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Research Article

Spherical Harmonics $Y_l^m(\theta,\phi)$: Positive and Negative Integer Representations of $\mathrm{su}(1,1)$ for $l-m$ and $l+m$

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The azimuthal and magnetic quantum numbers of spherical harmonics $Y_l^m(\theta,\phi)$ describe quantization corresponding to the magnitude and z-component of angular momentum operator in the framework of realization of $\mathrm{su}(2)$ Lie algebra symmetry. The azimuthal quantum number l allocates to itself an additional ladder symmetry by the operators which are written in terms of l. Here, it is shown that simultaneous realization of both symmetries inherits the positive and negative $(l-m)$- and $(l+m)$-integer discrete irreducible representations for $\mathrm{su}(1,1)$ Lie algebra via the spherical harmonics on the sphere as a compact manifold. So, in addition to realizing the unitary irreducible representation of $\mathrm{su}(2)$ compact Lie algebra via the $Y_l^m(\theta,\phi)$'s for a given l, we can also represent $\mathrm{su}(1,1)$ noncompact Lie algebra by spherical harmonics for given values of $l-m$ and $l+m$.

1. Introduction

The set of principal, azimuthal, magnetic, and spin quantum numbers describe the unique quantum state of a single electron for any system in which the potential depends only on the radial coordinate. The labels l and m of the usual complex spherical harmonics $Y_l^m(\theta,\phi)$ are the second and the third numbers of this set. Spherical surface harmonics are an orthonormal set of vibration solutions for eigenvalue equation of the Laplace-Beltrami operator on the sphere S^2 as a compact Riemannian manifold. They also form the wave functions which represent the orbital angular momentum operator $\mathbf{L}=\mathbf{r}\times\mathbf{p}=-i\mathbf{r}\times\nabla$ ($\hbar=1$) and have a wide range of applications in theoretical and applied physics [1–4]. The three components of the angular momentum operator, that is L_x, L_y, and L_z, are Killing vector fields that generate the rotations about x-, y-, and z-axes, respectively. All spherical harmonics with the given quantum number l form a unitary irreducible representation of $\mathrm{su}(2)\cong\mathrm{so}(3)$ Lie algebra. In fact, they can be seen as representations of the $SO(3)$ symmetry group of rotations about a point and its double-cover $SU(2)$. It can also be noted that the spherical harmonics $Y_l^m(\theta,\phi)$ are just the independent components of symmetric traceless tensors of rank l. The properties of the spherical harmonics are well known and may be found in many texts and papers (e.g., see [5–9]).

In spite of the fact that the problem of quantization of particle motion on a sphere is 80 years old, there still exist some open questions concerning the symmetry properties of the bound states. The aim of this work is to introduce new symmetries based on the quantization of both azimuthal and magnetic numbers l and m of the usual spherical harmonics $Y_l^m(\theta,\phi)$. In order to provide the necessary background and also to attribute a quantization relation for azimuthal quantum number l, here, we present some basic facts about the spherical harmonics [2, 3]. In Section 2, with the application of angular momentum operator, we review realization of the unitary irreducible representations of $\mathrm{su}(2)$ Lie algebra on the sphere in terms of spherical harmonics by shifting m only. In Section 3, the representations of the ladder symmetry with respect to azimuthal quantum number l are constructed in terms of a pair of ladder operators, and its corresponding quantization relation is also expressed as an operator identity originated from solubility in the framework of supersymmetry and shape invariance theories. In Section 4, these results are applied to show that $\mathrm{su}(1,1)$ Lie algebra can also be represented irreducibly by using spherical harmonics. Finally, in Section 5 we discuss the results and make some final comments.

2. The Unitary Irreducible Representations of su(2) Lie Algebra via Orbital Angular Momentum Operator

This section covers the standard and the well-known formalism of su(2) commutation relations in order to encounter spherical harmonics. In what follows, we describe points on S^2 using the parametrization ($x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $x = r\cos\theta$) where $0 \leq \theta < \pi$ is the polar (or colatitude) angle and $0 \leq \phi < 2\pi$ is the azimuthal (or longitude) angle. For a given l with the lower bound $l \geq 0$, we define the $(2l+1)$-dimensional Hilbert space

$$\mathscr{H}_l \coloneqq \operatorname{span}\left\{Y_l^m(\theta,\phi)\right\}_{-l\leq m\leq l}, \tag{1}$$

with the spherical harmonics as bases:

$$\begin{aligned} Y_l^m(\theta,\phi) &= \frac{(-1)^m}{2^l\Gamma(l+1)} \\ &\cdot \sqrt{\frac{(2l+1)\Gamma(l+m+1)}{4\pi\Gamma(l-m+1)}}\left(\frac{e^{i\phi}}{\sin\theta}\right)^m \\ &\cdot \left(\frac{1}{\sin\theta}\frac{d}{d\theta}\right)^{l-m}(\sin\theta)^l. \end{aligned} \tag{2}$$

