Hakl and Zamora *Boundary Value Problems* 2014, 2014:113
http://www.boundaryvalueproblems.com/content/2014/1/113

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Fredholm-type theorem for boundary value problems for systems of nonlinear functional differential equations

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Dedicated to Professor Ivan Kiguradze.

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Abstract
A Fredholm-type theorem for boundary value problems for systems of nonlinear functional differential equations is established. The theorem generalizes results known for the systems with linear or homogeneous operators to the case of systems with positively homogeneous operators.
MSC: 34K10

Keywords: functional-differential equations; boundary value problems; existence of solutions

1 Statement of the problem

Consider the system of functional-differential equations

$$u_i'(t) = p_i(u_1,\dots,u_n)(t) + f_i(u_1,\dots,u_n)(t) \quad \text{for a.e. } t \in [a,b] \ (i = 1,\dots,n) \tag{1}$$

together with the boundary conditions

$$\ell_i(u_1,\dots,u_n) = h_i(u_1,\dots,u_n) \quad (i = 1,\dots,n). \tag{2}$$

Here, $p_i, f_i : C([a,b];\mathbb{R}^n) \to L([a,b];\mathbb{R})$ are continuous operators satisfying Carathéodory conditions, *i.e.* for every $r > 0$ there exists $q_r \in L([a,b];\mathbb{R}_+)$ such that

$$\sum_{i=1}^n \big(\big|p_i(u_1,\dots,u_n)(t)\big| + \big|f_i(u_1,\dots,u_n)(t)\big|\big) \le q_r(t) \quad \text{for a.e. } t \in [a,b], \sum_{i=1}^n \|u_i\|_C \le r,$$

and $\ell_i, h_i : C([a,b];\mathbb{R}^n) \to \mathbb{R}$ are continuous functionals which are bounded on every ball by a constant, *i.e.* for every $r > 0$ there exists $M_r > 0$ such that

$$\sum_{i=1}^n \big(\big|\ell_i(u_1,\dots,u_n)\big| + \big|h_i(u_1,\dots,u_n)\big|\big) \le M_r \quad \text{whenever } \sum_{i=1}^n \|u_i\|_C \le r.$$

Furthermore, we assume that p_i and ℓ_i satisfy the following condition: there exist positive real numbers λ_{ij} and μ_i such that $\lambda_{ij}\lambda_{jm} = \lambda_{im}$ whenever $i,j,m \in \{1,\dots,n\}$, and for every

$c > 0$ and $(u_k)_{k=1}^n \in C([a,b];\mathbb{R}^n)$ we have

$$cp_i(u_1,\dots,u_n)(t) = p_i\big(c^{\lambda_{i1}}u_1,\dots,c^{\lambda_{in}}u_n\big)(t) \quad \text{for a.e. } t \in [a,b], \tag{3}$$

$$c^{\mu_i}\ell_i(u_1,\dots,u_n) = \ell_i\big(c^{\lambda_{i1}}u_1,\dots,c^{\lambda_{in}}u_n\big). \tag{4}$$

Remark 1 From the above-stated assumptions it follows that $\lambda_{ii} = 1$, $\lambda_{ij} = 1/\lambda_{ji}$ for every $i,j \in \{1,\dots,n\}$.

In the case when p_i and ℓ_i are linear bounded operators and $f_i(\cdot,\dots,\cdot)(t) \equiv q_i(t)$, $h_i(\cdot,\dots,\cdot) \equiv c_i$, the relationship between the existence of a solution to problem (1), (2) and the existence of only the trivial solution to its corresponding homogeneous problem, so-called Fredholm alternative, is well known; for more details see *e.g.* [1–8] and references therein.

In 1966, Lasota established the Fredholm-type theorem in the case when p_i and ℓ_i are homogeneous operators (see [9]). Recently, Fredholm-type theorems in the case when p_i and ℓ_i are positively homogeneous operators were established by Kiguradze, Půža, Stavroulakis in [10] and also by Kiguradze, Šremr in [11].

In this paper we unify the ideas used in [11] and [9] to obtain a new Fredholm-type theorem for the case when p_i and ℓ_i are positively homogeneous operators. The consequences of the obtained result for particular cases of problem (1), (2) are formulated at the end of the paper.

The following notation is used throughout the paper.

