{mod}\,A$ is called a torsion pair, if the following conditions are satisfied: (1) $\mathrm{Hom}_A(M,N) = 0$ for all $M \in \mathscr{T}$, $N \in \mathscr{F}$; (2) $\mathrm{Hom}_A(M,-)|_{\mathscr{F}} = 0$ implies $M \in \mathscr{T}$; (3) $\mathrm{Hom}_A(-,N)|_{\mathscr{T}} = 0$ implies $N \in \mathscr{F}$. A torsion pair $(\mathscr{T},\mathscr{F})$ is called splitting if each indecomposable A-module lies either in $\mathscr{T}$ or in $\mathscr{F}$.

A module T is called a tilting module if the following three conditions are satisfied: (1) $\mathrm{pd}\,T \le 1$; (2) $\mathrm{Ext}^1_A(T,T) = 0$;

(3) there exists a short exact sequence: $0 \to A \to T_1 \to T_2 \to 0$, with $T_1, T_2 \in \operatorname{add} T$.

It is well known that ${}_AT$ induces a torsion pair $(\mathcal{T}_T, \mathcal{F}_T)$ in mod A, and a torsion pair $(\mathcal{X}_T, \mathcal{Y}_T)$ in mod B. T is said to be separating if the induced torsion pair $(\mathcal{T}_T, \mathcal{F}_T)$ in mod A is splitting and said to be splitting if the induced torsion pair $(\mathcal{X}_T, \mathcal{Y}_T)$ in mod B is splitting.

The following lemma is crucial in this paper.

Lemma 2. *Let A be a finite dimensional algebra and T a tilting A-module. Let $F = \mathrm{Hom}_A(T, -)$, $G = \mathrm{Ext}_A^1(T, -)$. Then the following assertions hold.*

(1) *For any $M \in \mathcal{T}_T$,*

  (a) *there exists a constant $c > 0$, such that* $\dim_k M \leq c \dim_k FM$.

  (b) *there exists a constant $\lambda > 0$, such that* $\dim_k FM \leq \lambda \dim_k M$.

(2) *For any $M \in \mathcal{F}_T$,*

  (a) *there exists a constant $c' > 0$, such that* $\dim_k M \leq c' \dim_k GM$.

  (b) *there exists a constant $\lambda' > 0$, such that* $\dim_k GM \leq \lambda' \dim_k M$.

Proof. The proof is similar to [3, Lemma 3.1, Lemma 3.3]. There exists a short exact sequence of the form $0 \to A \to T_1 \to T_2 \to 0$ where $T_1, T_2 \in \operatorname{add} T$ since T is a tilting A-module. Denote by t the number of indecomposable summands of $T_1 \oplus T_2$. Given a module M, apply $\mathrm{Hom}_A(-, M)$ to the short exact sequence above to obtain the long exact sequence:

$$0 \longrightarrow \mathrm{Hom}_A(T_2, M) \longrightarrow \mathrm{Hom}_A(T_1, M) \longrightarrow \mathrm{Hom}_A(A, M) \longrightarrow \mathrm{Ext}_A^1(T_2, M) \longrightarrow \mathrm{Ext}_A^1(T_1, M) \longrightarrow 0. \quad (*)$$

(1)

  (a) Assume that M is a torsion module. Then $\mathrm{Ext}_A^1(T_2, M) = 0$ in the long exact sequence because T_2 is in add T and M is torsion. We obtain the short exact sequence $0 \to \mathrm{Hom}_A(T_2, M) \to \mathrm{Hom}_A(T_1, M) \to \mathrm{Hom}_A(A, M) \to 0$. Noting that $\mathrm{Hom}_A(A, M) \cong M$, we have $\dim_k M \leq t \dim_k \mathrm{Hom}(T, M) = t \dim_k FM$.

  (b) For any finitely generated A-module M, we have $\dim_k FM \leq \dim_k T \cdot \dim_k M$ [3, Remark 3.2], we set $\lambda = \dim_k T$, and then the assertion follows immediately.