Also, the infinite dimensional Hilbert space $\mathscr{H} = L^2(S^2, d\Omega(\theta,\phi))$ is defined as a direct sum of finite dimensional subspaces: $\mathscr{H} = \oplus_{l=0}^{+\infty}\mathscr{H}_l$. We must emphasize that the bases of $\mathscr{H}$ are independent spherical harmonics with different values for both indices l and m. The spherical harmonics as the bases of $\mathscr{H}$ constitute an orthonormal set with respect to the following inner product over the sphere S^2:

$$\int_{S^2} Y_l^{m*}(\theta,\phi)\,Y_{l'}^{m'}(\theta,\phi)\,d\Omega(\theta,\phi) = \delta_{ll'}\delta_{mm'}. \tag{3}$$

Therefore, similar to the Fourier expansion, they can be used to expand any arbitrary square integrable function of latitude and longitude angles. $d\Omega(\theta,\phi) = d\cos\theta d\phi$ is the natural invariant measure (area) on the sphere S^2. The following proposition is an immediate consequence of the raising and lowering relations of the index m of the associated Legendre functions [2, 4, 10].

Proposition 1. *Let one introduce three differential generators L_+, L_-, and L_z on the sphere S^2, corresponding to the orbital angular momentum operator $\mathbf{L}$ as*

$$\begin{aligned} L_\pm &= e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right), \\ L_z &= -i\frac{\partial}{\partial\phi}. \end{aligned} \tag{4}$$

They satisfy the commutation relations of su(2) Lie algebra as follows:

$$\begin{aligned} \left[L_+, L_-\right] &= 2L_z, \\ \left[L_z, L_\pm\right] &= \pm L_\pm. \end{aligned} \tag{5}$$

L_z is a self-adjoint operator, and two operators L_+ and L_- are Hermitian conjugate of each other with respect to the inner product (3). Each of the Hilbert subspaces $\mathscr{H}_l$ realizes an l-integer unitary irreducible representation of su(2) Lie algebra as

$$L_+Y_l^{m-1}(\theta,\phi) = \sqrt{(l-m+1)(l+m)}\,Y_l^m(\theta,\phi), \tag{6a}$$

$$L_-Y_l^{m}(\theta,\phi) = \sqrt{(l-m+1)(l+m)}\,Y_l^{m-1}(\theta,\phi), \tag{6b}$$

$$L_zY_l^m(\theta,\phi) = mY_l^m(\theta,\phi). \tag{6c}$$

The Hilbert subspace $\mathscr{H}_l$ contains the lowest and highest bases

$$Y_l^{\mp l}(\theta,\phi) = \frac{\sqrt{\Gamma(2l+2)}}{\sqrt{\pi}2^{l+1}\Gamma(l+1)}(\sin\theta)^l\frac{e^{\mp il\phi}}{(-1)^{l/2\mp l/2}} \tag{7}$$

with the lowest and highest weights $-l$ and l, respectively. They are annihilated by the operators L_- and L_+: $L_-Y_l^{-l}(\theta,\phi) = 0$ and $L_+Y_l^l(\theta,\phi) = 0$. Meanwhile, an arbitrary basis belonging to each of the Hilbert subspaces $\mathscr{H}_l$ can be calculated by an algebraic method as follows:

$$\begin{aligned} &Y_l^m(\theta,\phi) \\ &\quad= \sqrt{\frac{\Gamma(l\mp m+1)}{\Gamma(2l+1)\Gamma(l\pm m+1)}}(L_\pm)^{l\pm m}\,Y_l^{\mp l}(\theta,\phi) \\ &\qquad -l\leq m\leq l. \end{aligned} \tag{8}$$

Also, the Casimir operator corresponding to the generators (4), that is,

$$\mathbf{L}^2_{su(2)} = L_+L_- + L_z^2 - L_z, \tag{9}$$

is a self-adjoint operator and has a $(2l+1)$-fold degeneracy on $\mathscr{H}_l$ as

$$\mathbf{L}^2_{su(2)}Y_l^m(\theta,\phi) = l(l+1)\,Y_l^m(\theta,\phi) \quad -l\leq m\leq l. \tag{10}$$

Obviously, the representation of the su(2) Lie algebra in the Hilbert space $\mathscr{H}$ via (6a)–(6c) is reducible.