$\mathbb{N}$ is the set of all natural numbers;

$\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0,+\infty)$;

$\mathbb{R}^n$ is the linear space of vectors $x = (x_i)_{i=1}^n$ with the elements $x_i \in \mathbb{R}$ endowed with the norm

$$\|x\| = \sum_{i=1}^n |x_i|;$$

$C([a,b];\mathbb{R}^n)$ is the Banach space of continuous vector-valued functions $u = (u_i)_{i=1}^n : [a,b] \to \mathbb{R}^n$ with the norm

$$\|u\|_C = \sum_{i=1}^n \max\big\{\big|u_i(t)\big| : t \in [a,b]\big\};$$

$AC([a,b];\mathbb{R}^n)$ is the set of absolutely continuous vector-valued functions $u : [a,b] \to \mathbb{R}^n$;

$L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to \mathbb{R}$ with the norm

$$\|p\|_L = \int_a^b \big|p(s)\big|\,ds;$$

$L([a,b];\mathbb{R}_+) = \{p \in L([a,b];\mathbb{R}) : p(t) \geq 0 \text{ for a.e. } t \in [a,b]\}$;

if Ω is a set then $\operatorname{meas}\Omega$, $\operatorname{int}\Omega$, $\overline{\Omega}$, and $\partial\Omega$ denotes the measure, interior, closure, and boundary of the set Ω, respectively.

By a solution to (1), (2) we understand a function $(u_i)_{i=1}^n \in AC([a,b];\mathbb{R}^n)$ satisfying (1) almost everywhere in $[a,b]$ and (2).

Notation 1 Define, for every $i \in \{1,\dots,n\}$, the following functions:

$$q_i(t,\rho) \stackrel{\text{def}}{=} \sup\{|f_i(u_1,\dots,u_n)(t)| : \|u_k\|_C \le \rho^{\lambda_{ik}}, k = 1,\dots,n\} \quad \text{for a.e. } t \in [a,b],$$
$$\eta_i(\rho) \stackrel{\text{def}}{=} \sup\{|h_i(u_1,\dots,u_n)| : \|u_k\|_C \le \rho^{\frac{\lambda_{ik}}{\mu_i}}, k = 1,\dots,n\}.$$

2 Main result

Theorem 1 *Let*

$$\lim_{\rho\to+\infty} \int_a^b \frac{q_i(s,\rho)}{\rho}\,ds = 0, \qquad \lim_{\rho\to+\infty} \frac{\eta_i(\rho)}{\rho} = 0 \quad (i = 1,\dots,n). \tag{5}$$

If the problem

$$u_i'(t) = (1-\delta)p_i(u_1,\dots,u_n)(t) - \delta p_i(-u_1,\dots,-u_n)(t) \quad \text{for a.e. } t \in [a,b]\ (i = 1,\dots,n), \tag{6}$$

$$(1-\delta)\ell_i(u_1,\dots,u_n) - \delta\ell_i(-u_1,\dots,-u_n) = 0 \quad (i = 1,\dots,n) \tag{7}$$

has only the trivial solution for every $\delta \in [0,1/2]$, then problem (1), (2) *has at least one solution.*

The proof of Theorem 1 is based on the following result by Krasnosel'skii (see [12, Theorem 41.3, p.325]). We will formulate it in a form suitable for us.

Theorem 2 *Let X be a Banach space, $\Omega \subseteq X$ be a symmetric*[a] *bounded domain with $0 \in \operatorname{int}\Omega$. Let, moreover, $A : \overline{\Omega} \to \overline{\Omega}$ be a compact*[b] *continuous operator which has no fixed point on $\partial\Omega$. If, in addition,*

$$\frac{A(x)-x}{\|A(x)-x\|} \ne \frac{A(-x)+x}{\|A(-x)+x\|} \quad \text{for } x \in \partial\Omega$$

then A has a fixed point in Ω, i.e. there exists $x_0 \in \Omega$ such that $x_0 = A(x_0)$.

Furthermore, to prove Theorem 1 we will need the following lemma.