(2)

  (a) Assume that M is torsion-free. In the long exact sequence $(*)$ above, we now have $\mathrm{Hom}_A(T_1, M) = 0$ since M is torsion-free. We thus have the short exact sequence $0 \to \mathrm{Hom}_A(A, M) \to \mathrm{Ext}_A^1(T_2, M) \to \mathrm{Ext}_1^1(T_1, M) \to 0$. Noting that $\mathrm{Hom}_A(A, M) \cong M$, we have $\dim_k M \leq \dim_k \mathrm{Ext}_A^1(T_2, M) \leq \dim_k \mathrm{Ext}_A^1(T^t, M) \leq t \dim_k \mathrm{Ext}_A^1(T, M) = t \dim_k GM$.

  (b) For any finitely generated A-module M, we have $\dim_k GM \leq \dim_k T \cdot \dim_k M \cdot (\dim_k A)^2$ [3, Remark 3.2], we set $\lambda' = \dim_k T \cdot (\dim_k A)^2$, and then the assertion follows immediately. □

Let T be a tilting A-module, $B = \mathrm{End}_A(T)$. Denote by τ_A, τ_B the Auslander-Reiten translation in mod A and mod B, respectively. Let $M \in \operatorname{mod} A$; the τ_A-complexity of M is defined as follows:

$$c_{\tau_A}(M) = \inf \left\{ t \in \mathbb{N} \mid \exists \lambda \in \mathbb{R} \text{ such that } \dim_k \tau_A^i(M) \leq \lambda i^{t-1} \text{ for } i \gg 0 \right\}. \quad (1)$$

When no such $t \in \mathbb{N}$ exists, we say that τ_A-complexity of M is infinite and write $c_{\tau_A}(M) = \infty$. And τ-complexity of the algebra A is defined to be the supreme of τ_A-complexities of all the finitely generated A-modules, which will be denoted by c_{τ_A}.

Proof of Theorem 1.

Step 1 ($c_{\tau_A} \leq c_{\tau_B}$). We will show that, for each indecomposable A-module M, there exists $N \in \operatorname{mod} B$ such that $c_{\tau_A}(M) = c_{\tau_B}(N)$.

Case 1 ($0 \neq M \in \mathcal{F}_T$). In this case, by [4, Chapter VI, Proposition 1.7], for any $i > 0$, $\tau_A^i M \in \mathcal{F}_T$.

Case 1.1. There exists an integer $i_0 > 0$, such that $\tau_A^{i_0}(M) = 0$. Then $c_{\tau_A}(M) = c_{\tau_A}(\tau_A^{i_0} M) = 0$. Let $N = 0 \in \operatorname{mod} B$; then $c_{\tau_A}(M) = c_{\tau_B}(N)$.

Case 1.2. For any integer $i > 0$, $\tau_A^i M \neq 0$. In this case, we will show that $c_{\tau_A}(M) = c_{\tau_B}(GM)$ and the assertion holds. By [4, Chapter VI, Lemma 5.3(b)], $\tau_B^i(GM) \cong G(\tau_A^i M)$. Set $c_0 = c_{\tau_A}(M)$ and $d_0 = c_{\tau_B}(G(M))$, respectively. By definition, there exist constants $\lambda > 0$ and $\lambda' > 0$ such that $\dim_k \tau_A^i M \leq \lambda i^{c_0 - 1}$ and $\dim_k \tau_B^i GM \leq \lambda' i^{d_0 - 1}$. By Lemma 2(2)(b), there exists a constant c such that $\dim_k \tau_B^i GM = \dim_k G(\tau_A^i M) \leq c \dim_k \tau_A^i M \leq c \lambda i^{c_0 - 1}$. Therefore $d_0 \leq c_0$. On the other hand, by Lemma 2(2)(a), there exists a constant c' such that $\dim_k \tau_A^i M \leq c' \dim_k \tau_B^i(GM) \leq c' \lambda' i^{d_0 - 1}$. It implies that $c_0 \leq d_0$, and hence $c_{\tau_A}(M) = c_{\tau_B}(GM)$.