A given unitary irreducible representation is characterized by the index l. The spherical harmonics $Y_l^m(\theta,\phi)$, via their m index, describe quantization corresponding to commutation relations of the three components of orbital angular momentum operator. $L_z = -i(\partial/\partial\phi)$ is always a Killing vector field which corresponds to an angular momentum about the body-fixed z-axis. The Casimir operator $\mathbf{L}^2_{\mathrm{su}(2)}$ along with the Cartan subalgebra generator L_z describes the Hamiltonian of a free particle on the sphere with dynamical symmetry group SU(2) and $(2l+1)$-fold degeneracy for the energy spectrum. It must be emphasized that the spherical harmonics and their mathematical structure, as given by Proposition 1, are playing a more visible and important role in different branches of physics. Proposition 1 implies that the spherical harmonics are created by orbital angular momentum operator. Schwinger has developed the realization of this proposition in the framework of creation and annihilation operators of two-dimensional isotropic oscillator [11].

3. Ladder Symmetry for the Azimuthal Quantum Number l

It is evident that simultaneous realization of laddering relations with respect to two different parameters l and m of the associated Legendre functions gives us the possibility to represent laddering relations with respect to the azimuthal quantum number l of spherical harmonics. Representation of such ladder symmetry by the spherical harmonics $Y_l^m(\theta,\phi)$ with the same m but different l induces a new splitting on the Hilbert space $\mathscr{H}$:

$$\mathscr{H} = \bigoplus_{m=-\infty}^{+\infty} \mathscr{H}_m \quad \text{with } \mathscr{H}_m := \operatorname{span}\left\{Y_l^m(\theta,\phi)\right\}_{l\geq |m|}. \tag{11}$$

The following proposition provides an alternative characterization of the mathematical structure of spherical harmonics.

Proposition 2. *Let one define two first-order differential operators on the sphere S^2:*

$$J_\pm(l) = \pm\sin\theta\frac{\partial}{\partial\theta} + l\cos\theta. \tag{12}$$

They satisfy the following operator identity in the framework of shape invariance theory:

$$J_-(l+1)\,J_+(l+1) - J_+(l)\,J_-(l) = 2l+1. \tag{13}$$

$J_\pm(l\pm 2)$ are the adjoint of the operators $J_\mp(l)$ with respect to the inner product (3); that is, one has $J_\mp^\dagger(l) = J_\pm(l\pm 2)$. Each of the Hilbert subspaces $\mathscr{H}_m$ realizes the semi-infinite raising and lowering relations with respect to l as

$$J_+(l)\,Y_{l-1}^m(\theta,\phi) = \sqrt{\frac{2l-1}{2l+1}(l-m)(l+m)}Y_l^m(\theta,\phi) \qquad l\geq |m|+1, \tag{14a}$$

$$J_-(l)\,Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{2l-1}(l-m)(l+m)}Y_{l-1}^m(\theta,\phi) \qquad l\geq |m|. \tag{14b}$$

The lowest bases, that is,

$$Y_{\pm m}^m(\theta,\phi) = \frac{(-1)^{-m/2\mp m/2}}{2^{\pm m}\Gamma(1\pm m)}\sqrt{\frac{\Gamma(2\pm 2m)}{4\pi}}e^{im\phi}(\sin\theta)^{\pm m}, \tag{15}$$

belonging to the Hilbert subspaces $\mathscr{H}_m$ with $m \geq 0$ and $m\leq 0$, are, respectively, annihilated by $J_-(m)$ and $J_-(-m)$ as $J_-(m)Y_m^m(\theta,\phi) = 0$ and $J_-(-m)Y_{-m}^m(\theta,\phi) = 0$. Meanwhile, an arbitrary basis belonging to each of the Hilbert subspaces $\mathscr{H}_m$ with $m\geq 0$ and $m\leq 0$ can be calculated by the algebraic method:

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)\,\Gamma(1\pm 2m)}{(1\pm 2m)\,\Gamma(l-m+1)\,\Gamma(l+m+1)}}J_+(l) \cdot J_+(l-1)\cdots J_+(1\pm m)\,Y_{\pm m}^m(\theta,\phi). \tag{16}$$