Lemma 1 *Let, for every $\delta \in [0,1/2]$, problem* (6), (7) *has only the trivial solution. Then there exists $r > 0$ such that for any $(u_i)_{i=1}^n \in AC([a,b];\mathbb{R}^n)$ and any $\delta \in [0,1/2]$, the a priori estimate*

$$\sum_{k=1}^n \|u_k\|_C^{\lambda_{k1}} \le r \sum_{i=1}^n \left(\|\widetilde{f}_i\|_L^{\lambda_{i1}} + |\widetilde{h}_i|^{\frac{\lambda_{i1}}{\mu_i}}\right) \tag{8}$$

holds, where

$$\widetilde{f}_i(t) \stackrel{\text{def}}{=} u_i'(t) - (1-\delta)p_i(u_1,\dots,u_n)(t) + \delta p_i(-u_1,\dots,-u_n)(t)$$
$$\text{for a.e. } t \in [a,b] \ (i = 1,\dots,n),$$
$$\widetilde{h}_i \stackrel{\text{def}}{=} (1-\delta)\ell_i(u_1,\dots,u_n) - \delta\ell_i(-u_1,\dots,-u_n) \quad (i = 1,\dots,n).$$

Proof Suppose on the contrary that for every $m \in \mathbb{N}$ there exist $(u_{im})_{i=1}^n \in AC([a,b];\mathbb{R}^n)$ and $\delta_m \in [0,1/2]$ such that

$$\sum_{k=1}^n \|u_{km}\|_C^{\lambda_{k1}} > m\sum_{i=1}^n \bigl(\|\widetilde{f}_{im}\|_L^{\lambda_{i1}} + |\widetilde{h}_{im}|^{\frac{\lambda_{i1}}{\mu_i}}\bigr), \tag{9}$$

where

$$\widetilde{f}_{im}(t) \stackrel{\text{def}}{=} u_{im}'(t) - (1-\delta_m)p_i(u_{1m},\dots,u_{nm})(t) + \delta_m p_i(-u_{1m},\dots,-u_{nm})(t)$$
$$\text{for a.e. } t \in [a,b] \ (i = 1,\dots,n), \tag{10}$$
$$\widetilde{h}_{im} \stackrel{\text{def}}{=} (1-\delta_m)\ell_i(u_{1m},\dots,u_{nm}) - \delta_m\ell_i(-u_{1m},\dots,-u_{nm}) \quad (i = 1,\dots,n). \tag{11}$$

Put

$$\rho_m = \sum_{k=1}^n \|u_{km}\|_C^{\lambda_{k1}} \quad \text{for } m \in \mathbb{N}, \tag{12}$$
$$v_{im}(t) = \frac{u_{im}(t)}{\rho_m^{\lambda_{1i}}} \quad \text{for } t \in [a,b], m \in \mathbb{N}. \tag{13}$$

Then

$$\sum_{i=1}^n \|v_{im}\|_C^{\lambda_{i1}} = 1 \quad \text{for } m \in \mathbb{N} \tag{14}$$

and from (10) and (11), in view of (3), (4), (12), and (13), we get

$$\frac{\widetilde{f}_{im}(t)}{\rho_m^{\lambda_{1i}}} = v_{im}'(t) - (1-\delta_m)p_i(v_{1m},\dots,v_{nm})(t) + \delta_m p_i(-v_{1m},\dots,-v_{nm})(t)$$
$$\text{for a.e. } t \in [a,b] \ (i = 1,\dots,n; m \in \mathbb{N}), \tag{15}$$
$$\frac{\widetilde{h}_{im}}{\rho_m^{\lambda_{1i}\mu_i}} = (1-\delta_m)\ell_i(v_{1m},\dots,v_{nm}) - \delta_m\ell_i(-v_{1m},\dots,-v_{nm}) \quad (i = 1,\dots,n; m \in \mathbb{N}). \tag{16}$$

On the other hand, from (9) and (12) we have

$$\sum_{i=1}^n \left(\left\|\frac{\widetilde{f}_{im}}{\rho_m^{\lambda_{1i}}}\right\|_L^{\lambda_{i1}} + \left|\frac{\widetilde{h}_{im}}{\rho_m^{\lambda_{1i}\mu_i}}\right|^{\frac{\lambda_{i1}}{\mu_i}}\right) < \frac{1}{m} \quad \text{for } m \in \mathbb{N}, \tag{17}$$

whence, according to [13, Corollary IV.8.11] it follows that

$$\lim_{\operatorname{meas} E\to 0}\int_E \frac{\widetilde{f}_{im}(s)}{\rho_m^{\lambda_{1i}}}\,ds = 0 \quad \text{uniformly for } m\in\mathbb{N}\ (i=1,\dots,n). \tag{18}$$

Therefore, (14), (15), and (18) imply that the sequences $\{v_{im}\}_{m=1}^{+\infty}$ $(i=1,\dots,n)$ are uniformly bounded and equicontinuous. Thus, according to Arzelà-Ascoli theorem, without loss of generality we can assume that there exist $(v_{i0})_{i=1}^n \in C([a,b];\mathbb{R}^n)$ and $\delta_0\in[0,1/2]$ such that

$$\lim_{m\to+\infty}\delta_m=\delta_0,\qquad \lim_{m\to+\infty}\|v_{im}-v_{i0}\|_C=0 \quad (i=1,\dots,n). \tag{19}$$