Case 2 ($0 \neq M \in \mathcal{T}_T$).

Case 2.1. For any $i \geq 0$, $\tau_A^i M \notin \mathcal{F}_T$. In this case, $0 \neq \tau_A^i M \in \mathcal{T}_T$. By [4, Chapter VI, Lemma 5.3(a)], we have that

$\tau_B^i(FM) \cong F(\tau_A^i M)$ for any $i \geq 0$. By Lemma 2(1) and the similar argument of Case 1, we have $c_{\tau_A}(M) = c_{\tau_B}(F(M))$.

Case 2.2. There exists an integer $i_0 > 0$, such that $\tau_A^{i_0}(M) \in \mathcal{F}_T$. Let i_0 be the minimal integer such that $\tau_A^{i_0}(M) \in \mathcal{F}_T$. And then $0 \neq \tau_A^{i_0-1}(M) \in \mathcal{T}_T$. Since T is separating and splitting, we obtain that $\tau_A^{i_0-1}(M) \in \operatorname{add} T$ by [4, Chapter VI, Proposition 1.11]. And we have the following two subcases:

(i) $\tau_A^{i_0}(M) = 0$. In this case, $c_{\tau_A}(M) = 0$. Set $N = 0 \in \operatorname{mod} B$; then $c_{\tau_A}(M) = c_{\tau_B}(N)$.

(ii) $0 \neq \tau_A^{i_0}(M) \in \mathcal{F}_T$. By Case 1, there exists $N \in \operatorname{mod} B$, such that $c_{\tau_A}(\tau_A^{i_0}M) = c_{\tau_B}(N)$, and then $c_{\tau_A}(M) = c_{\tau_A}(N)$.

Step 2 ($c_{\tau_A} \geq c_{\tau_B}$). We will show that for any indecomposable B-module X, there exists $N \in \operatorname{mod} A$ such that $c_{\tau_B}(X) = c_{\tau_A}(N)$, and the assertion holds.

Case 1 ($0 \neq X \in \mathcal{X}_T$). In this case, there exists $0 \neq N \in \mathcal{F}_T$ such that $X \cong G(N)$. If N belongs to Case 1.1 in Step 1, then denote i_0 by the minimal positive integer such that $\tau_A^{i_0}N = 0$. Then $\tau_A^{i_0-1}N \cong P \neq 0$, where P is an indecomposable projective A-module and does not belong to $\operatorname{add} T$ (if P belongs to $\operatorname{add} T$, then $P \in \mathcal{T}_T \cap \mathcal{F}_T$, contradiction!). Now, by [4, Chapter VI, Lemma 5.3(b)] we have that $\tau_B^{i_0-1}(X) \cong \operatorname{Ext}_A^1(T, \tau_A^{i_0-1}N) \cong \operatorname{Ext}_A^1(T, P)$. Therefore, by [4, Chapter VI, Lemma 4.9], $\tau_B^{i_0}(X) \cong \tau_B \operatorname{Ext}_A^1(T, P) \cong \operatorname{Hom}_A(T, E(\operatorname{top} P))$, where $E(\operatorname{top} P)$ denotes the injective hull of $\operatorname{top} P$. Clearly, $0 \neq E(\operatorname{top} P) \in \mathcal{T}_T$. If $E(\operatorname{top} P)$ belongs to Case 2.1 in Step 1, then $c_{\tau_A}(E(\operatorname{top} P)) = c_{\tau_B}(F(E(\operatorname{top} P))) = c_{\tau_B}(\tau_B^{i_0}(X)) = c_{\tau_B}(X)$. Otherwise, $E(\operatorname{top} P)$ belongs to Case 2.2 in Step 1; then there exists $j_0 > 0$ such that $\tau_A^{j_0-1}(E(\operatorname{top} P)) \in \operatorname{add} T$. By [4, Chapter VI, Lemma 5.3(a)], we know $\tau_B^{j_0-1}(\tau_B^{i_0}(X)) \cong \tau_B^{j_0-1}(\operatorname{Hom}_A(T, E(\operatorname{top} P))) \cong \operatorname{Hom}_A(T, \tau_A^{j_0-1}E(\operatorname{top} P))$. Therefore, $\tau_B^{j_0+i_0-1}(X)$ is a projective B-module. It implies that $c_{\tau_B}(X) = 0$.