Proof. The proof follows immediately from the raising and lowering relations of the index l of the associated Legendre functions [2]. ☐

According to the $-l \leq m \leq +l$ limitation obtained from the commutation relations of su(2), $2l+1$ must be an odd and even nonnegative integer for the orbital and spin angular momenta, respectively. Although the relation (13) is identically satisfied for any constant number l, however, it is represented only via the nonnegative integers l (odd positive integer values for $2l+1$) of spherical harmonics $Y_l^m(\theta,\phi)$. This is an essential difference with respect to the spin angular momentum. In fact, the relation (13) distinguishes the orbital angular momentum from the spin one. It also implies that the number of independent components of spherical harmonics of a given irreducible representation l of su(2) Lie algebra, that is, $2l+1$, is derived by the shift operators corresponding to the azimuthal quantum number l. If we take the adjoint of (13), we obtain $J_-(l-1)J_+(l+3) - J_+(l+2)J_-(l-2) = 2l+1$, which is identically satisfied. Thus, Proposition 2 presents a symmetry structure, called ladder symmetry with respect to the azimuthal quantum number l of spherical harmonics. Note that, indeed, the identical equality (13) has been originated from a brilliant theory in connection with geometry and physics named supersymmetry. In other words, although contrary to L_+ and L_- the two operators $J_+(l)$ and $J_-(l)$ do not contribute in a set of closed commutation relations, however, the operator identity (13) for them can be interpreted as a quantization relation in the framework of shape invariance symmetry (for reviews about supersymmetric quantum mechanics and shape invariance, see [12–17]). Thus, the operators $J_+(l)$ and $J_-(l)$ describe quantization of the azimuthal quantum number l which, in turn, lead to the presentation of a different algebraic technique from (8), in order to create the spherical harmonics $Y_l^m(\theta,\phi)$, according to (16). Furthermore, spherical harmonics belonging to the Hilbert subspaces $\mathscr{H}_l$ have parity $(-1)^l$, since $\mathbf{L}$ commutes with the parity operator. Thus, the operators $J_+(l)$ and $J_-(l)$ can be interpreted as the interchange operators of parity: $J_+(l) : \mathscr{H}_{l-1} \to \mathscr{H}_l$ and $J_-(l) : \mathscr{H}_l \to \mathscr{H}_{l-1}$.

4. Positive and Negative Integer Irreducible Representations of u(1, 1) for $l \mp m$

The laddering equations (6a) and (6b) as well as (14a) and (14b), which describe shifting the indices m and l separately, lead to the derivation of two new types of simultaneous ladder symmetries with respect to both azimuthal and magnetic quantum numbers of spherical harmonics. Our proposed ladder operators for simultaneous shift of l and m are of first-order differential type, contrary to [2]. They lead to a new perspective on the two quantum numbers l and m in connection with realization of $\mathrm{u}(1,1)$ (consequently, $\mathrm{su}(1,1)$) Lie algebra which in turn is accomplished by all spherical harmonics $Y_l^m(\theta,\phi)$ with constant values for $l-m$ and $l+m$, separately. First, it should be pointed out that the Hilbert space $\mathscr{H}$ can be split into the infinite direct sums of infinite dimensional Hilbert subspaces in two different ways as follows:

$$\mathscr{H} = \left(\bigoplus_{j=0}^{\infty} \mathscr{H}^{+}_{d=2j+1}\right) \oplus \left(\bigoplus_{k=1}^{\infty} \mathscr{H}^{+}_{d=2k}\right)$$
$$\text{with} \begin{cases} \mathscr{H}^{+}_{d=2j+1} = \operatorname{span}\left\{Y^m_{m+2j}(\theta,\phi)\right\}_{m\geq -j} \\ \mathscr{H}^{+}_{d=2k} = \operatorname{span}\left\{Y^m_{m+2k-1}(\theta,\phi)\right\}_{m\geq 1-k}, \end{cases} \tag{17a}$$

$$\mathscr{H} = \left(\bigoplus_{j=0}^{\infty} \mathscr{H}^{-}_{s=2j+1}\right) \oplus \left(\bigoplus_{k=1}^{\infty} \mathscr{H}^{-}_{s=2k}\right)$$
$$\text{with} \begin{cases} \mathscr{H}^{-}_{s=2j+1} = \operatorname{span}\left\{Y^m_{-m+2j}(\theta,\phi)\right\}_{m\leq j} \\ \mathscr{H}^{-}_{s=2k} = \operatorname{span}\left\{Y^m_{-m+2k-1}(\theta,\phi)\right\}_{m\leq k-1}. \end{cases} \tag{17b}$$

The constant values for the expressions $l-m$ and $l+m$ of spherical harmonics have been labeled by $d-1$ and $s-1$, respectively.