Furthermore, (15)-(17) yield $(v_{i0})_{i=1}^n\in AC([a,b];\mathbb{R}^n)$ and show that it is a solution to (6), (7). However, (14) and (19) result in

$$\sum_{i=1}^n \|v_{i0}\|_C^{\lambda_{i1}}=1,$$

which contradicts our assumptions. □

Proof of Theorem 1 Let $X=C([a,b];\mathbb{R}^n)\times\mathbb{R}^n$ and for $x\in X$, *i.e.* $x=(u,\alpha)=((u_i)_{i=1}^n,(\alpha_i)_{i=1}^n)$, define the norm

$$\|x\|=\|u\|_C+\|\alpha\|.$$

Then $(X,\|\cdot\|)$ is a Banach space. Let the operators $T,F,A:X\to X$ be defined as follows:

$$T(x)\stackrel{\text{def}}{=}\left(\left(u_i(a)+\alpha_i+\int_a^t p_i(u_1,\dots,u_n)(s)\,ds\right)_{i=1}^n,\left(\alpha_i+\ell_i(u_1,\dots,u_n)\right)_{i=1}^n\right), \tag{20}$$

$$F(x)\stackrel{\text{def}}{=}\left(\left(\int_a^t f_i(u_1,\dots,u_n)(s)\,ds\right)_{i=1}^n,\left(-h_i(u_1,\dots,u_n)\right)_{i=1}^n\right), \tag{21}$$

$$A(x)\stackrel{\text{def}}{=}T(x)+F(x), \tag{22}$$

and consider the operator equation

$$x=A(x). \tag{23}$$

It can easily be seen that problem (1), (2), and (23) are equivalent in the following sense: if $x=(u,\alpha)$ is a solution to (23), then $\alpha_i=0$ $(i=1,\dots,n)$ and $(u_i)_{i=1}^n$ is a solution to (1), (2); and *vice versa* if $(u_i)_{i=1}^n$ is a solution to (1), (2), then $x=(u,0)$ is a solution to (23).

Let $r>0$ be such that the conclusion of Lemma 1 is valid. According to (5) we can choose $\rho_0>0$ such that

$$\frac{1}{\rho_0}\sum_{i=1}^n\left(\left\|q_i\left(\cdot,\rho_0^{\lambda_{1i}}\right)\right\|_L^{\lambda_{i1}}+\left|\eta_i\left(\rho_0^{\lambda_{1i}\mu_i}\right)\right|^{\frac{\lambda_{i1}}{\mu_i}}\right)<\frac{1}{r}. \tag{24}$$

Let, moreover,

$$\Omega=\left\{x\in X:\sum_{k=1}^n\left(\|u_k\|_C^{\lambda_{k1}}+|\alpha_k|\right)<\rho_0\right\}. \tag{25}$$

Now we will show that the operator A has a fixed point in Ω. According to Theorem 2 it is sufficient to show that

$$A(x) - x \neq \nu\big(A(-x) + x\big) \quad \text{for } x \in \partial\Omega, \nu \in (0,1].$$

Assume on the contrary that there exist $x_0 = ((u_{i0})_{i=1}^n, (\alpha_{i0})_{i=1}^n) \in \partial\Omega$ and $\nu_0 \in (0,1]$ such that

$$A(x_0) - x_0 = \nu_0\big(A(-x_0) + x_0\big). \tag{26}$$

Then from (26), in view of (20)-(22) we obtain

$$x_0 = (1-\delta_0)T(x_0) - \delta_0 T(-x_0) + (1-\delta_0)F(x_0) - \delta_0 F(-x_0),$$

where $\delta_0 = \nu_0/(1+\nu_0) \in (0,1/2]$, *i.e.*

$$\begin{aligned} u_{i0}(t) = u_{i0}(a) + \alpha_{i0} + (1-\delta_0)\int_a^t p_i(u_{10},\dots,u_{n0})(s)\,ds \\ - \delta_0\int_a^t p_i(-u_{10},\dots,-u_{n0})(s)\,ds + (1-\delta_0)\int_a^t f_i(u_{10},\dots,u_{n0})(s)\,ds \\ - \delta_0\int_a^t f_i(-u_{10},\dots,-u_{n0})(s)\,ds \quad \text{for } t\in[a,b]\ (i=1,\dots,n), \end{aligned} \tag{27}$$

$$\begin{aligned} \alpha_{i0} = \alpha_{i0} + (1-\delta_0)\ell_i(u_{10},\dots,u_{n0}) - \delta_0\ell_i(-u_{10},\dots,-u_{n0}) \\ - (1-\delta_0)h_i(u_{10},\dots,u_{n0}) + \delta_0 h_i(-u_{10},\dots,-u_{n0}) \quad (i=1,\dots,n). \end{aligned} \tag{28}$$