Case 2 ($X \in \mathcal{Y}_T$). In this case, there exists $0 \neq N \in \mathcal{T}_T$ such that $Y \cong \operatorname{Hom}_A(T, N)$.

If N belongs to Case 2.1 in Step 1, then $c_{\tau_B}(X) = c_{\tau_B}(F(N)) = c_{\tau_A}(N)$.

Otherwise N belongs to Case 2.2 in Step 1; that is, there exists a positive integer i_0 such that $\tau_A^{i_0}(N) \in \mathcal{F}_T$ and $\tau_A^{i_0-1}(N) \notin \mathcal{F}_T$. In this case, $\tau_A^{i_0-1}N \in \operatorname{add} T$. By [4, Chapter VI, Lemma 5.3(a)], $\tau_B^{i_0-1}(X) \cong \operatorname{Hom}_A(T, \tau_A^{i_0-1}N)$. Therefore, $\tau_B^{i_0-1}(X)$ is a projective B-module, and $c_{\tau_B}(X) = 0$. □

Proposition 3 (See [2, Proposition 3.1]). *Let A be a finite dimensional hereditary algebra. Then the following assertions hold.*

(1) *A is of finite representation type if and only if $c_{\tau_A} = 0$.*

(2) *A is of tame representation type if and only if $c_{\tau_A} = 2$.*

(3) *A is of wild representation type if and only if $c_{\tau_A} = \infty$.*

Combining Theorem 1 and Proposition 3, we have the following corollary.

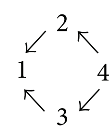
Corollary 4. *Let A be a finite dimensional hereditary algebra and T a separating and splitting A-module, with $B = \operatorname{End}_A(T)$. If B is a hereditary algebra, then A and B are of the same representation type.*

Remark 5. Let A be a finite dimensional hereditary algebra and T a APR-tilting module. It is well known that $B = \operatorname{End}(T)$ is a hereditary algebra. Therefore, they are of the same representation type.

3. Two Examples

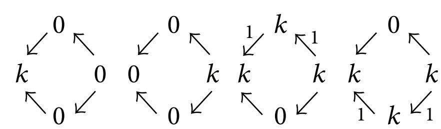
In this section, we shall give two examples to illustrate our result.

Example 1. Let A be the path algebra of the Euclidean quiver Q:



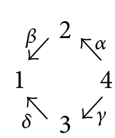
(2)

Consider the indecomposable A-modules: T_1, T_2, T_3, and T_4, respectively:



(3)

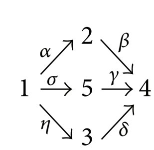
Then $T = T_1 \oplus T_2 \oplus T_3 \oplus T_4$ is a tilting module, and $B = \operatorname{End}_A(T)$ is the algebra given by the following quiver



(4)

bound by $\alpha\beta = 0$, $\gamma\delta = 0$. This example was given as Example 4.8(a) in [4, Chapter VIII]. Clearly, T is splitting, but it is not separating. A is a hereditary algebra and of tame representation type; then $c_{\tau_A} = 2$. However, B is of finite representation type and of finite global dimension, $c_{\tau_B} = 0$.

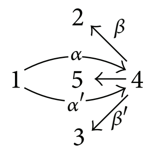
Example 2. Let A be the algebra (over a field k) given by the quiver



(5)

with relations $\alpha\beta = \sigma\gamma = \eta\delta$. This example was also given in [1].

Let ${}_AT = \tau^{-1}S_4 \oplus P$ be the APR-tilting module coresponding to the vertex 4. Then ${}_AT$ is separating, but it isn't splitting. By [5], $B = \mathrm{End}_A(T)$ is given by the quiver



(6)

with relations $\alpha\beta = 0$, $\alpha'\beta' = 0$. Since A is of finite global dimension and finite representation type, $c_{\tau_A} = 0$, however, B is of infinite representation type, $c_{\tau_B} > 0$.