Proposition 3. *Let one define two new first-order differential operators on the sphere S^2:*

$$K^d_{\pm} = e^{\pm i\phi}\left(\pm\cos\theta\frac{\partial}{\partial\theta} + i\left(\frac{1}{\sin\theta}+\sin\theta\right)\frac{\partial}{\partial\phi} - \left(d-\frac{1}{2}\pm\frac{1}{2}\right)\sin\theta\right). \tag{18}$$

They, together with the generators $K_z = L_z = -i(\partial/\partial\phi)$ and 1, satisfy the commutation relations of $u(1,1)$ Lie algebra

$$\left[K^d_+, K^d_-\right] = -8K_z - 4d + 2,$$
$$\left[K_z, K^d_{\pm}\right] = \pm K^d_{\pm}. \tag{19}$$

$K^{d\pm 2}_{\pm}$ are the adjoint of the operators $K^d_{\mp}$ with respect to the inner product (3); that is, one has $K^{d\dagger}_{\mp} = K^{d\pm 2}_{\pm}$. Each of the Hilbert subspaces $\mathscr{H}^+_d$ realizes separately $(d-1)$-integer irreducible positive representations of $u(1,1)$ Lie algebra as (It must be pointed out that, by defining $S^d_z \coloneqq L_z + d/2 - 1/4$ and $S^d_{\pm} \coloneqq K^d_{\pm}/2$, the $u(1,1)$ Lie algebra (19) can be considered as commutation relations corresponding to the $su(1,1)$ Lie algebra: $[S^d_+, S^d_-] = -2S^d_z$ and $[S^d_z, S^d_{\pm}] = \pm S^d_{\pm}$. This means that 1 is a trivial center for the semisimple Lie algebra $u(1,1)$. In [18], a short review on the three different real forms h_4, $u(2)$, and $u(1,1)$ of $gl(2,c)$ Lie algebra has been presented. There, their differences in connection with the structure constants and their representation spaces have also been pointed out.)

$$K^d_+ Y^{m-1}_{m+d-2}(\theta,\phi) = \sqrt{\frac{2m+2d-3}{2m+2d-1}(2m+d-2)(2m+d-1)}\, Y^m_{m+d-1}(\theta,\phi), \tag{20a}$$

$$K^d_- Y^{m}_{m+d-1}(\theta,\phi) = \sqrt{\frac{2m+2d-1}{2m+2d-3}(2m+d-2)(2m+d-1)}\, Y^{m-1}_{m+d-2}(\theta,\phi), \tag{20b}$$

$$K_z Y^m_{m+d-1}(\theta,\phi) = m Y^m_{m+d-1}(\theta,\phi). \tag{20c}$$

Also, the Casimir operator corresponding to the generators K^d_+, K^d_-, and K_z,

$$\mathbf{K}^{d^2}_{u(1,1)} = K^d_+ K^d_- - 4K_z^2 - 2(2d-3)K_z, \tag{21}$$

has an infinite-fold degeneracy on the Hilbert subspace $\mathscr{H}^+_d$ as

$$\mathbf{K}^{d^2}_{u(1,1)} Y^m_{m+d-1}(\theta,\phi) = (d-1)(d-2)\, Y^m_{m+d-1}(\theta,\phi). \tag{22}$$

The Hilbert subspaces $\mathscr{H}^+_{d=2j+1} = \mathscr{D}^+(-j)$ and $\mathscr{H}^+_{d=2k} = \mathscr{D}^+(1-k)$ with j and k as nonnegative and positive integers contain, respectively, the following lowest bases:

$$Y^{-j}_j(\theta,\phi) = \frac{1}{2^j\Gamma(j+1)}\sqrt{\frac{\Gamma(2j+2)}{4\pi}}\, e^{-ij\phi}(\sin\theta)^j, \tag{23a}$$

$$Y^{1-k}_k(\theta,\phi) = \frac{1}{2^{k+1/2}\Gamma(k+1)} \cdot \sqrt{\frac{k\Gamma(2k+2)}{\pi}}\, e^{i(1-k)\phi}(\sin\theta)^{k-1}\cos\theta. \tag{23b}$$