Now from (27) and (28) it follows that $(u_i)_{i=1}^n \in AC([a,b];\mathbb{R}^n)$,

$$\alpha_{i0} = 0 \quad (i=1,\dots,n), \tag{29}$$

$$\begin{aligned} u_{i0}'(t) = (1-\delta_0)p_i(u_{10},\dots,u_{n0})(t) - \delta_0 p_i(-u_{10},\dots,-u_{n0})(t) \\ + (1-\delta_0)f_i(u_{10},\dots,u_{n0})(t) - \delta_0 f_i(-u_{10},\dots,-u_{n0})(t) \\ \text{for a.e. } t\in[a,b]\ (i=1,\dots,n), \end{aligned} \tag{30}$$

$$\begin{aligned} (1-\delta_0)\ell_i(u_{10},\dots,u_{n0}) - \delta_0\ell_i(-u_{10},\dots,-u_{n0}) \\ = (1-\delta_0)h_i(u_{10},\dots,u_{n0}) - \delta_0 h_i(-u_{10},\dots,-u_{n0}) \quad (i=1,\dots,n). \end{aligned} \tag{31}$$

Moreover, since $x_0 \in \partial\Omega$, on account of (25) and (29) we have

$$\rho_0 = \sum_{k=1}^n \|u_{k0}\|_C^{\lambda_{k1}}. \tag{32}$$

Now the equality (32), according to Notation 1, implies

$$\begin{aligned} \big|(1-\delta_0)f_i(u_{10},\dots,u_{n0})(t) - \delta_0 f_i(-u_{10},\dots,-u_{n0})(t)\big| \le q_i\big(t,\rho_0^{\lambda_{1i}}\big) \\ \text{for a.e. } t\in[a,b]\ (i=1,\dots,n), \end{aligned} \tag{33}$$

$$\big|(1-\delta_0)h_i(u_{10},\dots,u_{n0}) - \delta_0 h_i(-u_{10},\dots,-u_{n0})\big| \le \eta_i\big(\rho_0^{\lambda_{1i}\mu_i}\big) \quad (i=1,\dots,n). \tag{34}$$

Therefore, in view of Lemma 1, with respect to (30)-(34) we obtain

$$\rho_0 \le r \sum_{i=1}^{n} \left(\left\| q_i\left(\cdot, \rho_0^{\lambda_{1i}}\right) \right\|_L^{\lambda_{i1}} + \left| \eta_i\left(\rho_0^{\lambda_{1i}\mu_i}\right) \right|^{\frac{\lambda_{i1}}{\mu_i}} \right).$$

However, the latter inequality contradicts (24). □

3 Corollaries

If the operators p_i and ℓ_i are homogeneous, *i.e.* if moreover

$$p_i(-u_1, \dots, -u_n)(t) = -p_i(u_1, \dots, u_n)(t)$$
$$\text{for a.e. } t \in [a,b], (u_k)_{k=1}^n \in C\big([a,b]; \mathbb{R}^n\big) \ (i = 1, \dots, n), \tag{35}$$

$$\ell_i(-u_1, \dots, -u_n) = -\ell_i(u_1, \dots, u_n), \quad (u_k)_{k=1}^n \in C\big([a,b]; \mathbb{R}^n\big) \ (i = 1, \dots, n), \tag{36}$$

then from Theorem 1 we obtain the following assertion.