Competing Interests

The authors declare that they have no competing interests.

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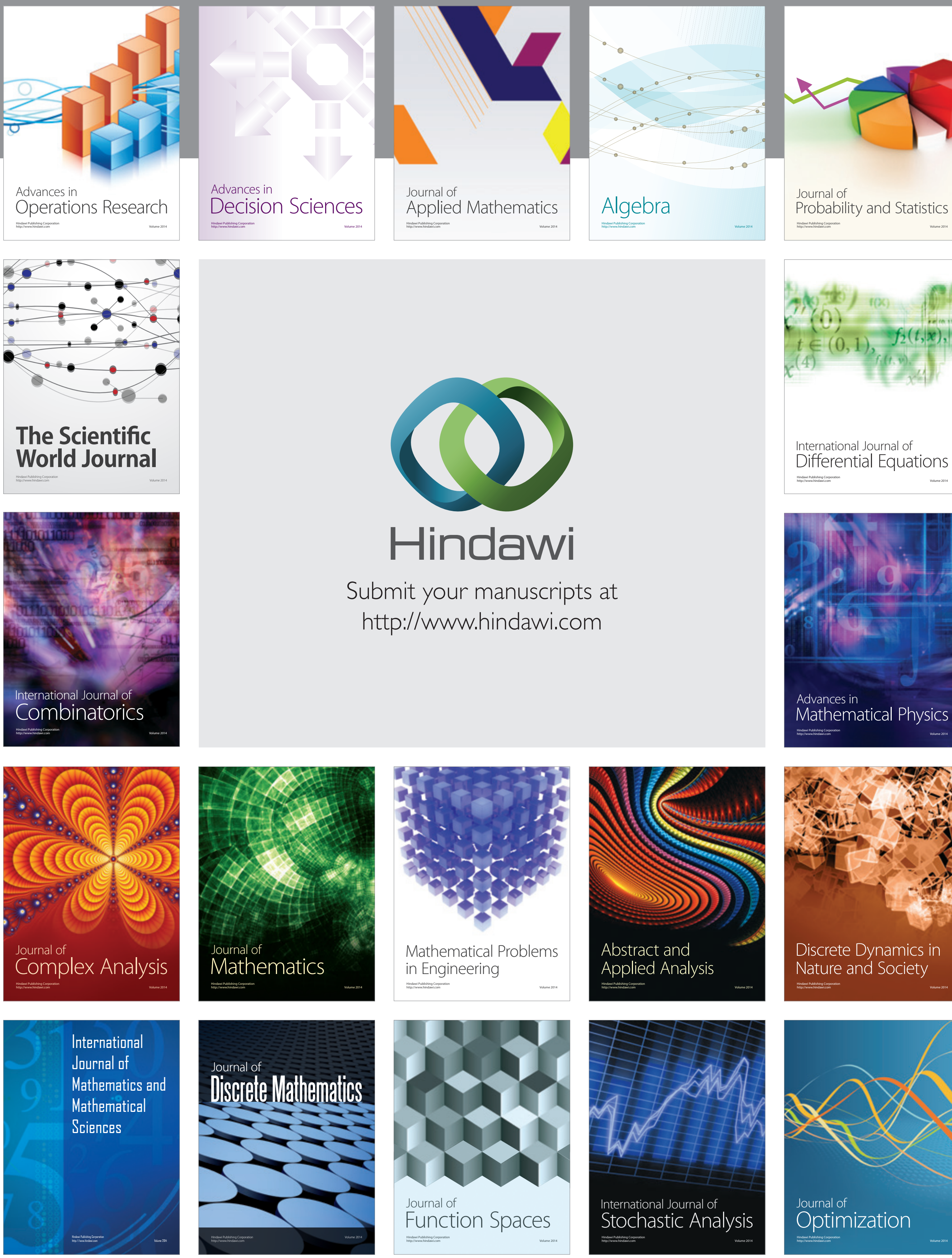
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