They are annihilated as $K^{2j+1}_- Y^{-j}_j(\theta,\phi) = 0$ and $K^{2k}_- Y^{1-k}_k(\theta,\phi) = 0$ and also have the lowest weights $-j$ and $1-k$. Meanwhile, the arbitrary bases of the Hilbert subspaces

$\mathscr{H}^{+}_{d=2j+1}$ and $\mathscr{H}^{+}_{d=2k}$ can be, respectively, calculated by the algebraic methods as

$$Y^{m}_{m+2j}(\theta,\phi) = \frac{\left(K^{2j+1}_{+}\right)^{m+j} Y^{-j}_{j}(\theta,\phi)}{\sqrt{(2j+1)\Gamma(2m+2j+1)/(2m+4j+1)}} \quad m \geq -j, \tag{24a}$$

$$Y^{m}_{m+2k-1}(\theta,\phi) = \frac{\left(K^{2k}_{+}\right)^{m+k-1} Y^{1-k}_{k}(\theta,\phi)}{\sqrt{(2k+1)\Gamma(2m+2k)/(2m+4k-1)}} \quad m \geq 1-k. \tag{24b}$$

Proof. The relations (18), (20a), and (20b) can be followed from the realization of laddering relations with respect to both azimuthal and magnetic quantum numbers l and m, simultaneously and agreeably. It is sufficient to consider that two new differential operators

$$\begin{aligned} A_{\pm,\pm}(l) &:= \pm\left[L_{\pm}, J_{\pm}(l)\right] \\ &= e^{\pm i\phi}\left(\pm\cos\theta\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} - l\sin\theta\right) \end{aligned} \tag{25}$$

satisfy the simultaneous laddering relations with respect to l and m as

$$A_{+,+}(l)\,Y^{m-1}_{l-1}(\theta,\phi) = \sqrt{\frac{2l-1}{2l+1}(l+m-1)(l+m)}Y^{m}_{l}(\theta,\phi), \tag{26a}$$

$$A_{-,-}(l)\,Y^{m}_{l}(\theta,\phi) = \sqrt{\frac{2l+1}{2l-1}(l+m-1)(l+m)}Y^{m-1}_{l-1}(\theta,\phi). \tag{26b}$$

The relations (26a) and (26b) are obtained from (6a), (6b), (14a), and (14b). The relations (19) and (20c) are directly followed. The adjoint relation between the operators can be easily checked by means of the inner product (3). The commutativity of operators K^{d}_{+}, K^{d}_{-}, and K_z with $\mathbf{K}^{d^2}_{\mathrm{u}(1,1)}$ results from (19). The eigenequation (22) follows immediately from the representation relations (20a)–(20c). The relation (20b) implies that $Y^{-j}_{j}(\theta,\phi)$ and $Y^{1-k}_{k}(\theta,\phi)$ are the lowest bases for the Hilbert subspaces $\mathscr{H}^{+}_{2j+1}$ and $\mathscr{H}^{+}_{2k}$, respectively. Then, with repeated application of the raising relation (20a), one may obtain the arbitrary representation bases of u(1, 1) Lie algebra as (24a) and (24b). □

Although the commutation relations (19) are not closed with respect to taking the adjoint, however, their adjoint relations $[K^{d+2}_{+}, K^{d-2}_{-}] = -8K_z - 4d + 2$ and $[K_z, K^{d\mp2}_{\mp}] = \mp K^{d\mp2}_{\mp}$ are identically satisfied.

Proposition 4. *Let one define two new first-order differential operators on the sphere S^2 as*

$$\begin{aligned} I^{s}_{\pm} = e^{\pm i\phi}&\left(\pm\cos\theta\frac{\partial}{\partial\theta} + i\left(\frac{1}{\sin\theta} + \sin\theta\right)\frac{\partial}{\partial\phi}\right. \\ &\left. + \left(s - \frac{1}{2} \mp \frac{1}{2}\right)\sin\theta\right). \end{aligned} \tag{27}$$

They, together with the generators $I_z = L_z = -i(\partial/\partial\phi)$ and 1, satisfy the commutation relations of $u(1,1)$ Lie algebra as

$$\begin{aligned} \left[I^{s}_{+}, I^{s}_{-}\right] &= -8I_z + 4s - 2, \\ \left[I_z, I^{s}_{\pm}\right] &= \pm I^{s}_{\pm}. \end{aligned} \tag{28}$$