Corollary 1 *Let* (5), (35), *and* (36) *be fulfilled. If the problem*

$$u_i'(t) = p_i(u_1, \dots, u_n)(t) \quad \text{for a.e. } t \in [a,b] \ (i = 1, \dots, n), \tag{37}$$

$$\ell_i(u_1, \dots, u_n) = 0 \quad (i = 1, \dots, n) \tag{38}$$

has only the trivial solution then problem (1), (2) *has at least one solution.*

For a particular case when p_i are defined by

$$p_i(u_1, \dots, u_n)(t) \stackrel{\text{def}}{=} \widetilde{p}_i(t) \big| u_{i+1}\big(\tau_i(t)\big) \big|^{\lambda_i} \operatorname{sgn} u_{i+1}\big(\tau_i(t)\big)$$
$$\text{for a.e. } t \in [a,b] \ (i = 1, \dots, n-1), \tag{39}$$

$$p_n(u_1, \dots, u_n)(t) \stackrel{\text{def}}{=} \widetilde{p}_n(t) \big| u_1\big(\tau_n(t)\big) \big|^{\lambda_n} \operatorname{sgn} u_1\big(\tau_n(t)\big) \quad \text{for a.e. } t \in [a,b], \tag{40}$$

where $\widetilde{p}_i \in L([a,b]; \mathbb{R})$ and $\tau_i : [a,b] \to [a,b]$ are measurable functions, we have the following assertion.

Corollary 2 *Let* (5), (36), (39), *and* (40) *be fulfilled. Let, moreover,*

$$\prod_{i=1}^{n} \lambda_i = 1,$$

and let problem (37), (38) *have only the trivial solution. Then problem* (1), (2) *has at least one solution.*

Namely, for a two-dimensional system of ordinary equations and a particular case of boundary conditions we get the following.

Corollary 3 *Let* $\lambda_1 \lambda_2 = 1$, *and let*

$$u_1' = \widetilde{p}_1(t) |u_2|^{\lambda_1} \operatorname{sgn} u_2, \qquad u_2' = \widetilde{p}_2(t) |u_1|^{\lambda_2} \operatorname{sgn} u_1,$$

$$u_1(a) - c_1 u_1(b) = 0, \qquad u_2(a) - c_2 u_2(b) = 0$$

with $\widetilde{p}_1, \widetilde{p}_2 \in L([a,b];\mathbb{R})$, $c_1, c_2 \in \mathbb{R}$ *have only the trivial solution. Then the problem*

$$u_1' = \widetilde{p}_1(t)|u_2|^{\lambda_1} \operatorname{sgn} u_2 + f_1(t), \qquad u_2' = \widetilde{p}_2(t)|u_1|^{\lambda_2} \operatorname{sgn} u_1 + f_2(t),$$
$$u_1(a) - c_1 u_1(b) = h_1, \qquad u_2(a) - c_2 u_2(b) = h_2$$

has at least one solution for every $f_1, f_2 \in L([a,b];\mathbb{R})$ *and* $h_1, h_2 \in \mathbb{R}$.

The particular case of the system discussed in Corollary 3 is so-called second-order differential equation with λ-Laplacian. Therefore, in the case when $\widetilde{p}_1 \equiv 1$, Corollary 3 yields the following.

Corollary 4 *Let the problem*

$$\big(\Phi_\lambda\big(u'(t)\big)\big)' = p(t)\Phi_\lambda\big(u(t)\big),$$
$$u(a) - c_1 u(b) = 0, \qquad u'(a) - c_2 u'(b) = 0$$

with $p \in L([a,b];\mathbb{R})$, $\Phi_\lambda(x) = |x|^\lambda \operatorname{sgn} x$, $c_1, c_2 \in \mathbb{R}$ *have only the trivial solution. Then the problem*

$$\big(\Phi_\lambda\big(u'(t)\big)\big)' = p(t)\Phi_\lambda\big(u(t)\big) + f(t),$$
$$u(a) - c_1 u(b) = h_1, \qquad u'(a) - c_2 u'(b) = h_2$$

has at least one solution for every $f \in L([a,b];\mathbb{R})$ *and* $h_1, h_2 \in \mathbb{R}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
RH and MZ obtained the results in a joint research. Both authors read and approved the final manuscript.

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Acknowledgements
Robert Hakl has been supported by RVO: 67985840. Manuel Zamora has been supported by Ministerio de Educación y Ciencia, Spain, project MTM2011-23652, and by a postdoctoral grant from University of Granada.

Endnotes
[a] If $x \in \Omega$ then $-x \in \Omega$.
[b] It transforms bounded sets into relatively compact sets.

Received: 30 January 2014 Accepted: 29 April 2014 Published: 13 May 2014

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10.1186/1687-2770-2014-113
Cite this article as: Hakl and Zamora: **Fredholm-type theorem for boundary value problems for systems of nonlinear functional differential equations.** *Boundary Value Problems* 2014, 2014:113