$I^{s\mp2}_{\pm}$ are the adjoint of the operators $I^{s}_{\mp}$ with respect to the inner product (3); that is, one has $I^{s\,\dagger}_{\mp} = I^{s\mp2}_{\pm}$. Each of the Hilbert subspaces $\mathscr{H}^{-}_{s}$ realizes separately $(s-1)$-integer irreducible positive representations of $u(1,1)$ Lie algebra as

$$I^{s}_{+}Y^{m-1}_{-m+s}(\theta,\phi) = \sqrt{\frac{-2m+2s+1}{-2m+2s-1}(-2m+s)(-2m+s+1)}Y^{m}_{-m+s-1}(\theta,\phi), \tag{29a}$$

$$I^{s}_{-}Y^{m}_{-m+s-1}(\theta,\phi) = \sqrt{\frac{-2m+2s-1}{-2m+2s+1}(-2m+s)(-2m+s+1)}Y^{m-1}_{-m+s}(\theta,\phi), \tag{29b}$$

$$I_zY^{m}_{-m+s-1}(\theta,\phi) = mY^{m}_{-m+s-1}(\theta,\phi). \tag{29c}$$

Also, the Casimir operator corresponding to the generators I^{s}_{+}, I^{s}_{-}, and I_z,

$$\mathbf{I}^{s2}_{u(1,1)} = I^{s}_{+}I^{s}_{-} - 4I^{2}_{z} + 2(2s+1)I_z, \tag{30}$$

has an infinite-fold degeneracy on the Hilbert subspace $\mathscr{H}^{-}_{s}$ as

$$\mathbf{I}^{s2}_{u(1,1)}Y^{m}_{-m+s-1}(\theta,\phi) = s(s+1)Y^{m}_{-m+s-1}(\theta,\phi). \tag{31}$$

The Hilbert subspaces $\mathscr{H}^{-}_{s=2j+1} = \mathscr{D}^{-}(j)$ and $\mathscr{H}^{-}_{s=2k} = \mathscr{D}^{-}(k-1)$ with j and k as nonnegative and positive integers contain, respectively, the following highest bases:

$$Y^{j}_{j}(\theta,\phi) = \frac{(-1)^j}{2^j\Gamma(j+1)}\sqrt{\frac{\Gamma(2j+2)}{4\pi}}e^{ij\phi}(\sin\theta)^j, \tag{32a}$$

$$\begin{aligned} Y^{k-1}_{k}(\theta,\phi) &= \frac{(-1)^{k-1}}{2^{k-1/2}\Gamma(k+1)} \\ &\cdot\sqrt{\frac{(2k+1)\Gamma(2k)}{2\pi}}e^{i(k-1)\phi}(\sin\theta)^{k-1}\cos\theta. \end{aligned} \tag{32b}$$

They are annihilated as $I^{2j+1}_{+}Y^{j}_{j}(\theta,\phi) = 0$ and $I^{2k}_{+}Y^{k-1}_{k}(\theta,\phi) = 0$ and also have the highest weights j and $k-1$. Meanwhile, *the arbitrary bases of the Hilbert subspaces $\mathscr{H}^-_{s=2j+1}$ and $\mathscr{H}^-_{s=2k}$ can be, respectively, calculated by the algebraic methods as*

$$Y^m_{2j-m}(\theta,\phi) = \frac{\left(I^{2j+1}_-\right)^{j-m} Y^j_j(\theta,\phi)}{\sqrt{(2j+1)\,\Gamma(2j-2m+1)/(4j-2m+1)}} \quad m \le j, \tag{33a}$$

$$Y^m_{2k-m-1}(\theta,\phi) = \frac{\left(I^{2k}_-\right)^{k-m-1} Y^{k-1}_k(\theta,\phi)}{\sqrt{(2k+1)\,\Gamma(2k-2m)/(4k-2m-1)}} \quad m \le k-1. \tag{33b}$$

Proof. The proof is quite similar to the proof of Proposition 3. So, we have to take into account that the two new differential operators

$$A_{\mp,\pm}(l) \coloneqq \mp\left[L_\pm, J_\mp(l)\right] = e^{\pm i\phi}\left(\pm\cos\theta\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi} + l\sin\theta\right) \tag{34}$$

are represented by spherical harmonics whose corresponding laddering equations shift both the azimuthal and magnetic quantum numbers l and m simultaneously and inversely:

$$A_{-,+}(l+1)\,Y^{m-1}_{l+1}(\theta,\phi) = \sqrt{\frac{2l+3}{2l+1}(l-m+1)(l-m+2)}Y^m_l(\theta,\phi), \tag{35a}$$

$$A_{+,-}(l+1)\,Y^m_l(\theta,\phi) = \sqrt{\frac{2l+1}{2l+3}(l-m+1)(l-m+2)}Y^{m-1}_{l+1}(\theta,\phi). \tag{35b}$$

□

Here, again the adjoint of commutation relations (28) becomes $[I^{s-2}_+, I^{s+2}_-] = -8I_z + 4s - 2$ and $[I_z, I^{s\pm2}_\mp] = \mp I^{s\pm2}_\mp$, which are identically satisfied.

Thus, all unitary and irreducible representations of su(2) of dimensions $2l+1$ with the nonnegative integers l can carry the new kind of irreducible representations for u(1, 1). The new symmetry structures presented in the two recent propositions, the so-called positive and negative discrete representations of u(1, 1), in turn, describe the simultaneous quantization of the azimuthal and magnetic quantum numbers. Therefore, the Hilbert spaces of all spherical harmonics not only represent compact Lie algebra su(2) by ladder operators shifting m for a given l, but also represent the noncompact Lie algebra u(1, 1) by simultaneous shift operators of both quantum labels l and m for given values $l - m$ and $l + m$.

5. Concluding Remarks

For a given azimuthal quantum number l, quantization of the magnetic number m is customarily accomplished by representing the operators L_+, L_-, and L_3 on the sphere with the commutation relations su(2) compact Lie algebra, in a $(2l+1)$-dimensional Hilbert subspace $\mathscr{H}_l$. Furthermore, for a given magnetic quantum number m, quantization of the azimuthal number l is accomplished by representing the operators $J_+(l)$ and $J_-(l)$ on the sphere S^2 with the identity relation (13), in an infinite-dimensional Hilbert subspace $\mathscr{H}_m$.

Dealing with these issues together, simultaneous quantization of both azimuthal and magnetic numbers l and m is accomplished by representing two bunches of operators $\{K^d_+, K^d_-, K_3, 1\}$ and $\{I^s_+, I^s_-, I_3, 1\}$ on the sphere with their corresponding commutation relations of u(1, 1) noncompact Lie algebra, in the infinite-dimensional Hilbert subspaces $\mathscr{H}^+_d$ and $\mathscr{H}^-_s$, respectively. For given values $d = l - m + 1$ and $s = l + m + 1$, they are independent of each other, the so-called positive and negative $(l-m)$- and $(l+m)$-integer irreducible representations, respectively. As the spherical harmonics are generated from $Y^{\mp l}_l(\theta,\phi)$ by the operators $L_\pm$, they are also generated from $Y^{-j}_j(\theta,\phi)$ and $Y^{1-k}_k(\theta,\phi)$ by K^{2j+1}_+ and K^{2k}_+, as well as from $Y^j_j(\theta,\phi)$ and $Y^{k-1}_k(\theta,\phi)$ by I^{2j+1}_- and I^{2k}_-, respectively. Therefore, not only $Y^m_l(\theta,\phi)$'s with the given value for l represent su(2) Lie algebra, but also $Y^m_l(\theta,\phi)$'s with the given values for subtraction and summation of the both quantum numbers l and m represent separately u(1, 1) (hence, su(1, 1)) Lie algebra as well. In other words, two different real forms of sl(2, c) Lie algebra, that is, su(2) and su(1, 1), are represented by the space of all spherical harmonics $Y^m_l(\theta,\phi)$. This happens because the quantization of both quantum numbers l and m are considered jointly. Indeed, we have

$$\text{Propositions 1 and 2} \Longleftrightarrow \text{Propositions 3 and 4}. \tag{36}$$

We point out that the idea of this paper may find interesting applications in quantum devices. For instance, coherent states of the SU(1, 1) noncompact Lie group have been defined by Barut and Girardello as eigenstates of the ladder operators [19], and by Perelomov as the action of the displacement operator on the lowest and highest bases [20, 21]. So, our approach to the representation of su(1, 1) noncompact Lie algebra provides the possibility of constructing two different types of coherent states of su(1, 1) on compact manifold S^2 [22]. Also, realization of the additional symmetry named su(2) Lie algebra for Landau levels and bound states of a free particle on noncompact manifold AdS_2 can be found based on the above considerations.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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